BULLETIN

DE

L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

VOLUME VII
NUMÉRO 1

VARSOVIE 1959

BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume VII, No. 1 — 1959

APPLIED MECHANICS

Thermal Stresses in Orthotropic Plates

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Presented on September 8, 1958

In many cases of engineering practice we meet orthogonally anisotropic ("orthotropic") plates, showing different elastic and thermal properties in two orthogonal directions. By E_1 and E_2 we denote Young's moduli in the direction of the x_1 and x_2 axis, respectively, by $\nu = \nu_{12}$ — Poisson's ratio and by $G = G_{12}$ the shear modulus. Finally α_1 and α_2 denote the coefficients of thermal expansion and λ_1, λ_2 coefficients of thermal conductivity in the direction of the x_1 and x_2 axes, respectively The heat equation for an orthotropic plate has the form

$$\lambda_1 \frac{\partial^2 T}{\partial x_1^2} + \lambda_2 \frac{\partial^2 T}{\partial x_2^2} - c\varrho \frac{\partial T}{\partial t} = -W, \tag{1}$$

where c is the specific heat ϱ — the density and W — the rate of heat generated pro unity of volume and time. The relations between stress and strain in the plane state of stress are [1],

$$\begin{cases} \varepsilon_{11} = a_{11}\sigma_{11} + a_{12}\sigma_{12} + \alpha_1 T, \\ \varepsilon_{22} = a_{21}\sigma_{11} + a_{22}\sigma_{22} + \alpha_2 T, \\ \varepsilon_{12} = a_{66}\sigma_{12}, \quad a_{12} = a_{21}, \end{cases} \tag{2}$$

where

$$a_{11} = \frac{1}{E_1}, \quad a_{22} = \frac{1}{E_2}, \quad a_{12} = a_{21} = -\frac{\nu_1}{E_1}, \quad a_{66} = \frac{1}{2G}.$$

Substituting the strains in the compatibility equation

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}, \tag{3}$$

we have

$$\left(a_{11}\frac{\partial^2}{\partial x_2^2} + a_{12}\frac{\partial^2}{\partial x_1^2}\right)\sigma_{11} + \left(a_{12}\frac{\partial^2}{\partial x_2^2} + a_{22}\frac{\partial^2}{\partial x_1^2}\right)\sigma_{22} - \\ - 2a_{66}\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} + \left(\alpha_1 \frac{\partial^2}{\partial x_2^2} + \alpha_2 \frac{\partial^2}{\partial x_1^2}\right) T = 0. \tag{4}$$

Let us express the stresses by means of the Airy function

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 F}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2}, \tag{5}$$

and substitute them in (4). After some simple transformations we obtain the differential equation

$$\frac{\partial^4 F}{\partial x_1^4}\varkappa^4 + 2\eta\varkappa^2 \frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 F}{\partial x_2^4} + E_1\left(a_1 \frac{\partial^2}{\partial x_2^2} + a_2 \frac{\partial^2}{\partial x_1^2}\right) T = 0, \tag{6}$$

where

$$\varkappa^4 = \frac{E_1}{E_2}, \quad 2\eta\varkappa^2 = E_1\left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right).$$

Let us compose the solution of Eq. (6) of two components Φ and Ψ, where the function Ψ is a particular integral of Eq. (6). It therefore satisfies equation

$$\frac{\partial^4 \Psi}{\partial x_1^4}\varkappa^4 + 2\eta\varkappa^2 \frac{\partial^4 \Psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Psi}{\partial x_2^4} + E_1\left(a_1 \frac{\partial^2}{\partial x_2^2} + a_2 \frac{\partial^2}{\partial x_1^2}\right) T = 0, \tag{7}$$

the function Φ satisfying the quasi-biharmonic homogeneous equation

$$\frac{\partial^4 \Phi}{\partial x_1^4}\varkappa^4 + 2\eta\varkappa^2 \frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = 0 \tag{8}$$

and the boundary conditions.

The resulting stresses σ_{ij} will be obtained from the equations

$$\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij} = \left(\nabla^2 \delta_{ij} - \frac{\partial^2}{\partial x_i \partial x_j}\right)(\Psi + \Phi) \qquad i, j = 1, 2.$$

The procedure just described is particularly convenient in the case of boundary conditions expressed in stresses.

If the boundary conditions are given in displacements, the following method is preferable.

