BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume VI, Numéro 6

VARSOVIE 1958

BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume VI, No. 6 — 1958

APPLIED MECHANICS

The State of Stress in an Elastic Slab Due to a Steady Heat Source

by

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Presented on September 8, 1958

Let us consider a steady concentrated heat source of intensity W acting at a point $(0, \xi')$ in a homogeneous elastic slab of thickness h (Fig. 1). Let us assume that the temperature in the planes $z' = 0, h$ is equal to zero. The temperature field will be obtained by solving the differential equation

$$(1) \quad \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z'^2} = -\frac{W}{\lambda}\frac{\delta(r)}{2\pi r}\,\delta(z'-\zeta')$$

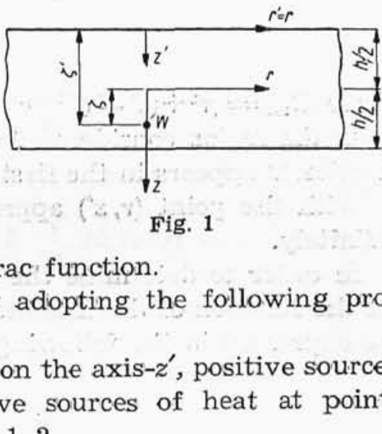

Fig. 1

with the boundary conditions

$$(2) \qquad T(r, 0) = 0, \quad T(r, h) = 0,$$

where T denotes the temperature, λ — the coefficient of heat transfer and δ — the Dirac function.

The conditions (2) may be satisfied by adopting the following procedure.

We consider in the infinite elastic space, on the axis-z', positive sources of heat at points $(0, \zeta \pm 2nh)$ and negative sources of heat at points $(0, -\zeta \pm 2nh)$ — in the case considered $n = 1, 2, \ldots,$

Then, the values of the temperature in the planes $z' = 0$, $z' = +h$, $z' = +2h$ will be zero. Assuming that the Eq. (1) concerns the infinite space with heat sources located as above, we may represent the right-hand member of Eq. (1) in the form of a Fourier series and a Hankel integral:

$$(3) \qquad \frac{W}{2\pi\lambda r}\,\delta(r)\,\delta(z'-\zeta') = \frac{W}{\pi h\lambda}\int_0^\infty \alpha J_0(\alpha r)\,d\alpha \sum_{n=1}^{\infty} \sin\alpha_n\zeta' \sin\alpha_n z', \qquad \alpha_n = \frac{n\pi}{h}.$$

The temperature is expressed in the following way:

$$(4) \qquad T = \sum_{n=1}^{\infty} \sin\alpha_n z \int_0^\infty C_n(\alpha)\, J_0(\alpha_n r)\, d\alpha.$$

Substituting Eqs. (3) and (4) into (1), we obtain

$$C_n(a) = \frac{W}{\pi h \lambda} \sin a_n \zeta' \frac{a}{a_n^2 + a^2}.$$

Therefore,

$$(5) \qquad T(r, z') = \frac{W}{\pi h \lambda} \sum_{n=1}^{\infty} \sin a_n \zeta' \sin a_n z' \int_0^{\infty} a J_0(ar) (a_n^2 + a^2)^{-1} \, da$$

or

$$(6) \qquad T(r, z') = \frac{W}{\pi h \lambda} \sum_{n=1}^{\infty} K_0(a_n r) \sin a_n \zeta' \sin a_n z', \qquad 0 < z' < h,$$

where $K_0(a_n r)$ is a modified Bessel function of the third kind and zero order. The last sum may be represented in a somewhat different form

$$(7) \qquad T(r, z') = \frac{W}{4\pi\lambda} \Bigg[R_1^{-1} - R_2^{-1} + \sum_{n=1}^{\infty} \{ [R_1^2 + 4nh(nh - z' + \zeta')]^{-1/2} -$$
$$- [R_2^2 + 4nh(hn - z' - \zeta')]^{-1/2} \} + \sum_{n=1}^{\infty} \{ [R_1^2 + 4nh(nh + z' - \zeta')]^{-1/2} -$$
$$- [R_2^2 + 4nh(nh + z' + \zeta')]^{-1/2} \} \Bigg],$$

where $R_{1,2} = [r^2 + (z' \mp \zeta')^2]^{1/2}$.

