BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume VI, Numéro 6

VARSOVIE 1958

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume VI, No. 6 — 1958

APPLIED MECHANICS

Non-Steady State Thermal Stresses in an Infinite Cylinder of Rectangular or Circular Cross-Section

by

W. NOWACKI

Presented on September 4, 1958

Let us consider an infinite cylinder whose cross-section has the form of a rectangle with sides a and b (Fig. 1). Let the temperature T_0 of the cylinder be uniform $T_0 > 0$ at a time $t = 0$. For $t > 0$, let the temperature at the lateral surface be $T = 0$. Such a state will be realized if the cylinder with temperature T_0 is immersed at the time $t = 0$ in a medium with a lower $(T = 0)$ temperature.

The temperature field is described by the differential equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{1}{\varkappa}\frac{\partial}{\partial t}\right) T = 0, \quad \varkappa = \frac{\lambda}{c\varrho}, \tag{1}$$

with the boundary conditions

$$\begin{cases} T\left(-\frac{a}{2}, x_2, t\right) = 0, & T\left(\frac{a}{2}, x_2, t\right) = 0, \\ T\left(x_1, -\frac{b}{2}, t\right) = 0, & T\left(x_1, \frac{b}{2}, t\right) = 0, \end{cases} \tag{2}$$

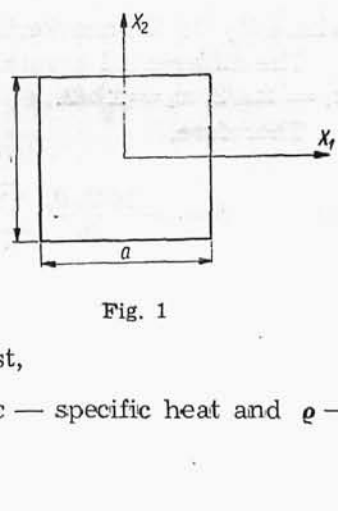

Fig. 1

and the initial condition

$$T(x_1, x_2, 0) = T_0 = \text{const}, \tag{3}$$

where λ is the coefficient of heat conduction, c — specific heat and ϱ — density.

The double series

$$T = \frac{16\, T_0}{ab} \sum_{n,m}^{\infty} \frac{(-1)^{(n+m-2)/2}}{\alpha_n \beta_m} e^{-(\alpha_n^2 + \beta_m^2)\varkappa t} \cos \alpha_n x_1 \cos \beta_m x_2, \tag{4}$$

$$\alpha_n = \frac{n\pi}{a}, \quad \beta_m = \frac{m\pi}{b}, \quad n, m = 1, 3, 5, \ldots, \infty$$

is the solution of Eq. (1) with the conditions (2) and (3).

To find the state of stress σ_{ij}, we shall use the potential of thermoelastic displacement Φ and the Airy function F. The function Φ is related to displacements by the equations

$$\bar{u}_1 = \frac{\partial \Phi}{\partial x_1}, \quad \bar{u}_2 = \frac{\partial \Phi}{\partial x_2}. \tag{5}$$

By substituting the displacements into the displacement equations of the theory of elasticity, we reduce them to the one single equation [1]

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\Phi = \vartheta_0 T, \quad \vartheta_0 = \frac{1+\nu}{1-\nu}\alpha_t, \tag{6}$$

where ν is Poisson's ratio and α_t — the coefficient of thermal dilatation.

The stress components are given by the relations [1]

$$\begin{cases} \sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij} = \left(\dfrac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)(2G\Phi - F), \quad i, j = 1, 2, \\ \sigma_{33} = \bar{\sigma}_{33} + \bar{\bar{\sigma}}_{33} = \nabla^2(\nu F - 2G\Phi), \end{cases} \tag{7}$$

where δ_{ij} is Kronnecker's delta.

The differential equation (6) will be solved by assuming that $\Phi = 0$ for $x_1 = \pm a/2$, $x_2 = \pm b/2$.