We solve the system of Eqs. (2) for stresses

$$\begin{cases} \sigma_{11} = A_{11}\varepsilon_{11} + A_{12}\varepsilon_{22} - \beta_1 T, \\ \sigma_{22} = A_{21}\varepsilon_{11} + A_{22}\varepsilon_{22} - \beta_2 T, \\ \sigma_{12} = 2A_{66}\varepsilon_{12}, \end{cases} \tag{9}$$

where

$$A_{11} = \frac{E_1^2}{E_1 - \nu_1^2 E_2}, \quad A_{22} = \frac{E_1 E_2}{E_1 - \nu_1^2 E_2}, \quad A_{12} = \frac{E_1 E_2 \nu_1}{E_1 - \nu_1^2 E_2}, \quad A_{66} = G,$$

$$\beta_1 = \frac{E_1^2(a_1 + a_2\nu_2)}{E_1 - \nu_1^2 E_2}, \quad \beta_2 = \frac{E_1 E_2(a_2 + a_1\nu_1)}{E_1 - \nu_1^2 E_2}, \quad E_1\nu_2 = E_2\nu_1.$$

Then, we substitute (9) in the equilibrium conditions

$$\sum_j \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad i, j = 1, 2 \tag{10}$$

and express the strains in terms of displacements

$$2\,\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \qquad i, j = 1, 2. \tag{11}$$

Thus, we obtain the system of equations

$$\begin{cases} A_{11} \dfrac{\partial^2 u_1}{\partial x_1^2} + A_{66} \dfrac{\partial^2 u_1}{\partial x_2^2} + (A_{12} + A_{66}) \dfrac{\partial^2 u_2}{\partial x_1 \partial x_2} - \beta_1 \dfrac{\partial T}{\partial x_1} = 0, \\ (A_{12} + A_{66}) \dfrac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{66} \dfrac{\partial^2 u_2}{\partial x_1^2} + A_{22} \dfrac{\partial^2 u_2}{\partial x_2^2} - \beta_2 \dfrac{\partial T}{\partial x_2} = 0. \end{cases} \tag{12}$$

Let us join to this Eq. (1)

$$\lambda_1 \frac{\partial^2 T}{\partial x_1^2} + \lambda_2 \frac{\partial^2 T}{\partial x_2^2} - c\varrho \frac{\partial T}{\partial t} = -W. \tag{13}$$

The system of Eqs. (12), (13) may be expressed in the operational form

$$\sum_{j=1}^{j=3} L_{ij} u_j = -W \delta_{3i} \qquad i = 1, 2, 3, \tag{14}$$

where

$$L_{11} = A_{11} \frac{\partial^2}{\partial x_1^2} + A_{66} \frac{\partial^2}{\partial x_2^2}, \qquad L_{22} = A_{66} \frac{\partial^2}{\partial x_1^2} + A_{22} \frac{\partial^2}{\partial x_2^2},$$

$$L_{12} = L_{21} = (A_{12} + A_{66}) \frac{\partial^2}{\partial x_1 \partial x_2}, \qquad L_{13} = -\beta \frac{\partial}{\partial x_1}, \qquad L_{23} = -\beta \frac{\partial}{\partial x_2},$$

$$L_{33} = \lambda_1 \frac{\partial^2}{\partial x_1^2} + \lambda_2 \frac{\partial^2}{\partial x_2^2} - c\varrho \frac{\partial}{\partial t}, \qquad L_{31} = 0, \qquad L_{32} = 0$$

it being assumed that $u_3 = T$.

The functions u_i $(i = 1, 2, 3)$ may be expressed by means of the three functions χ_i $(i = 1, 2, 3)$, as follows:

$$u_1 = \begin{vmatrix} \chi_1, & L_{12}, & L_{13} \\ \chi_2, & L_{22}, & L_{23} \\ \chi_3, & 0, & L_{33} \end{vmatrix} \qquad u_2 = \begin{vmatrix} L_{11}, & \chi_1, & L_{13} \\ L_{21}, & \chi_2, & L_{23} \\ 0, & \chi_3, & L_{33} \end{vmatrix} \qquad u_3 = \begin{vmatrix} L_{11}, & L_{12}, & \chi_1 \\ L_{21}, & L_{22}, & \chi_2 \\ 0, & 0, & \chi_3 \end{vmatrix}, \tag{15}$$

