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The State of Stress in an Elastic Semi-Space Due to an Instantaneous Source of Heat

by

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Consider an isotropic homogeneous elastic semi-space bounded by the plane $z=0$. Let a concentrated instantaneous source of heat located at the origin of the co-ordinate system act in this plane. It will provoke in the elastic semi-space considered a temperature field T and a state of stress (σ_{ij}), variable in function of the co-ordinates and time. Our solution is to be considered as a determination of the Green function for a more general problem — that of heat sources constituting continuous functions of time and distributed over the region Γ of the plane $z=0$. In the case of continuous time — variable sources showing no jump-like changes — the state of stress can be treated as quasi-static. We shall assume therefore that the inertia terms in the basic equations of the theory of elasticity can be disregarded. In addition, we assume that the plane $z=0$ is free from stresses and that the stress components should vanish at infinity at every time t. Two thermal boundary conditions will be discussed. First it will be assumed that the $z=0$ plane is thermally insulated ($\partial T/\partial z=0$), and then that $z=0$ is $T=0$ over that plane (except for the point where the heat source is located).

1. An elastic semi-space thermally insulated at the plane $z=0$

If in an infinite elastic space an instantaneous source of heat is supposed to act, the temperature field will be described by the following equation [1]:

$$T=\frac{W}{(\pi\vartheta)^{3/2}}e^{-R^2/\vartheta}; \qquad \vartheta=4\varkappa t; \qquad R=(x^2+y^2+z^2)^{1/2}, \tag{1.1}$$

where in the Eq. (1.1) $W=Q/\varrho c$, Q denoting the heat quantity emitted by the source per unit time $\varkappa=\lambda/\varrho c$, λ denoting the coefficient of heat conduction, ϱ — density and c — specific heat.

It is easy to observe that for $z = 0$ we have $\partial T/\partial z = 0$. Thus, the Eq. (1.1) determines at the same time the temperature field for an elastic semi-space thermally insulated at the plane $z = 0$.

In order to determine the stress components $\bar{\sigma}_{ij}$ in an infinite elastic space we shall use the potential of thermo-elastic displacement Φ. This function is connected with the displacement components by the following equations

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z}, \tag{1.2}$$

and, with the temperature field [2] by the equation

$$\nabla^2 \Phi = \frac{1+\nu}{1-\nu} a_t T, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}; \tag{1.3}$$

ν is Poisson's ratio and a_t the coefficient of linear thermal dilatation. The knowledge of the function Φ enables us to determine the stress components $(\bar{\sigma}_{ij})$ from the equations

$$\bar{\sigma}_{ij} = 2G \left(\frac{\partial^2 \Phi}{\partial i \, \partial j} - \nabla^2 \Phi \, \delta_{ij} \right) \quad i, j = x, y, z, \tag{1.4}$$

where δ_{ij} is Kronecker's delta and G — the modulus of elasticity in shear.

Let us observe that the temperature field T can be represented in cylindrical co-ordinates by the following Fourier-Hankel integral:

$$T(r, z, t) = \frac{W}{2\pi^2} \int_0^\infty \int_0^\infty \alpha J_0(\alpha r) \exp[-\varkappa t (\alpha^2 + \beta^2)] \cos \beta z \, d\alpha \, d\beta. \tag{1.5}$$

Applying to the above equation the Laplace transformation, we obtain

$$L(T) = T^* = \int_0^\infty e^{-pt} T(r, z, t) \, dt = \frac{W}{2\pi^2 \varkappa} \int_0^\infty \int_0^\infty \frac{\alpha J_0(\alpha r) \cos \beta z \, d\alpha \, d\beta}{\alpha^2 + \beta^2 + p/\varkappa}. \tag{1.6}$$