Therefore,

$$\Phi = -\frac{16 T_0 \vartheta_0}{ab} \sum_{n,m}^{\infty} \frac{(-1)^{(n+m-2)/2} e^{-\varkappa t(\alpha_n^2 + \beta_m^2)}}{\alpha_n \beta_m (\alpha_n^2 + \beta_m^2)} \cos \alpha_n x_1 \cos \beta_m x_2 \tag{8}$$

$$n, m = 1, 3, \ldots, \infty.$$

Knowing the function Φ, we can determine the stresses $\bar{\sigma}_{ij}$. We obtain then

$$\bar{\sigma}_{11} = -2G\frac{\partial^2 \Phi}{\partial x_2^2} = -\frac{32 T_0 \vartheta_0}{ab} \sum_{n,m}^{\infty} \frac{(-1)^{(n+m-2)/2} \beta_m e^{-(\alpha_n^2 + \beta_m^2)\varkappa t}}{\alpha_n (\alpha_n^2 + \beta_m^2)} \cos \alpha_n x_1 \cos \beta_m x_2,$$

$$\bar{\sigma}_{22} = -2G\frac{\partial^2 \Phi}{\partial x_1^2} = -\frac{32 T_0 \vartheta_0}{ab} \sum_{n,m}^{\infty} \frac{(-1)^{(n+m-2)/2} \alpha_n e^{-(\alpha_n^2 + \beta_m^2)\varkappa t}}{\beta_m (\alpha_n^2 + \beta_m^2)} \cos \alpha_n x_1 \cos \beta_m x_2,$$

$$\bar{\sigma}_{33} = -2G\nabla^2 \Phi = -G\vartheta_0 T,$$

$$(9)\quad \bar{\sigma}_{12} = 2G \frac{\partial^2 \Phi}{\partial x_1 \partial x_2} =$$

$$= -\frac{32\, T_0 \vartheta_0}{ab} \sum_{n,m}^{\infty} \frac{(-1)^{(n+m-2)/2} e^{-(\alpha_n^2+\beta_m^2)\varkappa t}}{\alpha_n^2 + \beta_m^2} \sin \alpha_n x_1 \sin \beta_m x_2,$$

$$\bar{\sigma}_{13} = 0, \quad \bar{\sigma}_{23} = 0.$$

It may be seen that the normal stresses $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ vanish in the cross-sections $x_1 = \pm a/2$, $x_2 = \pm b/2$. The stress $\bar{\sigma}_{12}$ remains different from zero. We have

$$(10)\quad \begin{cases} \bar{\sigma}_{12}\left(\frac{a}{2}, x_2, t\right) = -\frac{32\, GT_0 \vartheta_0}{ab} \sum\limits_{m=1,3,\dots}^{\infty} (-1)^{(m-1)/2} e^{-\beta_m^2 \varkappa t} \varrho(\beta_m, t) \sin \beta_m x_2, \\ \bar{\sigma}_{12}\left(x_1, \frac{b}{2}, t\right) = -\frac{32\, GT_0 \vartheta_0}{ab} \sum\limits_{n=1,3,\dots}^{\infty} (-1)^{(n-1)/2} e^{-\alpha_n^2 \varkappa t} \varrho(\alpha_n, t) \sin \alpha_n x_1, \end{cases}$$

where

$$\varrho(\alpha_n, t) = \sum_{m=1,3,\dots}^{\infty} (\alpha_n^2 + \beta_m^2)^{-1} e^{-\beta_m^2 \varkappa t}, \qquad \varrho(\beta_m, t) = \sum_{n=1,3,\dots}^{\infty} (\alpha_n^2 + \beta_m^2)^{-1} e^{-\alpha_n^2 \varkappa t}.$$

To suppress the stresses $\bar{\sigma}_{12}$ at the lateral surface, we add to the stresses $\bar{\sigma}_{ij}$ stresses $\bar{\bar{\sigma}}_{ij}$ chosen so as to satisfy the following boundary conditions

$$(11)\quad \begin{cases} \bar{\sigma}_{12} + \bar{\bar{\sigma}}_{12} = 0 \quad \text{for} \quad x_1 = \frac{a}{2}, \quad x_2 = \frac{b}{2}, \\ \bar{\sigma}_{11} + \bar{\bar{\sigma}}_{11} = 0 \quad \text{for} \quad x_1 = \frac{a}{2}, \\ \bar{\sigma}_{22} + \bar{\bar{\sigma}}_{22} = 0 \quad \text{for} \quad x_2 = \frac{b}{2}. \end{cases}$$