In the region considered $0 < z' < h$, $0 < r < \infty$, a discontinuity of the function T appears in the first term of the expression (7).

With the point (r, z') approaching $(0, \xi')$ the quantity T increases indefinitely.

In order to determine the stress components it will be convenient to use the function of thermoelastic displacement Φ, where the displacements are expressed in the following manner:

$$(8) \qquad u_r = \frac{\partial \Phi}{\partial r}, \qquad u_{z'} = \frac{\partial \Phi}{\partial z'}.$$

The introduction of these relations into the displacement equations of the theory of elasticity reduces them to the one single equation

$$(9) \qquad \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z'^2} = \vartheta_0 T, \qquad \vartheta_0 = \frac{1+\nu}{1-\nu} a_t,$$

where ν is Poisson's ratio and a_t — the coefficient of thermal dilatation.

The solution of Eq. (9) will be assumed in the form

$$(10) \qquad \Phi = \sum_{n=1}^{\infty} \sin a_n z' \int_0^{\infty} E_n(a) J_0(ar) \, da.$$

It is assumed that $\Phi = 0$ for $z' = 0$ and $z' = h$.
Substituting Eqs. (5) into (9), we obtain $E_n(a)$. Therefore,

$$\Phi(r, z') = -\frac{W\vartheta_0}{\pi h \lambda} \sum_{n=1}^{\infty} \sin a_n \zeta' \sin a_n z' \int_0^{\infty} (a_n^2 + a^2)^{-2} a J_0(ar)\, da.$$

Bearing in mind that

$$\int_0^{\infty} (a_n + a^2)^{-2} a J_0(ar)\, da = \frac{r}{2a_n} K_{-1}(a_n r) = \frac{r}{2a_n} K_1(a_n r),$$

we find that

$$\Phi = -\frac{W\vartheta_0 r}{2\pi h \lambda} \sum_{n=1}^{\infty} a_n^{-1} K_1(a_n r) \sin a_n \zeta' \sin a_n z'. \tag{11}$$

Knowing the function Φ, we may determine the stresses and the displacements from the relations [1]

$$\overline{\sigma}_{ij} = 2G\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij} \nabla^2\right)\Phi \qquad i, j = 1, 2, 3. \tag{12}$$

Thus,

$$\left\{\begin{aligned}
\overline{\sigma}_{r'z'} &= 2G\frac{\partial^2 \Phi}{\partial r \partial z'} = P\sum_{n=1}^{\infty} a_n r K_0(a_n r) \sin a_n \zeta' \cos a_n z', \\
\overline{\sigma}_{r'r'} &= -2G\left(\frac{\partial^2 \Phi}{\partial z'^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}\right) = -P\sum_{n=1}^{\infty} [r a_n K_1(a_n r) + \\
&\qquad + K_0(a_n r)] \sin a_n \zeta' \sin a_n z', \\
\overline{\sigma}_{\varphi'\varphi'} &= -2G\left(\frac{\partial^2 \Phi}{\partial z'^2} + \frac{\partial^2 \Phi}{\partial r^2}\right) = -P\sum_{n=1}^{\infty} K_0(a_n r) \sin a_n \zeta' \sin a_n z', \\
\overline{\sigma}_{z'z'} &= -2G\left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}\right) = -P\sum_{n=1}^{\infty} [2K_0(a_n r) - \\
&\qquad - a_n r K_1(a_n r)] \sin a_n \zeta' \sin a_n z',
\end{aligned}\right. \tag{13}$$

where

$$P = \frac{GW\vartheta_0}{\pi h \lambda}.$$

It is also possible to separate the singular parts of the expressions for stresses, bearing in mind that the sum

$$\sum_{n=1}^{\infty} K_0(a_n r) \sin a_n \zeta' \sin a_n z'$$

is expressed by Eq. (7) of which both members may be multiplied by $\pi h \lambda / W$, and the sum

$$\sum_{n=1}^{\infty} a_n K_1(a_n r) \sin a_n \zeta' \sin a_n z' = -\frac{\partial}{\partial r} \sum_{n=1}^{\infty} K_0(a_n r) \sin a_n \zeta' \sin a_n z'.$$

It may be seen that, in the planes $z'=0$, $z'=h$ the normal stress $\bar{\sigma}_{z'z'}$ is equal to zero and the stress $\sigma_{r'z'}$ is different from zero. Therefore, over the state of stress σ_{ij} a state $\bar{\bar{\sigma}}_{ij}$ should be superposed, such that the following boundary conditions be satisfied at the boundaries $z'=0$, $z'=h$.