Applying the Laplace transformation to the Eq. (1.3), and expressing the function Φ^* by means of the Fourier-Hankel integral, we find that

$$L(\Phi) = \Phi^* = \tag{1.7}$$

$$= -\frac{1+\nu}{1-\nu} \frac{a_t W}{2\pi^2 \varkappa} \int_0^\infty \int_0^\infty \alpha J_0(\alpha r) \left[(\alpha^2 + \beta^2 + p/\varkappa)(\alpha^2 + \beta^2) \right]^{-1} \cos \beta z \, d\alpha \, d\beta$$

or, after integration,

$$(1.8') \qquad \Phi^* = -\frac{1+\nu}{1-\nu}\frac{\alpha_t W}{4\pi R}\left\{1-\exp\left[-\left(p\frac{R^2}{\varkappa}\right)^{1/2}\right]\right\}p^{-1}.$$

Performing the inverse transformation, we find that

$$(1.8'') \quad \Phi = -\frac{1+\nu}{1-\nu}\frac{\alpha_t W}{4\pi R}\operatorname{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right), \quad \text{where} \quad \operatorname{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right) = \left(\frac{2}{\sqrt{\pi}}\right)\int_0^{R/\sqrt{\vartheta}} e^{-\eta^2}\,d\eta.$$

Using the relations (1.4), we find that [3]

$$(1.9) \qquad \begin{aligned}
\bar{\sigma}_{rr} &= -2G\left(\frac{\partial^2\Phi}{\partial z^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right) = \\
&= -K\frac{1}{R^3}\left\{\left(2-\frac{3z^2}{R^2}\right)\operatorname{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{2e^{-R^2/\vartheta}\cdot R}{\sqrt{\pi\vartheta}}\left[2-\frac{3z^2}{R^2}\left(1+\frac{2}{3}\frac{R^2}{\vartheta}\right)\right]\right\},\\
\bar{\sigma}_{\varphi\varphi} &= -2G\left(\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial r^2}\right) = \frac{K}{R^3}\left[\operatorname{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{2R}{\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\left(1+\frac{2R^2}{\vartheta}\right)\right],\\
\bar{\sigma}_{zz} &= -2G\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right) = \\
&= -\frac{K}{R^3}\left\{\left(2-\frac{3r^2}{R^2}\right)\operatorname{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{2R}{\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\left[2-\frac{3r^2}{R^2}\left(1+\frac{2}{3}\frac{R^2}{\vartheta}\right)\right]\right\},\\
\bar{\sigma}_{rz} &= 2G\frac{\partial^2\Phi}{\partial r\,\partial z} = \frac{3Krz}{R^5}\left[\operatorname{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{2R}{\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\left(1+\frac{2}{3}\frac{R^2}{\vartheta}\right)\right],
\end{aligned}$$

where

$$K = \frac{1+\nu}{1-\nu}\frac{\alpha_t W G}{2\pi}.$$

Let us observe that the stress $\bar{\sigma}_{rz}$ vanishes in the plane $z=0$, the stress $\bar{\sigma}_{zz}$ remaining different from zero. In order to suppress the stress $\bar{\sigma}_{zz}$ in the $z=0$ plane the stress components $(\bar{\bar{\sigma}}_{ij})$ should be superposed over $(\bar{\sigma}_{ij})$. They will be obtained by solving the following three-dimensional problem: determine in an elastic semi-space the state of stress $\bar{\bar{\sigma}}_{ij}$, due to the action of the stress $-\bar{\sigma}_{zz}|_{z=0}$ acting in the plane $z=0$ bounding the elastic semi-space considered. In order to determine the state of stress $(\bar{\bar{\sigma}}_{ij})$ we shall use Love's function φ satisfying the biharmonic equation [4]

$$(1.10) \qquad \nabla^2\nabla^2\varphi = 0$$

with the boundary conditions

$$(1.11) \qquad \bar{\sigma}_{zz}+\bar{\bar{\sigma}}_{zz}\big|_{z=0}=0, \quad \bar{\bar{\sigma}}_{rz}\big|_{z=0}=0 \quad \text{and} \quad \varphi=0 \quad \text{at infinity.}$$