The stresses $\bar{\bar{\sigma}}_{ij}$ may be expressed, according to (7), in terms of the Airy function. This function should satisfy the biharmonic equation

$$\nabla^2 \nabla^2 F = 0.$$

The Airy function is assumed in the form of the simple series

$$(12)\quad F = \sum_{m=1,3,\dots}^{\infty} \beta_m^{-2} [A_m \cosh \beta_m x_1 + B_m \beta_m x_1 \sinh \beta_m x_1] \cos \beta_m x_2 +$$

$$+ \sum_{n=1,3,\dots}^{\infty} \alpha_n^{-2} [C_n \cosh \alpha_n x_2 + D_n \alpha_n x_2 \sinh \alpha_n x_2] \cos \alpha_n x_1.$$

Substituting F into the boundary conditions (11), we obtain the following system of equations

$$
(13)\quad
\begin{cases}
\text{(a)} \quad A_m \cosh \mu_m + B_m \mu_m \sinh \mu_m = 0, \\
\text{(b)} \quad C_n \cosh \delta_n + D_n \delta_n \sinh \delta_n = 0, \\
\text{(c)} \quad \sum_{m=1,3,\ldots}^{\infty} [(A_m + B_m) \sinh \mu_m + B_m \mu_m \cosh \mu_m] \sin \beta_m x_2 + \\
\qquad + \sum_{n=1,3,\ldots}^{\infty} [(C_n + D_n) \sinh \alpha_n x_2 + D_n \alpha_n x_2 \cosh \alpha_n x_2] (-1)^{(n-1)/2} = \\
\qquad = \frac{32\, G T_0 \vartheta_0}{ab} \sum_{m=1,3,\ldots}^{\infty} (-1)^{(m-1)/2} e^{-\beta_m^2 \varkappa t} \varrho(\beta_m, t) \sin \beta_m x_2, \\
\text{(d} \quad \sum_{m=1,3,\ldots}^{\infty} [(A_m + B_m) \sinh \beta_m x_1 + B_m \beta_m x_1 \cosh \beta_m x_1] (-1)^{(m-1)/2} + \\
\qquad + \sum_{n=1,3,\ldots}^{\infty} [(C_n + D_n) \sinh \delta_n + D_n \delta_n \cosh \delta_n] \sin \alpha_n x_1 = \\
\qquad = \frac{32\, G T_0 \vartheta_0}{ab} \sum_{n=1,3,\ldots}^{\infty} (-1)^{(n-1)/2} e^{-\alpha_n^2 \varkappa t} \varrho(\alpha_n, t) \sin \alpha_n x_1,
\end{cases}
$$

where

$$
\mu_m = \frac{\beta_m a}{2}, \qquad \delta_n = \frac{\alpha_n b}{2}.
$$

Expressing the following functions by means of trigonometric series,

$$
(14)\quad
\begin{cases}
\sinh \alpha_n x_2 = \sum_{m=1,3,\ldots}^{\infty} E_{nm} \sin \beta_m x_2, \quad \alpha_n x_2 \cosh \alpha_n x_2 = \sum_{m=1,3,\ldots}^{\infty} F_{nm} \sin \beta_m x_2, \\
\sinh \beta_m x_1 = \sum_{n=1,3,\ldots}^{\infty} G_{nm} \sin \alpha_n x_1, \quad \beta_m x_1 \cosh \beta_m x_1 = \sum_{n=1,3,\ldots}^{\infty} H_{nm} \sin \alpha_n x_1,
\end{cases}
$$