$$\bar{\sigma}_{r'z'}+\bar{\bar{\sigma}}_{r'z'}=0, \quad \bar{\sigma}_{z'z'}+\bar{\bar{\sigma}}_{z'z'}=0. \tag{14}$$

It will be more convenient for further considerations to represent the stresses $\bar{\sigma}_{r'z'}$ in the cross-sections $z'=0$, $z'=h$, using the expressions (11) for the function Φ

$$\left\{\begin{aligned} \bar{\sigma}_{r'z'}(r,0) &= 2G\frac{\partial^2 \Phi}{\partial r\,\partial z'}\bigg|_{z'=0} = 2P\int_0^{\infty} a^2 J_1(ar)\sum_{n=1}^{\infty}\frac{a_n \sin a_n\zeta'}{(a^2+a^2)^2}\,da,\\ \bar{\sigma}_{z'r'}(r,h) &= 2G\frac{\partial^2 \Phi}{\partial r\,\partial z'}\bigg|_{z'=h} = 2P\int_0^{\infty} a^2 J_1(ar)\sum_{n=1}^{\infty}\frac{a_n(-1)^n \sin a_n\zeta'}{(a_n^2+a^2)^2}\,da.\end{aligned}\right. \tag{15}$$

Bearing in mind that

$$\sum_{n=1}^{\infty}\frac{a_n \sin a_n\zeta'}{(a_n^2+a^2)^2}=\frac{h^3}{4}\eta_1(a,\zeta'),$$

$$\sum_{n=1}^{\infty}\frac{a_n(-1)^n \sin a_n\zeta'}{(a_n^2+a^2)^2}=\frac{h^3}{4}\eta_2(a,\zeta'),$$

where

$$\eta_1(a,\zeta')=\frac{a\zeta'\sinh\beta\cosh a(h-\zeta')-\beta\sinh a\zeta'}{\beta^2\sinh^2\beta},$$

$$\eta_2(a,\zeta')=\frac{a\zeta'\sinh\beta\cosh a\zeta'-\beta\cosh\beta\sinh a\zeta'}{\beta^2\sinh^2\beta}, \quad \beta=ah,$$

the stresses $\bar{\sigma}_{r'z'}(r,0)$, $\sigma_{z'z'}(r,h)$ may be expressed in the form of Hankel's integrals

$$\left\{\begin{aligned} \bar{\sigma}_{r'z'}(r,0) &= \frac{Ph^3}{2}\int_0^{\infty} a^2 J_1(ar)\,\eta_1(a,\zeta')\,da.\\ \bar{\sigma}_{r'z'}(r,0) &= \frac{Ph^3}{2}\int_0^{\infty} a^2 J_1(ar)\,\eta_2(a,\zeta')\,da.\end{aligned}\right. \tag{16}$$

In further considerations it will be convenient to replace the heat source by two systems of two sources each, of intensity equal to half the above value, located in a symmetric and anti-symmetric manner in relation to the plane $z'=h/2$ (Fig. 2 a, b).

Let us introduce a new system of co-ordinates r, z and consider the action of two heat sources located symmetrically in relation to the plane $z=0$. In the cross-section $z=h/2$, we obtain

$$(17)\quad \begin{cases} \bar{\sigma}_{rz}^{(s)} = \frac{Ph^3}{4} \int\limits_0^\infty a^2 J_1(ar) \left[\eta_2 \left(a, \frac{h}{2} + \zeta \right) + \eta_2 \left(a, \frac{h}{2} - \zeta \right) \right] da = \\ \qquad = \frac{Ph^3}{16} \int\limits_0^\infty a^2 J_1(ar) \varrho^{(s)}(\mu, \zeta)\, da, \\ \bar{\sigma}_{zz}^{(s)} = 0, \end{cases}$$

where

$$\varrho^{(s)}(\mu, \zeta) = \frac{a\zeta \cosh \mu \sinh a\zeta - \mu \sinh \mu \cosh a\zeta}{\mu^2 \cosh^2 \mu}, \quad \mu = \frac{ah}{2}.$$