After determining the function φ, the stress components (σ_{ij}) will be determined from the equations

$$
(1.12)\quad
\begin{aligned}
\bar{\sigma}_{rr} &= \frac{2G}{1-2\nu}\,\frac{\partial}{\partial z}\left(\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial r^2}\right), &\quad \bar{\sigma}_{\varphi\varphi} &= \frac{2G}{1-2\nu}\,\frac{\partial}{\partial z}\left(\nu \nabla^2 \varphi - \frac{1}{r}\,\frac{\partial \varphi}{\partial r}\right),\\
\sigma_{zz} &= \frac{2G}{1-2\nu}\,\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial z^2}\right], &\quad \bar{\sigma}_{rz} &= \frac{2G}{1-2\nu}\,\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial z^2}\right].
\end{aligned}
$$

The function φ will be assumed in the form

$$
(1.13)\quad \varphi = \int_0^\infty Z(\alpha, z, t)\, J_0(\alpha r)\, d\alpha, \quad \text{where} \quad Z(\alpha, z, t) = (C + D\alpha z)\, e^{-\alpha z}.
$$

From the boundary condition $\bar{\sigma}_{rz}|_{z=0} = 0$ it follows that $C = 2\nu D$.

The stress components (σ_{ij}) will be represented in the integral form

$$
(1.14)\quad
\begin{aligned}
\bar{\sigma}_{rr} &= \frac{2G}{1-2\nu}\int_0^\infty D(\alpha, t)\,\alpha^3 e^{-\alpha z}\left[(1-\alpha z) J_0(\alpha r) + (2\nu - 1 + \alpha z)\frac{J_1(\alpha r)}{\alpha r}\right] d\alpha,\\
\sigma_{\varphi\varphi} &= \frac{2G}{1-2\nu}\int_0^\infty D(\alpha, t)\,\alpha^3 e^{-\alpha z}\left[2\nu J_0(\alpha r) - (2\nu - 1 + \alpha z)\,\frac{J_1(\alpha r)}{\alpha r}\right] d\alpha,\\
\bar{\sigma}_{zz} &= \frac{2G}{1-2\nu}\int_0^\infty D(\alpha, t)\,\alpha^3 e^{-\alpha z}(1 + \alpha z)\, J_0(\alpha r)\, d\alpha,\\
\bar{\sigma}_{rz} &= \frac{2G}{1-2\nu}\, z \int_0^\infty D(\alpha, t)\,\alpha^4 e^{-\alpha z} J_1(\alpha r)\, d\alpha.
\end{aligned}
$$

The quantity $D(\alpha, t)$ (constituting a function of the parameter α and time t) will be determined from the first boundary condition of the group (1.11). Applying the inverse transformation to the function Φ^* (Eq. (1.7)), we obtain

$$
(1.15)\quad \Phi = -\frac{1+\nu}{1-\nu}\,\frac{\alpha_t W}{2\pi^2}\int_0^\infty\!\!\int_0^\infty \alpha J_0(\alpha r)\,(\alpha^2+\beta^2)^{-1}\exp\left[-\varkappa t(\alpha^2+\beta^2)\right]\cos\beta z\, d\alpha\, d\beta.
$$

Integrating with respect to β, we have

$$
(1.16)\quad \Phi = -\frac{1+\nu}{1-\nu}\,\frac{\alpha_t W}{8\pi}\int_0^\infty J_0(\alpha r)\left[e^{-\alpha z}\operatorname{Erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2} - \frac{z}{\sqrt{\vartheta}}\right) + e^{\alpha z}\operatorname{Erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2} + \frac{z}{\sqrt{\vartheta}}\right)\right] d\alpha.
$$

From the first boundary condition of the group (1.11) which can be expressed in the form

$$(1.17)\qquad -2G\left(\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}\right)_{z=0}+\frac{2G}{1-2\nu}\int_0^\infty D(\alpha,t)\,\alpha^3 J_0(\alpha r)\,d\alpha=0,$$

we obtain

$$(1.18)\qquad D(\alpha,t)=\frac{1+\nu}{1-\nu}\,\frac{\alpha_t W}{4\pi}(1-2\nu)\,\alpha^{-1}\,\mathrm{Erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2}\right).$$

