BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

VOLUME VII
NUMÉRO 4

VARSOVIE 1959

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume VII, No. 4 — 1959

APPLIED MECHANICS

Free Vibration and Buckling of a Rectangular Plate with All the Edges Clamped

by

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Presented on January 20, 1959

Introduction

The problem of free vibration and stability of a rectangular plate with all the edges clamped has an ample literature. The accurate solution of the stability problem was obtained by G. I. Taylor [1] and G. H. Faxen [2] and that of the free vibration problem by S. Iguchi [3]. Both authors obtain an infinite system of homogeneous linear equations where the coefficients of the unknowns are functions of an unknown parameter, determining the frequency of free vibration or the critical force.

The determinant of this system set equal to zero constitutes the vibration or buckling condition. The unknowns in the infinite system of equations may be treated as the Fourier coefficients of the clamped moments.

The present paper proposes a different way of solving the above problem. The starting point for the considerations is the deflection of a strip with its edges clamped and loaded by transversal support forces along evenly spaced lines (the distance being $2b$). These forces are chosen so that the deflection along the above lines is zero. This condition leads to Fredholm's integral equation of the first kind. The solution of this problem leads to an infinite system of equations, where the unknowns are the Fourier coefficients of the support reactions. The system of equations thus obtained is characterized by a good convergence.

1. An auxiliary problem

Let a plate strip simply supported on the edges be compressed in the x, y-plane by the forces $q_1 = \sigma_x h$ in the x direction and by the forces $q_2 = \sigma_y h$ in the y-direction. Let concentrated forces, normal to the plate and varying periodically with time ($P = 1 \cdot e^{i\omega t}$), act at the points $(\xi, \pm 2bk)$, $k = 0, 1, 2, ..., \infty$.

The strip undergoes forced vibration. The deflection should satisfy the differential equation

$$(1.1\,a)\qquad \nabla^2\nabla^2 w+\lambda_1^2\frac{\partial^2 w}{\partial x^2}+\lambda_2^2\frac{\partial^2 w}{\partial y^2}+\frac{\mu h}{N}\frac{\partial^2 w}{\partial t^2}=\frac{e^{i\omega t}}{N}\delta(x-\xi)\sum_{k=-\infty}^{k=+\infty}\delta(y+2bk),$$

with the boundary conditions:

$$(1.2)\qquad w=0,\nabla^2 w=0\quad\text{for}\quad x=0,a;-\infty<y<+\infty,$$

$$(1.3)\qquad \frac{\partial w}{\partial y}=0\quad\text{for}\quad y=\pm 2bk,\quad k=0,1,2,\ldots,\infty;\quad 0<x<a,$$

where $w(x,y,t)=e^{i\omega t}\cdot G(x,y)$ is the deflection, $G(x,y)$ — deflection amplitude, N — flexural rigidity, ω — frequency of forced vibration, μ — mass per unit area of the middle surface, δ — Dirac's function, t — time. In addition, we denote $\lambda_1^2=q_1/N$, $\lambda_2^2=q_2/N$.

Since $w(x,y,t)=e^{i\omega t}\cdot G(x,y)$, the Eq. (1.1) and the boundary conditions (1.2) (1.3) may be brought to the form

$$(1.4)\qquad \nabla^2\nabla^2 G+\lambda_1^2\frac{\partial^2 G}{\partial x^2}+\lambda_2^2\frac{\partial^2 G}{\partial y^2}-\varrho^2 G=\frac{1}{N}\delta(x-\xi)\sum_{k=-\infty}^{k=+\infty}\delta(y+2bk),$$

$$\varrho^2=\frac{\mu\omega^2 h}{N},$$

$$(1.5)\qquad G=0,\nabla^2 G=0,\quad\text{for}\quad x=0,a;\quad -\infty<y<+\infty,$$

$$(1.6)\qquad \frac{\partial G}{\partial y}=0\quad\text{for}\quad y=\pm 2bk,\quad k=0,1,2,\ldots,\infty;\quad 0<x<a.$$