We assume that the planes $z = \pm h/2$ are free from stress. The state of stress $\sigma_{ij}^{(s)}$ satisfies only some of the boundary conditions. To the state $\bar{\sigma}_{ij}^{(s)}$ a state of stress $\bar{\sigma}_{ij}^{(s)}$ should be added, such that the following boundary conditions be satisfied at the $z = \pm h/2$ planes

$$(18)\quad \bar{\sigma}_{zr}^{(s)} + \bar{\sigma}_{rz}^{(s)} = 0, \quad \bar{\sigma}_{zz}^{(s)} + \bar{\bar{\sigma}}_{zz}^{(s)} = 0.$$

The state of stress $\bar{\bar{\sigma}}_{ij}^{(s)}$ will be expressed by means of Love's function φ [2] satisfying the biharmonic equation. The state of stress $\bar{\bar{\sigma}}_{ij}$ is determined by the equations

$$(19)\quad \begin{cases} \bar{\sigma}_{rr} = \frac{2G}{1-2\nu} \frac{\partial}{\partial z} \left(\nu \nabla^2 - \frac{\partial^2}{\partial r^2} \right) \varphi, \\ \bar{\sigma}_{\varphi\varphi} = \frac{2G}{1-2\nu} \frac{\partial}{\partial z} \left(\nu \nabla^2 - \frac{1}{r} \frac{\partial}{\partial r} \right) \varphi, \\ \bar{\sigma}_{zz} = \frac{2G}{1-2\nu} \frac{\partial}{\partial z} \left[(2-\nu) \nabla^2 - \frac{\partial^2}{\partial z^2} \right] \varphi, \\ \bar{\sigma}_{rz} = \frac{2G}{1-2\nu} \frac{\partial}{\partial \nu} \left[(1-\nu) \nabla^2 - \frac{\partial^2}{\partial z^2} \right] \varphi. \end{cases}$$

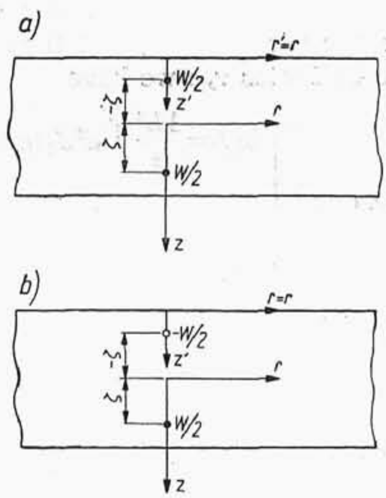

Fig. 2

The function $\varphi^{(s)}$ will be assumed in the form of Hankel's integral

$$(20)\quad \varphi^{(s)} = \int\limits_0^\infty a^{-2} J_0(ar) \left[A \sinh az + Baz \cosh az \right] da.$$

The quantities A, B will be determined from the boundary conditions (18). We obtain then

$$(21)\quad \begin{cases} A(a) = (1 - 2\nu - \mu \operatorname{tgh} \mu) B(a), \\ B(a) = -\frac{Ph^3}{16G} (1-2\nu) a \varrho^{(s)}(a, \zeta) \frac{\cosh \mu}{\sinh 2\mu + 2\mu}. \end{cases}$$

Using the Eqs. (19), we express the stress components $\bar{\bar{\sigma}}_{ij}^{(s)}$ by the integrals

$$(22)\quad \begin{cases} \bar{\bar{\sigma}}_{rr}^{(s)} = \dfrac{2G}{1-2\nu}\displaystyle\int_0^\infty \alpha B(\alpha)\{[\alpha z \sinh \alpha z + (2-\mu \operatorname{tgh}\mu)\cosh \alpha z]\, J_0(\alpha r) - \\ \qquad - [(2-2\nu-\mu\operatorname{tgh}\mu)\cosh \alpha z + \alpha z \sinh \alpha z]\, J_1(\alpha r)\}\, d\alpha, \\ \bar{\bar{\sigma}}_{\varphi\varphi}^{(s)} = \dfrac{2G}{1-2\nu}\displaystyle\int_0^\infty \alpha B(\alpha)\Big\{2\nu \cosh \alpha z J_0(\alpha r) + \\ \qquad + [(2-2\nu-\mu\operatorname{tgh}\mu)\cosh \alpha z + \alpha z \sinh \alpha z]\, \dfrac{J_1(\alpha r)}{\alpha r}\Big\}\, d\alpha, \\ \bar{\bar{\sigma}}_{zz}^{(s)} = -\dfrac{2G}{1-2\nu}\displaystyle\int_0^\infty \alpha B(\alpha)\,[\alpha z \sinh \alpha z - \mu \operatorname{tgh}\mu \cosh \alpha z]\, J_0(\alpha r)\, d\alpha, \\ \bar{\bar{\sigma}}_{rz}^{(s)} = \dfrac{2G}{1-2\nu}\displaystyle\int_0^\infty \alpha B(\alpha)\,[(1-\mu\operatorname{tgh}\mu)\sinh \alpha z + \alpha z \cosh \alpha z]\, J_1(\alpha r)\, d\alpha. \end{cases}$$