In consequence, the stress components $(\bar{\bar{\sigma}}_{ij})$ are determined on the basis of the Eqs. (1.14). The final form of the stress (σ_{ij}) will be obtained by superposing the states $(\bar{\sigma}_{ij})$ and $(\bar{\bar{\sigma}}_{ij})$. Unhappily, the stress components $(\bar{\bar{\sigma}}_{ij})$ cannot be represented in a closed form by means of known and tabulated functions.

Consider a heat source constituting a continuous function of time. In the period from $t=0$ to $t'=t$ let a heat quantity $W(t')\varrho c$ be emitted per unit of time. The temperature field and the stress components will take the form

$$(1.19)\qquad \begin{aligned} T(r,z,t)&=(\pi\varkappa)^{-3/2}\int_0^t \frac{W(t')\,e^{-\frac{R^2}{4\varkappa(t-t')}}}{(t-t')^{3/2}}\,dt',\\ \sigma_{ij}(r,z,t)&=\int_0^t W(t')\,\widehat{\sigma_{ij}}(r,z,t-t')\,dt',\end{aligned}$$

if $\widehat{\sigma_{ij}}$ denotes the stress due to an instantaneous unit source of heat.

Consider the particular case of $W(t)=W=\mathrm{const}$. Then, the temperature field takes the form

$$(1.20)\qquad T(r,z,t)=\frac{W}{4\pi\varkappa R}\left(1-\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)\right).$$

The function Φ can be expressed in the integral form

$$(1.21)\qquad \Phi=-\frac{1+\nu}{1-\nu}\,\alpha_t\,\frac{W}{2\pi^2\varkappa}\int_0^\infty\int_0^\infty \alpha J_0(\alpha r)\left[1-e^{-\varkappa t(\alpha^2+\beta^2)}\right](\alpha^2+\beta^2)^{-2}\cos\beta z\,d\alpha\,d\beta$$

or

$$(1.22)\qquad \Phi=\frac{1+\nu}{1-\nu}\,\alpha_t\,\frac{WR}{8\pi\varkappa}\left[1-\left(1+\frac{\vartheta}{2R^2}\right)\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{1}{R}\sqrt{\frac{\vartheta}{\pi}}\,e^{-R^2/\vartheta}\right].$$

The stress $\bar{\sigma}_{ij}$ can now be determined from the Eqs. (1.4).

The quantity $D(\alpha,t)$ will be found from the first boundary condition of the group (1.11)

$$(1.23)\qquad D(\alpha,t)=\frac{1+\nu}{1-\nu}\,\alpha_t\,\frac{W}{8\pi\varkappa}\,\frac{(1-2\nu)}{\alpha^3}\,(1-F(\alpha,t)),$$

where

$$F(\alpha,t)=\frac{4\alpha^3 e^{-\varkappa t\alpha^2}}{\pi}\int_0^\infty \frac{e^{-\varkappa t\beta^2}\,d\beta}{(\alpha^2+\beta^2)^2},$$

$$F(\alpha, t) = 2\alpha \sqrt{\frac{\varkappa t}{\pi}}\, e^{-\varkappa t \alpha^2} + (1 - 2\varkappa t \alpha^2)\,\mathrm{Erfc}\,(\alpha \sqrt{\varkappa t}).$$

In the limit case of a steady-state heat source (or, in other words for $t \to \infty$), we have

$$T_\infty(r,z) = \frac{W}{4\pi\varkappa R}, \quad \Phi_\infty(r,z) = \frac{1+\nu}{1-\nu}\,a_t\,\frac{WR}{8\pi\varkappa},$$

(1.24)

$$D_\infty(\alpha) = \frac{1+\nu}{1-\nu}\,a_t\,\frac{W}{4\pi\varkappa}\;\frac{1-2\nu}{\alpha^3}.$$

In this case the stress components (σ_{ij}) can be found in a closed form. This case, treated in detail by E. Melan and H. Parcus, [2], leads to an interesting result: the components $\sigma_{zz\infty}$, $\sigma_{rz\infty}$ are equal to zero at any point of the semi-infinite space. This is valid for sources distributed in an arbitrary way over the plane $z = 0$.