The solution of the Eq. (1.1) with the conditions (1.2) (1.3) in the region $0<y<2b$ is the function

$$(1.7)\qquad G(x,y;\xi)=\frac{1}{Na}\sum_{n=1,\ldots}^{\infty}\Gamma_n(y;\lambda_1,\lambda_2,\varrho)\sin\alpha_n\xi\sin\alpha_n x,\ \alpha_n=\frac{n\pi}{a},$$

where

$$(1.8)\qquad \begin{cases}\Gamma_n(y;\lambda_1,\lambda_2,\varrho)=\dfrac{1}{\chi_n^2-\varphi_n^2}\left(\dfrac{\cosh\varphi_n(y-b)}{\varphi_n\sinh\varphi_n b}-\dfrac{\cosh\chi_n(y-b)}{\chi_n\sinh\chi_n b}\right),\\ \varphi_n=\sqrt{\alpha_n^2-\dfrac{\lambda_2^2}{2}-\sqrt{\varrho^2+\lambda_1^2\alpha_n^2+\lambda_2^2\left(\dfrac{\lambda_2^2}{4}-\alpha_n^2\right)}},\\ \chi_n=\sqrt{\alpha_n^2-\dfrac{\lambda_2^2}{2}+\sqrt{\varrho^2+\lambda_1^2\alpha_n^2+\lambda_2^2\left(\dfrac{\lambda_2^2}{4}-\alpha_n^2\right)}}.\end{cases}$$

The surface $G(x, y; \xi)$ may also be represented by the series

$$(1.9) \qquad G(x, y; \xi) = \frac{1}{Nab} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\delta_m \sin \alpha_n \xi \sin \alpha_n x}{D_{nm}} \cos \beta_m y, \beta_m = \frac{m\pi}{b},$$

where

$$D_{nm} = (\alpha_n^2 + \beta_m^2)^2 - \lambda_1^2 \alpha_n^2 - \lambda_2^2 \beta_m^2 - \varrho^2, \quad D_{n0} = \alpha_n^4 - \lambda_1^2 \alpha_n^2 - \varrho^2,$$

$$\delta_m = \begin{cases} 1 & \text{for } m = 0, \\ 2 & \text{for } m = 1, 2, ..., \infty. \end{cases}$$

In subsequent considerations it will be more convenient to replace the system of forces $P = 1 \cdot e^{i\omega t}$ acting at the points $(\xi, \pm 2bk)$ by two systems of half its intensity, symmetric or antisymmetric in relation to the y_1-axis.

Let us denote by $G^{(s)}$ the amplitude of plate deflection due to symmetric forces and by $G^{(a)}$ the amplitude of plate deflection for the load represented in Fig. 1*b*. We have

$$(1.10) \qquad \begin{cases} G^{(s)}(x_1, y_1; \xi_1) = \dfrac{1}{Na} \displaystyle\sum_{n=1,3,...}^{\infty} \Gamma_n(y_1; \lambda_1, \lambda_2, \varrho) \cos \alpha_n \xi_1 \cos \alpha_n x_1, \\ G^{(a)}(x_1, y_1; \xi_1) = \dfrac{1}{Na} \displaystyle\sum_{n=1,3,...}^{\infty} \Gamma_n(y_1; \lambda_1, \lambda_2, \varrho) \sin \alpha_n \xi_1 \sin \alpha_n x_1, \end{cases}$$

$$(1.11) \qquad \begin{cases} G^{(s)}(x_1, y_1; \xi_1) = \dfrac{1}{Nab} \displaystyle\sum_{n=1,3,...}^{\infty} \sum_{m=0,1,...}^{\infty} \dfrac{\delta_m}{D_{mn}} \cos \beta_m y_1 \cos \alpha_n \xi_1 \cos \alpha_n x_1, \\ G^{(a)}(x_1, y_1; \xi_1) = \dfrac{1}{Nab} \displaystyle\sum_{n=1,3,...}^{\infty} \sum_{m=0,1,...}^{\infty} \dfrac{\delta_m}{D_{mn}} \cos \beta_m y_1 \sin \alpha_n \xi_1 \sin \alpha_n x_1. \end{cases}$$

We shall determine the deflection surface of a plate strip loaded by the forces $1/2\, e^{i\omega t}$ symmetric or antisymmetric in relation to the y_1-axis, assuming that the edges are clamped. Let us denote the deflection by

$$w^*(x, y, t) = e^{i\omega t} \cdot G^*(x, y; \xi).$$

The function G^* should satisfy the equation

$$(1.12) \qquad \nabla^2 \nabla^2 G^* + \lambda_1^2 \frac{\partial^2 G^*}{\partial x^2} + \lambda_2^2 \frac{\partial^2 G^*}{\partial y^2} - \varrho^2 G^* = \frac{1}{N} \delta(x-\xi) \sum_{k=-\infty}^{k=+\infty} \delta(y + 2bk)$$

and the boundary conditions

$$(1.13) \qquad \begin{cases} G^* = 0, \ \dfrac{\partial G^*}{\partial x} & \text{for} \quad x = 0, a, \\ \dfrac{\partial G^*}{\partial y} = 0 & \text{for} \quad y = \pm 2bk, \quad k = 0, 1, 2, ... \infty. \end{cases}$$

