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APPLIED MECHANICS

Thermal Stress Propagation in Visco-Elastic Bodies (I)

by

W. NOWACKI

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Let an instantaneous source of heat act in an infinite visco-elastic space with initial temperature $T = 0$. The action of this source will result in a temperature and stress field. Let us assume that our visco-elastic medium, of linear characteristic, is isotropic and homogeneous. Let us assume also that the deformations are small and that the physical constants are independent of temperature.

The action of the instantaneous heat source will cause dynamic effects. The inertia forces in the equations of equilibrium will be taken into account.

We shall consider visco-elastic bodies, where the relations between the state of stress and strain and the temperature field are given by the following equations [1], [2]:

$$(1.1)\qquad \sigma_{ij}^{(1)}(x_r, t) = 2\int_0^t \mu(t-\tau)\frac{\partial}{\partial\tau}\,\varepsilon_{ij}^{(1)}(x_r,\tau)\,d\tau + \\ + \delta_{ij}\int_0^t \left\{\lambda(t-\tau)\frac{\partial}{\partial\tau}\Theta^{(1)}(x_r,\tau) - [3\lambda(t-\tau)+2\mu(t-\tau)]\,\alpha_t\frac{\partial}{\partial\tau}T(x_r,\tau)\right\}d\tau,$$

$$(1.2)\qquad P_1(D)\,P_3(D)\,\sigma_{ij}^{(2)}(x_r,t) = P_3(D)\,P_2(D)\,\varepsilon_{ij}^{(2)}(x_r,t) + \\ + \delta_{ij}\left\{\frac{1}{3}\,[P_1(D)\,P_4(D) - P_2(D)\,P_3(D)]\,\Theta^{(2)}(x_r,t) - P_1(D)\,P_4(D)\,\alpha_t\,T(x_r,t)\right\}.$$

The relations (1.1) were obtained by M. A. Biot [1] and generalized by D. S. Berry [3] to three-dimensional visco-elastic problems. To these relations temperature terms are added. $\lambda(t), \mu(t)$ are relaxation functions which, in the case of a perfectly elastic body, reduce to the Lamé constants. The operators $P_i\,(D)$ $(i = 1, 2, 3, 4)$ in the Eqs. (1.2) are:

$$(1.3)\qquad P_i(D) = \sum_{n=0}^{N_i} a_i^{(n)}\,D^n, \qquad a_i^{(N_i)} \neq 0\,,$$

where $D^n = \partial^n/\partial t^n$ denotes the n-th derivative with respect to time t. In the Eqs. (1.1) and (1.2), a_t denotes the coefficient of thermal dilatation and δ_{ij} Kronnecker's delta. Let us substitute (1.1), (1.2) in the equations of equilibrium

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \varrho \frac{\partial^2 u_i}{\partial t^2}. \tag{1.4}$$

Expressing the strains in terms of displacement by means of the relation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.5}$$

we obtain the following displacement equations

$$\int_0^t \left\{\mu(t-\tau)\frac{\partial}{\partial \tau}\nabla^2 u_i^{(1)} + [\lambda(t-\tau)+\mu(t-\tau)]\frac{\partial}{\partial \tau}\frac{\partial \Theta^{(1)}}{\partial x_i}\right\}d\tau - \varrho\frac{\partial^2 u_i^{(1)}}{\partial t^2} = \\ = a_t\int_0^t [3\lambda(t-\tau)+2\mu(t-\tau)]\frac{\partial}{\partial \tau}\frac{\partial T}{\partial x_i}d\tau, \tag{1.6}$$

$$P_2(D)P_3(D)\nabla^2 u_i^{(2)} + \frac{1}{3}[2P_4(D)P_1(D)+P_2(D)P_3(D)]\frac{\partial \Theta^{(2)}}{\partial x_i} - \\ - 2P_1(D)P_3(D)\varrho\frac{\partial^2 u_i^{(2)}}{\partial t^2} = 2P_4(D)P_1(D)a_t\frac{\partial T}{\partial x_i}. \tag{1.7}$$