Let us observe that in the case of a continuous source of heat the functions T, Φ and σ_{ij}, D can be represented in the form

$$\text{(1.25)} \qquad T = T_\infty - T_1, \quad \Phi = \Phi_\infty - \Phi_1, \quad D = D_\infty - D_1, \quad \sigma_{ij} = \sigma_{ij,\infty} - \sigma_{ij,1},$$

where the functions $T_1, \Phi_1, D_1\, \sigma_{ij,1}$ depend on time and on the co-ordinates, while the quantities T_∞, Φ_∞, D_∞, $\sigma_{ij,\infty}$ are independent of time. For the stresses σ_{rz}, σ_{zz} we obtain

$$\text{(1.26)} \qquad \sigma_{rz} = -\sigma_{rz,1}, \quad \sigma_{zz} = -\sigma_{zz,1}.$$

These stresses vanish for $t = \infty$, taking for a certain value t_0 their extremal values.

2. An elastic semi-space in which the plane $z=0$ is kept at constant temperature $T=0$

The solution of this problem can be obtained in a direct manner from the preceding case. Let an instantaneous heat dipole of flow intensity W act in an infinite elastic space. Then, using the Eqs. (1.1), we obtain

$$\text{(2.1)} \qquad T(r,z,t) = -\frac{W}{(\pi\vartheta)^{3/2}}\,\frac{\partial}{\partial z}\left(e^{-R^2/\vartheta}\right) = \frac{2Wz}{(\pi\vartheta)^{3/2}\,\vartheta}\,e^{-R^2/\vartheta}.$$

It is seen that the condition $z = 0$ is satisfied in the plane $T = 0$.

Using the Eq. (1.8″), we obtain

$$\text{(2.2)} \qquad \Phi(r,z,t) = -\frac{1+\nu}{1-\nu}\,\frac{a_t W}{4\pi}\,\frac{\partial}{\partial z}\left(R^{-1}\,\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)\right) =$$

$$= \frac{1+\nu}{1-\nu}\,a_t\,\frac{W}{4\pi}\,\frac{z}{R^3}\left[\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right) - \frac{2R}{\sqrt{\pi\vartheta}}\,e^{-R^2/\vartheta}\right].$$

The knowledge of the function Φ enables us to determine the stress component $\bar{\sigma}_{ij}$ on the basis of the Eqs. (1.4). They can also be obtained in a direct manner from the Eqs. (1.9) by performing the operation $-\partial/\partial z$. We obtain in a successive manner

$$
\begin{aligned}
(2.3)\qquad \bar{\sigma}_{rr} &= -\frac{2Kz}{R^5}\Bigg\{3\left(4-\frac{5z^2}{R^2}\right)\left(\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{2e^{-R^2/\vartheta}R}{\sqrt{\pi\vartheta}}\right)-\\
&\qquad -\frac{4R^3}{\vartheta\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\left[4-\frac{z^2}{R^2}\left(5+\frac{2R}{\vartheta}\right)\right]\Bigg\},\\
\bar{\sigma}_{\varphi\varphi} &= \frac{2Kz}{R^5}\left\{3\left(\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{2R}{\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\right)-\frac{4R^3}{\vartheta\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\left(1+\frac{2}{\vartheta}R^2\right)\right\},\\
\bar{\sigma}_{zz} &= -\frac{6Kz}{R^5}\Bigg\{\left(2-\frac{5z^2}{R^2}\right)\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)+\frac{3e^{-R^2/\vartheta}R}{\sqrt{\pi\vartheta}}\left(\frac{5r^2}{R^2}-2\right)+\\
&\qquad +\frac{4R^3}{3\vartheta\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\left[\frac{r^2}{R^2}\left(5+\frac{2}{\vartheta}R^2\right)-2\right]\Bigg\},\\
\bar{\sigma}_{rz} &= 6K\frac{r}{R^5}\Bigg\{\left(1-\frac{5z^2}{R^2}\right)\left(\mathrm{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right)-\frac{2Re^{-R^2/\vartheta}}{\sqrt{\pi\vartheta}}\right)-\\
&\qquad -\frac{4R}{3\vartheta\sqrt{\pi\vartheta}}e^{-R^2/\vartheta}\left(1-\frac{z^2}{R^2}\left(5+\frac{2}{\vartheta}R^2\right)\right)\Bigg\}.
\end{aligned}
$$