The deflection G^* will be composed of two parts $G^* = G + W$: the first being the function G and the second a function W chosen in such a way that the equation

$$\nabla^2 \nabla^2 W + \lambda_1^2 \frac{\partial^2 W}{\partial x^2} + \lambda_2^2 \frac{\partial^2 W}{\partial y^2} - \varrho^2 W = 0 \tag{1.14}$$

is satisfied with the boundary conditions

$$\begin{cases} W = 0, \quad \dfrac{\partial}{\partial x}(G + W) = 0 \quad \text{for} \quad x = 0, a, \\ \dfrac{\partial W}{\partial y} = 0 \quad \text{for} \quad y = \pm 2bk, \quad k = 0, 1, 2, \ldots, \infty. \end{cases} \tag{1.15}$$

The function $W^{(s)}$ for symmetric forces (Fig. 1a) is assumed in the form

$$W^{(s)}(x_1, y_1; \xi_1) = \sum_{m=0}^{\infty} A_m^{(s)} \cosh \gamma_m x_1 + B_m^{(s)} \cosh \eta_m x_1) \cos \beta_m y_1, \tag{1.16}$$

where

$$\gamma_m, \eta_m = \sqrt{\beta_m^2 - \frac{\lambda_1^2}{2} \pm \sqrt{\lambda_2^2 \beta_m^2 - \lambda_1^2 \left(\beta_m^2 - \frac{\lambda_1^2}{4}\right) + \varrho^2}}.$$

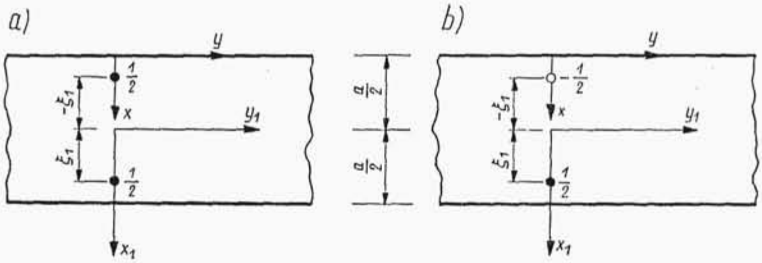

Fig. 1

From the first and the second condition (1.15) we obtain

$$A_m^{(s)} = \frac{1}{Nb} \frac{\varrho_m^{(s)}(\xi_1) \cdot \delta_m}{\Delta_m^{(s)} \cosh \dfrac{\gamma_m a}{2}}, \qquad B_m^{(s)} = -A_m^{(s)} \frac{\cosh \dfrac{\gamma_m a}{2}}{\cosh \dfrac{\eta_m a}{2}}. \tag{1.17}$$

We have

$$\Delta_m^{(s)} = \frac{a\gamma_m}{2} \operatorname{tgh} \frac{\gamma_m a}{2} - \frac{a\eta_m}{2} \operatorname{tgh} \frac{\eta_m a}{2}, \qquad \varrho_m^{(s)}(\xi_1) = \sum_{n=1,3,\ldots}^{\infty} \frac{a_n (-1)^{\frac{n-1}{2}}}{D_{nm}} \cos a_n \xi_1.$$

The surface $G^{*(s)}$ may be described by the equation

$$G^{*(s)}(x_1, y_1; \xi_1) = \frac{1}{Na} \sum_{n=1,3,\ldots}^{\infty} \Gamma_n(y_1; \lambda_1, \lambda_2, \varrho) \cos \alpha_n \xi_1 \cos \alpha_n x_1 + \tag{1.18}$$

$$+ \frac{1}{Nb} \sum_{m=0}^{\infty} \delta_m \varrho_m^{(s)}(\xi_1) F_m^{(s)}(x_1) \cos \beta_m y_1,$$

$$0 \leqslant y_1 \leqslant 2b,$$

where

$$F_m^{(s)} = \frac{1}{\Delta_m^{(s)}} \left(\frac{\cosh \gamma_m x_1}{\cosh \frac{\gamma_m a}{2}} - \frac{\cosh \eta_m x_1}{\cosh \frac{\eta_m a}{2}} \right).$$