In order to determine the particular integral of the Eqs. (1.6), (1.7) let us introduce the potential of thermoelastic strain Φ, where

$$u_i = \frac{\partial \Phi}{\partial x_i} \qquad l = 1, 2, 3. \tag{1.8}$$

Substituting (1.8) into (1.6) and (1.7), we obtain

$$\int_0^t [2\mu(t-\tau)+\lambda(t-\tau)]\frac{\partial}{\partial \tau}\nabla^2\Phi^{(1)}d\tau - \varrho\frac{\partial^2\Phi^{(1)}}{\partial t^2} = \\ = a_t\int_0^t [3\lambda(t-\tau)+2\mu(t-\tau)]\frac{\partial}{\partial \tau}T d\tau, \tag{1.9}$$

$$\frac{1}{3}[2P_2(D)P_3(D)+P_4(D)P_1(D)]\nabla^2\Phi^{(2)} - P_1(D)P_3(D)\varrho\frac{\partial^2\Phi^{(2)}}{\partial t^2} = \\ = P_1(D)P_4(D)a_t T. \tag{1.10}$$

Expressing also the relations (1.1) and (1.2) by means of Φ and using the Eqs. (1.9) and (1.10), we obtain

$$(1.11)\qquad \sigma_{ij}^{(1)} = \int\limits_0^t 2\mu(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right]\Phi^{(1)}\,d\tau + \delta_{ij}\,\varrho\,\frac{\partial^2\Phi^{(1)}}{\partial t^2},$$

$$(1.12)\qquad P_1(D)\,P_3(D)\,\sigma_{ij}^{(2)} = \\ = P_2(D)\,P_3(D)\left(\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right)\Phi^{(2)} + \delta_{ij}\,P_1(D)\,P_3(D)\,\varrho\,\frac{\partial^2\Phi^{(2)}}{\partial t^2}.$$

Let us apply Laplace's transformation to the relations (1.9), (1.10) and (1.11), (1.12), where

$$\overline{T}(x_r, p) = \int\limits_0^\infty e^{-pt}\,T(x_r, t)\,dt, \qquad \overline{\Phi}(x_r, p) = \int\limits_0^\infty e^{-pt}\,\Phi(x_r, t)\,dt,$$

$$\overline{\sigma}_{ij}(x_r, p) = \int\limits_0^\infty e^{-pt}\,\sigma_{ij}(x_r, t)\,dt;$$

we have

$$(1.13)\qquad \nabla^2\,\overline{\Phi}(x_r, p) - \sigma^2(p)\cdot p^2\,\overline{\Phi} = \vartheta(p)\,\overline{T}(x_r, p),$$

$$(1.14)\qquad \overline{\sigma}_{ij}(x_r, p) = 2\,G(p)\left(\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right)\overline{\Phi}(x_r, p) + \delta_{ij}\,\varrho\,p^2\,\overline{\Phi}(x_r, p),$$

where

$$\sigma^2(p) = p^{-1}\frac{\varrho}{2\mu(p) + \lambda(p)}, \qquad \vartheta(p) = \frac{3\lambda(p) + 2\mu(p)}{\lambda(p) + 2\mu(p)}\,a_t, \qquad G(p) = p\mu(p),$$

for a visco-elastic body for which the relation between the state of stress and that of strain is given by the Eqs. (1.1); and

$$\sigma^2(p) = \frac{3\,P_1(p)\,P_3(p)\,\varrho}{2\,P_2(p)P_3(p) + P_1(p)\,P_4(p)}, \qquad \vartheta(p) = \frac{3\,P_1(p)\,P_4(p)\,a_t}{2\,P_2(p)\,P_3(p) + P_1(p)\,P_4(p)},$$

$$G(p) = \frac{P_2(p)}{2\,P_1(p)}$$

for a visco-elastic body for which the Eqs. (1.2) are valid.

We assume in addition that, for a visco-elastic body for which the relations (1.2) hold, we have

$$\Phi^{(\beta-1)}_{(x_r,0)} = 0 \qquad \text{for} \quad \beta = 1, 2, \dots \max\left[(N_2+N_3),\ (N_1+N_4),\ (N_1+N_3+2)\right].$$

$$T^{(\gamma-1)}_{(x_r,0)} = 0 \qquad \text{for} \quad \gamma = 1, 2, \dots (N_1+N_4).$$

The initial conditions for Φ are, at the same time, initial conditions for the displacements and the stress σ_{ij}.