where

$$
E_{nm} = \frac{4\alpha_n}{b} \frac{(-1)^{(m-1)/2}}{\alpha_n^2 + \beta_m^2} \cosh \delta_n, \qquad G_{nm} = \frac{4\beta_m}{a} \frac{(-1)^{(n-1)/2}}{\alpha_n^2 + \beta_m^2} \cosh \mu_m,
$$

$$
F_{nm} = \frac{4\alpha_n}{b} \frac{(-1)^{(m-1)/2}}{\alpha_n^2 + \beta_m^2} \left[\delta_n \sinh \delta_n - \frac{\alpha_n^2 - \beta_m^2}{\alpha_n^2 + \beta_m^2} \cosh \delta_n \right],
$$

$$
H_{nm} = \frac{4\beta_m}{a} \frac{(-1)^{(n-1)/2}}{\alpha_n^2 + \beta_m^2} \left[\mu_m \sinh \mu_m - \frac{\beta_m^2 - \alpha_n^2}{\beta_m^2 + \alpha_n^2} \cosh \mu_m \right],
$$

we reduce the system of Eqs. (13) to the form:

$$
(15)\quad \begin{cases}
A_m t(\mu_m) + \dfrac{16}{b^2}\beta_m^2 (-1)^{(m-1)/2} \displaystyle\sum_{n=1,3,\dots}^{\infty} \dfrac{C_n (-1)^{(n-1)/2} \cosh^2 \delta_n}{(\alpha_n^2+\beta_m^2)^2 \sinh \delta_n} = \\
\qquad = -\dfrac{32\, T_0 G\vartheta_0}{ab} (-1)^{(m-1)/2} e^{-\beta_m^2 \varkappa t} \varrho(\beta_m, t), \\
C_n t(\delta_n) + \dfrac{16}{a^2}\alpha_n^2 (-1)^{(n-1)/2} \displaystyle\sum_{n=1,3,\dots}^{\infty} \dfrac{A_m (-1)^{(m-1)/2} \cosh^2 \mu_m}{(\alpha_n^2+\beta_m^2)^2 \sinh \mu_m} = \\
\qquad = -\dfrac{32\, T_0 G\vartheta_0}{ab} (-1)^{(n-1)/2} e^{-\alpha_n^2 \varkappa t} \varrho(\alpha_n, t), \\
B_m = -A_m \mu_m^{-1} \operatorname{ctgh} \mu_m, \quad D_n = -C_n \delta_n^{-1} \operatorname{ctgh} \delta_n, \quad n, m = 1, 3, \dots, \infty,
\end{cases}
$$

where

$$
t(\mu_m) = \frac{\sinh 2\mu_m + 2\mu_m}{2\mu_m \sinh \mu_m}, \qquad t(\delta_n) = \frac{\sinh 2\delta_n + 2\delta_n}{2\delta_n \sinh \delta_n}.
$$

We have obtained an infinite system of equations. Confining ourselves to r terms of the series (12), we obtain $2r$ equations for A_m and C_n. The quantities B_m and D_n will be found from the last two equations in (15). Thus, we have obtained an approximate equation for the function F. Using this function we shall determine the stress $\bar{\sigma}_{ij}$ from the equations

$$
(16)\quad \begin{cases}
\bar{\sigma}_{ij} = \left(\delta_{ij} \nabla^2 - \dfrac{\partial^2}{\partial x_i \partial x_j}\right) F, \quad i, j = 1, 2, \\
\bar{\sigma}_{33} = \nu \nabla^2 F.
\end{cases}
$$

Let us consider an infinite circular cylinder of radius a, and then let the temperature field be determined by the differential equation

$$
(17)\quad \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\varkappa}\frac{\partial}{\partial t}\right) T = 0,
$$

with the conditions

$$
(18)\quad T(r, 0) = T_0, \quad T(a, t) = 0 \quad \text{for} \quad t > 0.
$$

The series

$$
(19)\quad T(r, t) = \frac{2T_0}{a} \sum_{1}^{\infty} e^{-\alpha_n^2 \varkappa t} \frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n a)}
$$

is the solution of Eq. (17) with conditions (18), where $\alpha_n a$ are the roots of the equation $J_0(\alpha_n a) = 0$.