For $z=0$ the stress $\bar{\sigma}_{zz}|_{z=0}$ vanishes; the stress $\bar{\sigma}_{rz}|_{z=0}$ does not vanish, however. The additional stress component $\bar{\bar{\sigma}}_{ij}$ will be obtained by solving Love's equation (1.10) with the boundary conditions

$$(2.4)\qquad \bar{\bar{\sigma}}_{zz}|_{z=0}=0,\quad \bar{\sigma}_{rz}+\bar{\bar{\sigma}}_{rz}|_{z=0}=0\quad\text{and}\quad \varphi=0\quad\text{at infinity.}$$

We assume that the function φ has the form (1.13), where in view of the first boundary condition of the group (2.4), we put $C=-D(1-2\nu)$. The stress components $(\bar{\bar{\sigma}}_{ij})$ are described by the integrals

$$
\begin{aligned}
\bar{\bar{\sigma}}_{rr} &= \frac{2G}{1-2\nu}\int_0^\infty D(\alpha,t)\,\alpha^3e^{-\alpha z}\left[(2-\alpha z)J_0(\alpha r)+(2\nu-2+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha,\\
\bar{\bar{\sigma}}_{\varphi\varphi} &= \frac{2G}{1-2\nu}\int_0^\infty D(\alpha,t)\,\alpha^3e^{-\alpha z}\left[2\nu J_0(\alpha r)-(2\nu-2+\alpha z)\frac{J_1(\alpha r)}{\alpha r}\right]d\alpha,\\
(2.5)\qquad \bar{\bar{\sigma}}_{zz} &= \frac{2G}{1-2\nu}z\int_0^\infty D(\alpha,t)\,\alpha^4e^{-\alpha z}J_0(\alpha r)\,d\alpha,\\
\bar{\bar{\sigma}}_{rz} &= -\frac{2G}{1-2\nu}\int_0^\infty D(\alpha,t)\,\alpha^3e^{-\alpha z}(1-\alpha z)J_1(\alpha r)\,d\alpha.
\end{aligned}
$$

In order to determine the quantity $D(\alpha, t)$ it will be convenient to represent the function Φ in the integral form

$$(2.6) \qquad \Phi = -\frac{1+\nu}{1-\nu}\, a_t \frac{W}{2\pi^2 \varkappa} \int_0^\infty \int_0^\infty \alpha\beta J_0(\alpha r)(\alpha^2+\beta^2)^{-1} \exp[-\varkappa t(\alpha^2+\beta^2)] \times \times \sin\beta z \, d\alpha \, d\beta .$$

This expression will be obtained if the inverse Laplace transformation is applied to the function Φ^* from the Eq. (1.7). Then, the operation $-\partial/\partial z$ is performed. The function Φ can also be expressed by the equation

$$(2.7) \qquad \Phi = -\frac{1+\nu}{1-\nu}\, a_t \frac{W}{8\pi} \int_0^\infty \alpha J_0(\alpha r) \left[e^{-\alpha z} \operatorname{Erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2} - \frac{z}{\sqrt{\vartheta}}\right) - e^{-\alpha z} \operatorname{Erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2} + \frac{z}{\sqrt{\vartheta}}\right)\right] d\alpha .$$

The knowledge of the function Φ enables us to determine the stress components $(\bar{\sigma}_{ij})$ from the Eqs. (1.4).