For antisymmetric forces in relation to the y_1-axis (Fig. 1*b*) we obtain

$$W^{(a)}(x_1, y_1; \xi_1) = \sum_{m=0}^{\infty} (A_m^{(a)} \sinh \gamma_m x_1 + B_m^{(a)} \sinh \eta_m x_1) \cos \beta_m y_1. \tag{1.19}$$

The constants $A_m^{(a)}$, $B_m^{(a)}$ will be determined from the boundary conditions (1.15)

$$A_m^{(a)} = \frac{1}{Nb} \frac{\varrho_m^{(a)}(\xi_1) \delta_m}{\Delta_m^{(a)} \sinh \frac{\gamma_m a}{2}}, \quad B_m^{(a)} = -A_m^{(a)} \frac{\sinh \frac{\gamma_m a}{2}}{\sinh \frac{\eta_m a}{2}}, \tag{1.20}$$

where

$$\Delta_m^{(a)} = \frac{a\gamma_m}{2} \operatorname{ctgh} \frac{\gamma_m a}{2} - \frac{a\eta_m}{2} \operatorname{ctgh} \frac{\eta_m a}{2}, \quad \varrho_m^{(a)}(\xi_1) = \sum_{n=2,4,\ldots}^{\infty} \frac{\alpha_n (-1)^{\frac{n-2}{2}}}{D_{nm}} \sin \alpha_n \xi_1.$$

Therefore,

$$G^{*(a)}(x_1, y_1; \xi_1) = \frac{1}{Na} \sum_{n=2,4,\ldots}^{\infty} \Gamma_n(y_1; \lambda_1, \lambda_2, \varrho) \sin \alpha_n \xi_1 \sin \alpha_n x_1 + \tag{1.21}$$

$$+ \frac{1}{Nb} \sum_{m=0}^{\infty} \delta_m \varrho_m^{(a)}(\xi_1) F_m^{(a)}(x_1) \cos \beta_m y_1,$$

$$0 \leqslant y_1 \leqslant 2b,$$

where

$$F_m^{(a)}(x_1) = \frac{1}{\Delta_m^{(a)}} \left(\frac{\sinh \gamma_m x_1}{\sinh \frac{\gamma_m a}{2}} - \frac{\sinh \eta_m x_1}{\sinh \frac{\eta_m a}{2}} \right).$$

2. Free vibration and stability of a rectangular plate with its edges clamped

Let the loads $X(\xi)e^{i\omega t}$ act along the straight lines $y = \pm 2bk$, $k = 0, 1, 2, \ldots, \infty$ in the intervals $0 \leqslant x \leqslant a$. The deflection then takes the form

$$w(x, y, t) = e^{i\omega t} \int_0^a X(\xi) G^*(x, y; \xi) d\xi. \tag{2.1}$$

Let us choose a load $X(\xi)$, such that the deflection of the plate is zero along the lines $y = 0$ and $y = 2b$.

$$\int_0^a X(\xi) G^*(x, 0; \xi) d\xi = 0. \tag{2.2}$$

Thus, we have obtained a homogeneous Fredholm equation of the first kind. This equation is the condition of free vibration or buckling.

Let us consider first the case of symmetric deflection. In this case the Eq. (2.3) takes the form

$$\int_0^a X^{(s)}(\xi_1) G^{*(s)}(x_1, 0; \xi_1) d\xi_1 = 0. \tag{2.3}$$

Let us expand the function $F_m^{(s)}(x_1)$ appearing in the expression $G^{*(s)}$ into a Fourier series

$$F_m^{(s)}(x_1) = \sum_{j=1,3,\ldots}^{\infty} L_{mj}^{(s)} \cos a_j x_1, \qquad -\frac{a}{2} < x_1 < \frac{a}{2}. \tag{2.4}$$

We find that

$$L_{mj}^{(s)} = \frac{4}{a\Delta_m^{(s)}} a_j (-1)^{\frac{j-1}{2}} \cdot \frac{\eta_m^2 - \gamma_m^2}{(a_j^2 + \gamma_m^2)(a_j^2 + \eta_m^2)}.$$

Representing the sums in the expression $G^{*(s)}$, we bring them to the form

$$G^{*(s)}(x_1, 0; \xi_1) = \frac{1}{Na} \sum_{n=1,3,\ldots}^{\infty} \Big[\Gamma_n(0; \lambda_1, \lambda_2, \varrho) \cos a_n \xi_1 + \tag{2.5}$$
$$+ \frac{a}{b} \sum_{m=0}^{\infty} L_{mn}^{(s)} \delta_m \sum_{j=1,3,\ldots}^{\infty} H_{mj}^{(s)} \cos a_j \xi_1 \Big] \cos a_n x_1,$$

where

$$H_{mj}^{(s)} = \frac{a_j (-1)^{\frac{j-1}{2}}}{D_{jm}}, \quad D_{jm} \neq D_{mj}.$$