Introducing the function $\overline{\Psi}(x_r, p) = G(p)\overline{\Phi}(x_r, p)$, we can represent the function (1.14) in the form

$$\bar{\sigma}_{ij} = 2\left(\frac{\partial^2}{\partial x_i\, \partial x_j} - \delta_{ij}\nabla^2\right)\Psi(x_r, p) + \varrho p^2 \overline{\Phi}(x_r, p), \tag{1.15}$$

and the Eq. (1.13) as

$$\nabla^2 \Psi(x_r, p) - \chi^2(p)\, p^2\, \Psi(x_r, p) = h(p)\, \overline{T}(x_r, p), \tag{1.16}$$

where

$$\chi^2(p) = G(p)\,\sigma^2(p), \qquad h(p) = G(p)\,\vartheta(p).$$

Therefore, we should solve the Eqs. (1.13) and (1.16) and then determine the stress from the Eqs. (1.15). After performing the inverse Laplace transformation, we obtain the stresses $\sigma_{ij}(x_r, t)$. In the case of a concentrated instantaneous source of heat in an infinite elastic space we are concerned with spherical symmetry of the temperature field and the state of stress and strain. The temperature field is given by the equation

$$T(R, t) = \frac{Q}{(4\pi\varkappa t)^{3/2}} e^{-R^2/4\varkappa t}, \quad R = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \tag{1.17}$$

where $\varkappa = \lambda/c\varrho$ and λ is the coefficient of heat conduction, ϱ — density and c — specific heat. Next, $Q = W/\varrho c$, where W is the quantity of heat generated by the source of heat per unit of time and volume.

Performing Laplace's transformation in the Eq. (1.17), we obtain

$$\overline{T}(R, p) = \frac{Q}{4\pi R} e^{-R\sqrt{p/\varkappa}}. \tag{1.18}$$

In a system of cylindrical co-ordinates we can express the function $\overline{T}(R, p)$ in the form of the following Hankel-Fourier integral

$$T(r, z; p) = \frac{Q}{2\pi^2\varkappa} \int_0^\infty \int_0^\infty \frac{a J_0(ar) \cos\gamma z}{a^2 + \gamma^2 + p/\varkappa}\, da\, d\gamma. \tag{1.19}$$

The solution of the Eq. (1.13) may be represented in the form [4]:

$$\overline{\Phi}(r, z; p) = -\frac{Q\vartheta(p)}{2\pi^2\varkappa} \int_0^\infty \int_0^\infty \frac{a J_0(ar)\cos\gamma z}{(a^2 + \gamma^2 + p/\varkappa)(a^2 + \gamma^2 + p^2\sigma^2(p))}\, da\, d\gamma, \tag{1.20}$$

or

$$\overline{\Phi}(R, p) = -\frac{Q\vartheta(p)}{4\pi\varkappa p\left[p\sigma^2(p) - \frac{1}{\varkappa}\right] R} \left(e^{-R\sqrt{p/\varkappa}} - e^{-Rp\sigma(p)}\right). \tag{1.21}$$

Let us consider a visco-elastic body, where the relations (1.1) hold. Let us also assume that the functions $\lambda(t)$, $\mu(t)$ are expressed by the following simple exponential function and have the same relaxation time ε^{-1}

$$\lambda(t) = \lambda_0 e^{-\varepsilon t}, \qquad \mu(t) = \mu_0 e^{-\varepsilon t}.$$

In the general case considered we have

$$\vartheta(p) = \frac{3\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}\alpha_t = \vartheta_0 = \text{const.}, \quad \sigma^2(p) = \gamma\frac{p+\varepsilon}{p}, \quad \gamma = \frac{\varrho}{\lambda_0 + 2\mu_0},$$

therefore

$$G(p) = \frac{p\mu_0}{p+\varepsilon},$$

$$\Phi(R, p) = A\left(\frac{1}{p} - \frac{1}{p-\beta}\right)\left(e^{-R\sqrt{p/\varkappa}} - e^{-R\sqrt{\gamma}\sqrt{p(p+\varepsilon)}}\right),$$

$$A = \frac{Q\vartheta_0}{4\pi\varkappa R\gamma\beta}, \qquad \beta = \frac{1}{\varkappa\gamma} - \varepsilon.$$