Finally, the stresses $\sigma_{ij}^{(s)}$ will be found by superposition

$$(23)\qquad \sigma_{ij}^{(s)} = \bar{\sigma}_{ij}^{(s)} + \bar{\bar{\sigma}}_{ij}^{(s)}.$$

Let us consider the case of heat sources anti-symmetric in relation to the plane $z=0$; and then let us observe that in the new system of co-ordinates r, z we have

$$(24)\quad \begin{cases} \bar{\sigma}_{rz}^{(a)} = \dfrac{Ph^3}{4}\displaystyle\int_0^\infty \alpha^2 J_1(\alpha r)\left[\eta_2\left(\alpha, \dfrac{h}{2}+\zeta\right) - \eta_2\left(\alpha, \dfrac{h}{2}-\zeta\right)\right] d\alpha = \\ \qquad = \dfrac{Ph^3}{16}\displaystyle\int_0^\infty \alpha^2 J_1(\alpha r)\, \varrho^{(a)}(\mu,\zeta)\, d\alpha, \\ \bar{\sigma}_{zz}^{(a)} = 0, \end{cases}$$

where

$$\varrho^{(a)}(\mu,\zeta) = \frac{\alpha\zeta \sinh \mu \cosh \alpha\zeta - \mu \cosh \mu \sinh \alpha\zeta}{\mu^2 \sinh^2 \mu}.$$

Love's function $\varphi^{(a)}$ will be assumed in the form

$$(25)\qquad \varphi^{(a)} = \int_0^\infty \alpha^{-2} J_0(\alpha r)\,[C \cosh \alpha z + D\alpha z \sinh \alpha z]\, d\alpha.$$

The quantities C and D will be found from the boundary conditions (18)

$$(26)\quad \frac{2G}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi^{(a)}\left(r, \frac{h}{2}\right) + \frac{Ph^3}{16}\int_0^\infty \alpha^2 J_1(\alpha r)\, \varrho^{(a)}(\mu,\zeta)\, d\alpha = 0,$$

$$\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi^{(a)}\left(r, \frac{h}{2}\right) = 0.$$

Hence, we obtain

$$C(a) = (1 - 2\nu - \mu \operatorname{ctgh} \mu)\, D(a),$$

$$D(a) = -\frac{Ph^3}{16G}(1 - 2\nu)\, a\varrho^{(a)}(\mu, \zeta) \frac{\sinh \mu}{\sinh 2\mu - 2\mu}.$$

The stress components $\bar{\bar{\sigma}}_{ij}^{(a)}$ are expressed by the equations

$$(27)\quad \begin{cases} \bar{\bar{\sigma}}_{rr}^{(a)} = \dfrac{2G}{1-2\nu} \displaystyle\int_0^\infty aD(a) \left\{ \left[2\nu \sinh az J_0(ar) + az \cosh az \left(J_0(ar) - \right.\right.\right. \\ \qquad \left.\left. \left. - \dfrac{J_1(ar)}{ar} \right) \right] + (2 - 2\nu - \mu \operatorname{ctgh} \mu) \sinh az \left(J_0(ar) - \dfrac{J_1(ar)}{ar} \right) \right\} da, \\ \bar{\bar{\sigma}}_{\varphi\varphi}^{(a)} = \dfrac{2G}{1-2\nu} \displaystyle\int_0^\infty aD(a) \left\{ \left[2\nu \sinh az J_0(ar) + az \cosh az \dfrac{J_1(ar)}{ar} \right] + \right. \\ \qquad \left. + (2 - 2\nu - \mu \operatorname{ctgh} \mu) \sinh az \dfrac{J_1(ar)}{ar} \right\} da, \\ \bar{\bar{\sigma}}_{zz}^{(a)} = -\dfrac{2G}{1-2\nu} \displaystyle\int_0^\infty aD(a) J_0(ar) [az \cosh az - \mu \operatorname{ctgh} \mu \sinh az]\, da, \\ \bar{\bar{\sigma}}_{rz}^{(a)} = \dfrac{2G}{1-2\nu} \displaystyle\int_0^\infty aD(a) J_1(ar) [az \sinh az + (1 - \mu \operatorname{ctgh} \mu) \cosh az]\, da. \end{cases}$$