Assuming that the function $X^{(s)}(\xi_1)$ is expressed by the Fourier series

$$\sum_{k=1,3,\ldots}^{\infty} C_k^{(s)} \cos a_k \xi_1 \quad \text{for} \quad -\frac{a}{2} \leqslant \xi_1 \leqslant \frac{a}{2} \tag{2.6}$$

and performing the integrations required in the Eq. (2.3), we obtain the following system of equations

$$(2.7)\quad C_n^{(s)} \Gamma_n(0; \lambda_1, \lambda_2, \varrho) + \frac{a}{b} \sum_{j=1,3,\ldots}^{\infty} C_j^{(s)} \sum_{m=0}^{\infty} \delta_m H_{mj}^{(s)} L_{mn}^{(s)} = 0, \quad n = 1, 3, \ldots, \infty.$$

Setting the determinant of (2.6) equal to zero, we obtain an equation for the frequency of a rectangular plate clamped along the edges $x_1 = \pm a/2$ and $y_1 = 0,\ 2b$ and compressed by uniform forces q_1, q_2. In the particular case $q_1 = q_2 = 0$ we are concerned with free vibration of a rectangular plate without forces acting in its plane. Finally, for $\omega = 0$ we are concerned with the case of plate buckling. In view of a good convergence of the coefficients of $C_j^{(s)}$ it suffices to take only a few equations of the system (2.6).

For deflection antisymmetric in relation to the y_1-axis the Eq. (2.2) takes the form

$$(2.8)\quad \int_0^{a/2} X^{(a)}(\xi_1)\, G^{*(a)}(x_1, 0; \xi_1)\, d\xi_1 = 0.$$

Expanding the function $F_m^{(a)}(x_1)$ in a Fourier series

$$F_m^{(a)}(x_1) = \sum_{j=2,4,\ldots}^{\infty} L_{mj}^{(a)} \sin \alpha_j x_1,$$

where

$$L_{mj}^{(a)} = \frac{4}{a \Delta_m^{(a)}} \alpha_j (-1)^{\frac{j-2}{2}} \frac{\eta_m^2 - \gamma_m^2}{(\alpha_j^2 + \eta_m^2)(\alpha_j^2 + \gamma_m^2)},$$

we reduce the expression $G^{*(a)}(x_1, 0; \xi_1)$ to the form

$$(2.9)\quad G^{*(a)}(x_1, 0; \xi_1) = \frac{1}{Na} \sum_{n=2,4,\ldots}^{\infty} \Big[\Gamma_n(0; \lambda_1, \lambda_2, \varrho) \sin \alpha_n \xi_1 + \frac{a}{b} \sum_{m=0}^{\infty} L_{mn}^{(a)} \delta_m \sum_{j=2,4,\ldots}^{\infty} H_{mj}^{(a)} \sin \alpha_j \xi_1 \Big] \sin \alpha_n x_1,$$

where

$$H_{mj}^{(a)} = \frac{\alpha_j (-1)^{\frac{j-2}{2}}}{D_{jm}}.$$

Assuming that

$$X^{(a)}(\xi_1) = \sum_{k=2,4,\ldots}^{\infty} C_k^{(a)} \sin \alpha_k \xi_1, \qquad |\xi_1| < \frac{a}{2},$$

we reduce the Eq. (2.7) to the form

$$(2.10)\quad C_n^{(a)} \Gamma_n(0; \lambda_1, \lambda_2, \varrho) + \frac{a}{2} \sum_{j=2,4,\ldots}^{\infty} C_j^{(a)} \sum_{m=0}^{\infty} \delta_m H_{mj}^{(a)} L_{mn}^{(a)} = 0, \quad n = 2, 4, \ldots, \infty.$$

Setting the determinant of the system of equations (2.10) equal to zero, we obtain a criterion for free vibration (or buckling, for $\omega = 0$) of a rectangular plate clamped along the edges $x_1 = \pm a/2$, $y = 0, 2b$, where the form of vibration (or buckling, for $\omega = 0$) is antisymmetric in relation to the line $x_1 = 0$ and symmetric in relation to the line $y_1 = b$.

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