Performing the inverse Laplace transformation, we obtain

$$\Phi(R,t) = A\left[\operatorname{Erf}\left(\frac{R}{\sqrt{4\varkappa t}}\right) - F(R,t;\beta) - N(R,t) + K(R,t;\varepsilon,\beta)\right], \tag{1.22}$$

where the following notations are introduced

$$N(R,t) = \left[e^{-\frac{\varepsilon R\sqrt{\gamma}}{2}} + \frac{\varepsilon R\sqrt{\gamma}}{2}\int\limits_{R\sqrt{\gamma}}^{t} e^{-\frac{\varepsilon v}{2}} \frac{I_1\left(\frac{\varepsilon}{2}\sqrt{v^2 - R^2\gamma}\right)}{\sqrt{v^2 - R^2\gamma}}\,dv\right]\eta(t - R\sqrt{\gamma}),$$

$$F(R,t;\beta) = \frac{1}{2}e^{\beta t}\left[e^{-R\sqrt{\beta/\varkappa}}\operatorname{Erfc}\left(\frac{R}{\sqrt{4\varkappa t}} - \sqrt{\beta t}\right) + e^{R\sqrt{\beta/\varkappa}}\operatorname{Erfc}\left(\frac{R}{\sqrt{4\varkappa t}} + \sqrt{\beta t}\right)\right],$$

$$K(R,t;\varepsilon,\beta) = \int\limits_0^t h(R, t-\tau)\frac{\partial}{\partial\tau}g(\tau;\varepsilon,\beta)\,dt,$$

where

$$h(R,t) = e^{-\frac{\varepsilon}{2}t} I_0\left(\frac{\varepsilon}{2}\sqrt{t^2 - R^2\gamma}\right)\eta(t - R\sqrt{\gamma}),$$

$$g(t;\varepsilon,\eta) = \int\limits_0^t \frac{1}{\sqrt{\pi(t-\tau)}}\left(\frac{e^{-\varepsilon\tau}}{\sqrt{\pi\tau}} + \sqrt{\varepsilon+\beta}\,e^{\beta\tau}\operatorname{Erf}\sqrt{(\varepsilon+\beta)\tau}\right)d\tau,$$

$$\eta(t - R\sqrt{\gamma}) = \begin{cases} 0 & \text{for} \quad t < R\sqrt{\gamma} \\ 1 & \text{for} \quad t > R\sqrt{\gamma}. \end{cases}$$

In order to determine the stresses we shall use the function

$$\overline{\Psi}(R,t) = G(p)\,\overline{\Phi}(R,p),$$

$$\overline{\Psi}(R,p) = -A_1\left(\frac{1}{p-\beta} - \frac{1}{p+\varepsilon}\right)\left(e^{-R\sqrt{p/\varkappa}} - e^{-R\sqrt{\gamma}\cdot\sqrt{p(p+\varepsilon)}}\right), \tag{1.23}$$

where

$$A_1 = \frac{Q\,\vartheta_0\,\mu_0}{4\,\pi\varkappa R\gamma\,(\beta+\varepsilon)}.$$

Performing the inverse Laplace transformation, we obtain

$$\Psi(R,t) = A_1\,|F(R,t;-\varepsilon) - F(R,t;\beta) + K(R,t;\varepsilon,-\varepsilon) - K(R,t;\varepsilon,\beta)|, \tag{1.24}$$

where

$$F(R,t;-\varepsilon) = \frac{1}{2}\,e^{-\varepsilon t}\left[e^{-iR\sqrt{\varepsilon\varkappa}}\,\mathrm{Erfc}\left(\frac{R}{\sqrt{4\varkappa t}} - i\sqrt{\varepsilon t}\right) + \right.$$
$$\left. + e^{iR\sqrt{\varepsilon\varkappa}}\,\mathrm{Erfc}\left(\frac{R}{\sqrt{4\varkappa t}} + i\sqrt{\varepsilon t}\right)\right],$$
$$g(t;\varepsilon,-\varepsilon) = \int_0^t \frac{R^{-\varepsilon\tau}\,d\tau}{\pi\sqrt{\tau(t-\tau)}}.$$

A knowledge of the functions $\Psi(R,t)$ and $\Phi(R,t)$ enables us to determine the stresses $\sigma_{ij}(R,t)$ by means of the Eqs. (1.15).

If the intensity of the heat source is given by the function $Q_0(t)$, $t > 0$, then, using the function Ψ, we obtain for an instantaneous source the function $\Psi_0(R,t)$ from the equation

$$\Psi_0(R,t) = \frac{1}{Q}\int_0^t Q_0(\tau)\,\Psi(R,t-\tau)\,d\tau. \tag{1.25}$$

The corresponding stress will be found from the Eqs. (1.15) by replacing Ψ by Ψ_0.