or, after performing the operations prescribed,

$$
(16)\qquad \begin{cases}
u_1 = L_{33}\left(A_{66}\dfrac{\partial^2}{\partial x_1^2} + A_{22}\dfrac{\partial^2}{\partial x_2^2}\right)\chi_1 - L_{33}(A_{12}+A_{66})\dfrac{\partial^2 \chi_2}{\partial x_1 \partial x_2} + \\
\qquad\qquad + \beta_1 \dfrac{\partial}{\partial x_1}\left(A_{66}\dfrac{\partial^2}{\partial x_1^2} + \bar{A}_{22}\dfrac{\partial^2}{\partial x_2^2}\right)\chi_3, \\
u_2 = -L_{33}(A_{12}+A_{66})\dfrac{\partial^2 \chi_1}{\partial x_1 \partial x_2} + L_{33}\left(A_{11}\dfrac{\partial^2}{\partial x_1^2} + A_{66}\dfrac{\partial^2}{\partial x_2^2}\right) + \\
\qquad\qquad + \beta_2 \dfrac{\partial}{\partial x_2}\left(A_{66}\dfrac{\partial^2}{\partial x_2^2} + \bar{A}_{11}\dfrac{\partial^2}{\partial x_1^2}\right)\chi_3, \\
u_3 = T = A_{22}A_{66}\left(\dfrac{\partial^4}{\partial x_2^4} + 2\sigma\varkappa^2\dfrac{\partial^4}{\partial x_1^2 \partial x_2^2} + \varkappa^4\dfrac{\partial^4}{\partial x_1^4}\right)\chi_3,
\end{cases}
$$

where

$$
\bar{A}_{11} = A_{11} - \frac{\beta_1}{\beta_2}(A_{12}+A_{66}), \qquad \bar{A}_{22} = A_{21} - \frac{\beta_2}{\beta_1}(A_{11}+A_{66}),
$$

$$
\varkappa^4 = \frac{A_{11}}{A_{22}} = \frac{E_1}{E_2}, \qquad \sigma = \frac{A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{66}}{2A_{66}\sqrt{A_{11}A_{22}}}.
$$

The functions χ_i $(i=1,2,3)$ satisfy the operational equation

$$
(17)\qquad \begin{vmatrix} L_{11}, & L_{12}, & L_{13} \\ L_{21}, & L_{21}, & L_{23} \\ 0, & 0, & L_{33} \end{vmatrix} \chi_i = -W\delta_{3i} \qquad i=1,2,3,
$$

or

$$
A_{22}A_{66}L_{33}\left(\mu_1^2\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\mu_2^2\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\chi_i = -W\delta_{3i}, \qquad i=1,2,3,
$$

where

$$
\mu_{1,2}^2 = \varkappa^2 \begin{cases} \sigma \pm \sqrt{\sigma^2 - 1} & \text{for} \quad \sigma > 1, \\ \sigma & \text{for} \quad \sigma = 1, \\ \left(\sqrt{\dfrac{1+\sigma}{2}} \pm \sqrt{\dfrac{1-\sigma}{2}}\right)^2 & \text{for} \quad 0 < \sigma < 1. \end{cases}
$$

The functions χ_1, χ_2 satisfy the homogeneous equation and the function χ_3 — the non-homogeneous differential equation. The functions χ_1, χ_2 are B. G. Galerkin's functions generalized to the case of orthotropy [2].

The solution procedure is as follows. From the Eq. (17″) we determine, for $i=3$, the particular integral χ_3. By means of the functions χ_1, χ_2 we satisfy the given boundary conditions in displacements. Substituting the quantities u_i $(i=1,\ 2)$ in relations (9), we obtain the stresses.

The determination of the thermal stresses is particularly simple for an infinite orthotropic plate. In this particular case it is most convenient to use Eq. (6).

Let the temperature $T_0=$ const. be prescribed in the region of the rectangle of sides c, d in an infinite plate. Let the T temperature outside this region be zero.