From the second boundary condition of the groups (2.4) which may be represented in the form

$$(2.8) \qquad 2G \frac{\partial^2 \Phi}{\partial r\, \partial z}\bigg|_{z=0} - \frac{2G}{1-2\nu} \int_0^\infty D(\alpha, t)\, \alpha^3 J_1(\alpha r)\, d\alpha = 0$$

we obtain

$$D(\alpha, t) = \frac{1+\nu}{1-\nu}(1-2\nu)\, a_t \frac{W}{4\pi} \operatorname{Erfc}\left(\frac{\alpha\sqrt{\vartheta}}{2}\right).$$

Thus, the stress components $\bar{\bar{\sigma}}_{ij}$ are determined. Unfortunately, they are not expressed in a closed form. The final form of the stresses will be obtained by superposing $\bar{\sigma}_{ij}$ and $\bar{\bar{\sigma}}_{ij}$.

Consider finally the case of a continuous heat source of constant flow intensity W. We have

$$(2.9) \qquad T(r, z, t) = \frac{W}{4\pi\varkappa} \frac{z}{R^3} \left[1 - \operatorname{Erf}\left(\frac{R}{\sqrt{\vartheta}}\right) + \frac{4R^2}{\sqrt{\pi\vartheta}} e^{-R^2/\vartheta}\right]$$

and

$$(2.10) \quad \Phi(r, z, t) = -\frac{1+\nu}{1-\nu}\, a_t \frac{W}{8\pi\varkappa} \frac{z}{R}\left\{1 - \left(1 - \frac{\vartheta}{2R^2}\right) \operatorname{Erfc}\left(\frac{R}{\sqrt{\vartheta}}\right) - \frac{1}{R}\sqrt{\frac{\vartheta}{\pi}}\, e^{-R^2/\vartheta}\right\}.$$

Assuming the function Φ in the form

$$(2.11) \qquad \Phi = -\frac{1+\nu}{1-\nu}\, a_t \frac{W}{2\pi^2 \varkappa} \int_0^\infty \int_0^\infty \alpha\beta J_0(\alpha r)\left[1 - e^{-\varkappa t(\alpha^2+\beta^2)}\right] \times (\alpha^2+\beta^2)^{-2} \sin\beta z \, d\alpha \, d\beta ,$$

we can express the quantity $D(a, t)$ from the second of the boundary conditions (2.4), as

$$D(a,t) = \frac{1+\nu}{1-\nu}(1-2\nu)\,a_t\,\frac{W}{8\pi\varkappa}\,a^{-2}(1-F(a,t)), \tag{2.12}$$

where

$$F(a,t) = \frac{4a}{\pi}e^{-\varkappa t a^2}\int_0^\infty \frac{\beta^2 e^{-\varkappa t \beta^2}\,d\beta}{(a^2+\beta^2)^2} = (1+2a^2\varkappa t)\,\mathrm{Erfc}(a\sqrt{\varkappa t}) - 2a\,e^{-a^2\varkappa t}\sqrt{\frac{\varkappa t}{\pi}}.$$

In consequence, the stress components $(\bar{\sigma}_{ij})$ can be determined from the Eqs. (2.5). For a stationary heat source $t\to\infty$ we obtain, therefore,

$$T_\infty(r,z) = \frac{W}{4\pi\varkappa}\,\frac{z}{R^3}, \qquad \Phi_\infty(r,z) = -\frac{1+\nu}{1-\nu}\,a_t\,\frac{W}{8\pi\varkappa}\,\frac{z}{R}.$$

In this case the function φ and, in consequence, all stress components can be determined in a closed form. This case was treated in detail by E. Sternberg, [5].

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