Let us consider now the state of stress due to a point source of heat varying in a harmonic manner: $Q_0(t) = Q_0\,e^{i\omega t}$, where $Q_0 = \mathrm{const}$. The displacements and stresses will also vary in a harmonic manner. The relations (1.1), (1.2) take the form

$$\sigma_{ij}^{(1)}(x_r,t) = e^{i\omega t}\,\sigma_{ij}^{(1)*}(x_r) = \tag{1.26}$$
$$= i\omega\,e^{i\omega t}\{2\,\bar{\mu}(i\omega)\,\varepsilon_{ij}^{(1)*}(x_r) + \delta_{ij}\,[\bar{\lambda}(i\omega)\,\Theta^{(1)*} - (3\,\bar{\lambda}(i\omega) + 2\,\bar{\mu}(i\omega))\,a_t\,T^*(x_r)]\},$$

$$P_1(i\omega)\,P_3(i\omega)\,\sigma_{ij}^{(2)*}(x_r) = P_2(i\omega)\,P_3(i\omega)\,\varepsilon_{ij}^{(2)*}(x_r) + \tag{1.27}$$
$$+ \delta_{ij}\left[\frac{1}{3}(P_1(i\omega)\,P_4(i\omega) - P_2(i\omega)\,P_3(i\omega))\,\Theta^{(2)*}(x_r) - P_1(i\omega)\,P_4(i\omega)\,a_t\,T^*(x_r)\right],$$

where

$$\varepsilon_{ij}(x_r, t) = e^{i\omega t}\, \varepsilon_{ij}^*(x_r), \qquad T(x_r, t) = e^{i\omega t}\, T^*(x_r),$$

$$\lambda(i\omega) = \int_0^{\infty} e^{-i\omega t}\, \lambda(t)\, dt, \qquad \bar{\mu}(i\omega) = \int_0^{\infty} e^{-i\omega t}\, \mu(t)\, dt.$$

The Eqs. (1.9) or (1.10) will be reduced to the form

$$\nabla^2 \Phi^*(x_r) - \sigma^2(i\omega)(\omega i)^2 \Phi^*(x_r) = \vartheta(i\omega)\, T^*(x_r), \tag{1.28}$$

where the functions σ, ϑ will be assumed as in the Eq. (1.13), and p will be replaced by $i\omega$.

For the stress amplitudes $\sigma_{ij}^*(x_r)$ we obtain

$$\sigma_{ij}^* = 2\left(\frac{\partial^2}{\partial x_i\, \partial x_j} - \delta_{ij}\nabla^2\right)\Psi^* + \varrho(i\omega)^2 \Phi^*, \qquad \Psi^* = G(i\omega)\,\Phi^*. \tag{1.29}$$

In the particular case of a visco-elastic body, where the relations (1.1) are valid for the functions λ, μ with the same relaxation time ε^{-1}, we have

$$\bar{\mu}(i\omega) = \frac{\mu_0}{\varepsilon + i\omega}, \qquad \lambda(i\omega) = \frac{\lambda_0}{\varepsilon + i\omega}, \qquad \vartheta(i\omega) = \vartheta_0 = \text{const.}$$

$$\sigma^2(i\omega) = \gamma\, \frac{\varepsilon + i\omega}{i\omega}, \qquad \gamma = \frac{\varrho}{\lambda_0 + 2\mu_0}.$$

The functions Φ^* and Ψ^* will be obtained directly from the Eqs. (1.21) and (1.22) by replacing p by $i\omega$. Thus,

$$\Phi^*(R) = A\left(\frac{1}{i\omega} - \frac{1}{i\omega - \beta}\right)\left(e^{-R\sqrt{\frac{i\omega}{\varkappa}}} - e^{-R\sqrt{\gamma i\omega(i\omega+\varepsilon)}}\right), \tag{1.30}$$

$$\Psi^*(R) = -A_1\left(\frac{1}{i\omega - \beta} - \frac{1}{i\omega + \varepsilon}\right)\left(e^{-R\sqrt{\frac{i\omega}{\varkappa}}} - e^{-R\sqrt{\gamma i\omega(i\omega+\varepsilon)}}\right).$$

Knowing the functions Φ^* and Ψ^*, we can determine $\sigma_{ij}^* e^{i\omega t} = \sigma_{ij}(x_r, t)$ from the Eqs. (1.29).

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA, INSTITUTE OF BASIC TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES

(ZAKŁAD MECHANIKI OŚRODKÓW CIĄGŁYCH, INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI PAN)

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