The temperature field is represented by the Fourier integral

$$T=\frac{4T_0}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\alpha c\sin\beta d}{\alpha\beta}\cos\alpha x_1\cos\beta x_2\,d\alpha\,d\beta. \tag{18}$$

Representing the Airy function also by means of a Fourier integral

$$F=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty A(\alpha,\beta)\cos\alpha x_1\cos\beta x_2\,d\alpha\,d\beta, \tag{19}$$

and substituting (18) and (19) into (6), we obtain $A(\alpha,\beta)$ and the function F in the form

$$F=\frac{4T_0E_1}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\alpha c\sin\beta d}{\alpha\beta}\,\frac{a_1\beta^2+a_2\alpha^2}{\alpha^4\varkappa^4+2\eta\varkappa^2\alpha^2\beta^2+\beta^4}\cos\alpha x_1\cos\beta x_2\,d\alpha\,d\beta. \tag{20}$$

Bearing in mind that

$$\alpha^4\varkappa^4+2\eta\varkappa^2\alpha^2\beta^2+\beta^4=(\alpha^2\gamma_1^2+\beta^2)(\alpha^2\gamma_2^2+\beta^2),$$

where

$$\gamma_{1,2}^2=\varkappa^2\begin{cases}\eta\pm\sqrt{\eta^2-1} & \text{for}\quad \eta>0,\\ \eta & \text{for}\quad \eta=1,\\ \left(\sqrt{\dfrac{1+\eta}{2}}\pm\sqrt{\dfrac{1-\eta}{2}}\right)^2 & \text{for}\quad 0<\eta<1,\end{cases}$$

and introducing the notations

$$\gamma_3^2=\frac{a_2}{a_1},\qquad a_1=\frac{\gamma_1^2-\gamma_3^2}{\gamma_1^2-\gamma_2^2},\qquad a_2=\frac{\gamma_2^2-\gamma_3^2}{\gamma_2^2-\gamma_1^2},$$

we represent the function F in the form

$$F=\frac{4T_0a_1E_1}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\alpha c\sin\beta d}{\alpha\beta}\left[\frac{a_1}{\gamma_1^2\alpha^2+\beta^2}+\frac{a_2}{\gamma_2^2\alpha^2+\beta^2}\right]\cos\alpha x_1\cos\beta x_2\,d\alpha\,d\beta. \tag{21}$$

The stresses σ_{ij} will be determined by means of Eqs. (5). Thus, for instance, for stresses σ_{12} and σ_{11}, we obtain, after the operations indicated, the following closed expressions

$$(22) \quad \sigma_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2} =$$

$$= -\frac{T_0 a_1 E_1}{2\pi} \left\{ \frac{a_1}{\gamma_1} \ln \frac{[\gamma_1^2 (x_2 + d)^2 + (x_1 - c)^2] [\gamma_1^2 (x_2 - d)^2 + (x_1 + c)^2]}{[\gamma_1^2 (x_2 - d)^2 + (x_1 - c)^2] [\gamma_1^2 (x_2 + d)^2 + (x_1 + c)^2]} + \right.$$

$$\left. + \frac{a_2}{\gamma_2} \ln \frac{[\gamma_2^2 (x_2 + d)^2 + (x_1 - c)^2] [\gamma_2^2 (x_2 - d)^2 + (x_1 + c)^2]}{[\gamma_2^2 (x_2 - d)^2 + (x_1 - c)^2] [\gamma_2^2 (x_2 + d)^2 + (x_1 + c)^2]} \right\},$$

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_2^2} = -\frac{T_0 a_1 E_1}{2\pi} \left\{ a_1 \left[\mathrm{tg}^{-1} \frac{x_1 - c}{\gamma_1 (x_2 - d)} - \mathrm{tg}^{-1} \frac{x_1 - c}{\gamma_1 (x_2 + d)} - \right. \right.$$

$$\left. - \mathrm{tg}^{-1} \frac{x_1 + c}{\gamma_1 (x_2 - d)} + \mathrm{tg}^{-1} \frac{x_1 + c}{\gamma_1 (x_2 + d)} \right] + a_2 \left[\mathrm{tg}^{-1} \frac{x_1 - c}{\gamma_2 (x_2 - d)} - \right.$$

$$\left. \left. - \mathrm{tg}^{-1} \frac{x_1 - c}{\gamma_2 (x_2 + d)} - \mathrm{tg}^{-1} \frac{x_1 + c}{\gamma_2 (x_2 - d)} + \mathrm{tg}^{-1} \frac{x_1 + c}{\gamma_1 (x_2 + d)} \right] \right\},$$

$\mathrm{tg}^{-1} z = \mathrm{arc\,tg}\, z$.

These expressions are valid for $\eta > 1$.

It is evident that, if the "corner" of the rectangle is approached, the stresses σ_{12}, σ_{11} increase indefinitely, and the stresses σ_{11} show discontinuities in the cross-sections $x_2 = \pm d$.

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