By means of the function of thermoelastic displacement Φ the displacement equations of the theory of elasticity may be reduced to the form, [1]:

$$
(20)\quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) \Phi = \vartheta_0 T.
$$

Assuming that $\Phi(a, t) = 0$, the solution of Eq. (20) is

$$\Phi = -2 T_0 \vartheta_0 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \varkappa t} J_0(\alpha_n r)}{a \alpha_n^3 J_1(\alpha_n a)}. \tag{21}$$

A knowledge of the function Φ enables us to find the stresses $\bar{\sigma}_{ij}$. Thus,

$$\left\{\begin{aligned} \bar{\sigma}_{rr} &= -2G \frac{1}{r} \frac{\partial \Phi}{\partial r} = -4 T_0 G \vartheta_0 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \varkappa t}}{\alpha_n a J_1(\alpha_n a)} \frac{J_1(\alpha_n r)}{\alpha_n r}, \\ \bar{\sigma}_{\varphi\varphi} &= -2G \frac{\partial^2 \Phi}{\partial r^2} = -4 T_0 G \vartheta_0 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \varkappa t}}{\alpha_n a J_1(\alpha_n a)} \left(J_0(\alpha_n r) - \frac{J_1(\alpha_n r)}{\alpha_n r} \right), \\ \bar{\sigma}_{zz} &= -2G \nabla^2 \Phi = -2G \vartheta_0 T = \bar{\sigma}_{rr} + \bar{\sigma}_{\varphi\varphi}. \end{aligned}\right. \tag{22}$$

It may be seen that the stress $\bar{\sigma}_{rr}(a, t)$ is different from zero at the surface $r = a$. To suppress the stresses $\bar{\sigma}_{rr}(a, t)$, we add to the state of stress $\bar{\sigma}_{ij}$ the state of stress $\bar{\bar{\sigma}}_{ij}$ determined by the relations

$$\bar{\bar{\sigma}}_{rr} = \bar{\bar{\sigma}}_{\varphi\varphi} = 4 G T_0 \vartheta_0 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \varkappa t}}{(\alpha_n a)^2}, \quad \bar{\bar{\sigma}}_{zz} = \nu (\bar{\bar{\sigma}}_{rr} + \bar{\bar{\sigma}}_{\varphi\varphi}). \tag{23}$$

Finally,

$$\left\{\begin{aligned} \sigma_{rr} &= -4 G T_0 \vartheta_0 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \varkappa t}}{\alpha_n a J_1(\alpha_n a)} \left(\frac{J_1(\alpha_n r)}{\alpha_n r} - \frac{J_1(\alpha_n a)}{\alpha_n a} \right), \\ \sigma_{\varphi\varphi} &= -4 G T_0 \vartheta_0 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \varkappa t}}{\alpha_n a J_1(\alpha_n a)} \left(J_0(\alpha_n r) - \frac{J_1(\alpha_n r)}{\alpha_n r} - \frac{J_1(\alpha_n a)}{\alpha_n a} \right). \\ \sigma_{zz} &= -4 G T_0 \vartheta_0 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \varkappa t}}{\alpha_n a J_1(\alpha_n a)} \left(J_0(\alpha_n r) - 2\nu \frac{J_1(\alpha_n a)}{\alpha_n a} \right). \end{aligned}\right. \tag{24}$$

For $r = 0$, we have

$$\sigma_{rr}(0, t) = \sigma_{\varphi\varphi}(0, t).$$

It follows from Eqs. (24) that the maximum stress is reached for $t = 0$. With increasing time t the stresses decrease and tend to zero for $t \to \infty$.

The problem of sudden cooling of a rectangular plate or a circular plate may be solved in an analogous manner. For the solution of this problem, we can use the above results by replacing ϑ_0 by $\vartheta_0' = (1 + \nu) \alpha_t / h$, where h is the plate thickness, and by assuming that the stress σ_{zz} is equal to zero.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA, INSTITUTE OF BASIC TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] E. Melan, H. Parcus, *Wärmespannungen infolge stationärer Temperaturfelder*, Vienna, 1953.