The resultant stresses $\sigma_{ij}^{(a)}$ will be obtained by adding $\bar{\sigma}_{ij}^{(a)}$ to $\bar{\bar{\sigma}}_{ij}^{(a)}$.

The case considered above was that of the planes $z = \pm h/2$ is $T = 0$. There is no obstacle for assuming other boundary conditions.

If we arrange the heat sources, as shown in Fig. 3a, we obtain $\partial T/\partial z' = 0$ in the planes $z = 0, \pm h, \pm 2h, \pm 3h$ etc. If they are located as shown in Fig. 3b, we obtain $\partial T/\partial z' = 0$ in the planes $z' = 0$, $z' = \pm 2h$, $z' = \pm 4h$ and $T = 0$ in the planes $z' = \pm h, \pm 3h$ etc. Since such an arrangement of heat sources may be represented by a Fourier series in the direction of the z'-axis, it becomes evident that the right-hand member of the heat equation may be represented by a Fourier-Hankel series.

The solution procedure is, in these two cases, analogous to that for the boundary conditions $T = 0$ for $z' = 0, h$. It should be observed that, in the case of the boundary conditions $\partial T/\partial z' = 0$ for $z' = 0, h$, the stresses $\bar{\sigma}_{r'z'}$ vanish in the planes bounding the plate. The stresses $\bar{\sigma}_{z'z'}$ will be different from zero. For the boundary conditions represented in Fig. 3b, we have

$$\bar{\sigma}_{r'z'}(r, 0) = 0, \quad \bar{\sigma}_{z'z'}(r, 0) \neq 0, \quad \bar{\sigma}_{r'z'}(r, h) \neq 0, \quad \bar{\sigma}_{z'z'}(r, h) = 0.$$

Let us observe finally that a procedure analogous to the above may be used to solve the axially symmetric problem of determining the thermal stress due to the action of a uniformly distributed heat source (of intensity $W = \text{const.}$) along the circle of radius c, at the depth ξ' in the elastic slab. In this case the right-hand member of Eq. (1) takes a somewhat different form,

$$-\frac{2\,wc}{\lambda h}\sum_{n=1}^{\infty}\sin a_n\zeta'\sin a_n z'\int_0^{\infty}J_1(ac)\,J_0(ar)\,da. \tag{28}$$

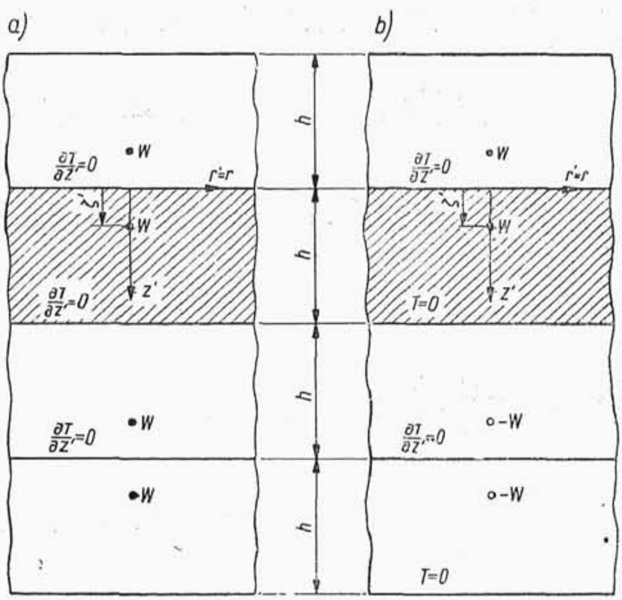

Fig. 3

Integrating along the z'-axis we may obtain the solution for heat sources uniformly distributed over the region of a cylinder of radius c and height $h_0 \leq h$.

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