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APPLIED MECHANICS

The State of Stress in an Elastic Space Due to a Source of Heat Varying Harmonically in Function of Time

by

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Let a concentrated source of heat of intensity variable in a harmonic manner act at the point A constituting the origin of the co-ordinate system. The action of this source will result in a time-variable temperature and stress field T and σ_{ij}, respectively, both varying also in a harmonic manner. Assume that the frequency of vibration of the heat source is insignificant, so that the phenomenon under consideration can be treated as quasi-static. The acceleration containing terms in displacement equations of the theory of elasticity will therefore be dropped.

The temperature field is determined by the equation

$$\nabla^2 T = \frac{1}{k} \frac{\partial T}{\partial t} - \frac{W}{\lambda} \delta(x)\,\delta(y)\,\delta(z), \tag{1.1}$$

where $k = \lambda/\varrho c$, λ is the coefficient of heat conduction, ϱ — density and c — specific heat. The symbol δ denotes the Dirac function.

In view of the harmonic character of the action of the source, we assume

$$T(x, y, z, t) = U(x, y, z)\, e^{i(\omega t - \varepsilon)}, \qquad W = W_0 e^{i(\omega t - \varepsilon)}. \tag{1.2}$$

The Eq. (1.1) may therefore be reduced to the form

$$\nabla^2 U - i\eta U = -\frac{W_0}{\lambda} \delta(x)\,\delta(y)\,\delta(z) \qquad \eta = \frac{\omega}{k}. \tag{1.3}$$

The solution of this equation in cylindrical co-ordinates, assuming $T = 0$ at infinity, is

$$U = \frac{W}{2\pi^2 \lambda} \int_0^\infty \int_0^\infty \alpha(\alpha^2 + \gamma^2 + i\eta)^{-1} J_0(\alpha r) \cos \gamma z \, d\alpha \, d\gamma, \qquad r = (x^2 + y^2)^{1/2}. \tag{1.4}$$

After integration, we have

$$(1.5)\qquad U=\frac{W_0}{4\pi\lambda}R^{-1}\exp(-R\sqrt{\eta i}),\qquad R=(x^2+y^2+z^2)^{1/2}.$$

Bearing in mind (1.2), we find that

$$(1.6)\qquad T=\frac{W_0}{4\pi\lambda}R^{-1}\exp\left[i(\omega t-\varepsilon)-R\sqrt{i\eta}\right].$$

The temperature field T will be obtained as the real part of the expression (1.6):*)

$$(1.7)\qquad T=\frac{W_0}{4\pi\lambda}R^{-1}\exp\left(-R\sqrt{\frac{\omega}{2k}}\right)\cdot\cos\left(\omega t-\varepsilon-R\sqrt{\frac{\omega}{2k}}\right).$$

It will be convenient for the determination of stress to use the potential of thermo-elastic stress Φ.

This function is related to the temperature field.

Thus, [1], we have

$$(1.8)\qquad \nabla^2\Phi=\frac{1+\nu}{1-\nu}a_t T,$$

where ν is Poisson's ratio and a_t — the coefficient of thermal dilatation. In view of the harmonic character of the source, we assume that

$$(1.9)\qquad \Phi(x,y,z,t)=\mathrm{Re}\{\Psi(x,y,z)e^{i(\omega t-\varepsilon)}\}.$$

Thus,

$$(1.10)\qquad \nabla^2\Psi=\frac{1+\nu}{1-\nu}a_t U.$$

Bearing in mind the Eq. (1.4), the solution of the above equation has the form

$$(1.11)\qquad \Psi=-\frac{W_0 a_t}{2\pi^2\lambda}\frac{1+\nu}{1-\nu}\int_0^\infty\int_0^\infty \alpha(\alpha^2+\gamma^2)^{-1}(\alpha^2+\gamma^2+i\eta)^{-1}\times J_0(\alpha r)\cos\gamma z\,d\alpha\,d\gamma,$$

or, after integration,

$$(1.12)\qquad \Psi=\frac{1+\nu}{1-\nu}\frac{a_t W_0}{4\pi\lambda\eta}i\left[1-\exp(-R\sqrt{i\eta})\right]R^{-1}.$$

The real part of the function Φ in the Eq. (1.9) will take the form

$$(1.13)\qquad \Phi=-\frac{1+\nu}{1-\nu}a_t\frac{W_0 k}{4\pi\lambda\omega}R^{-1}\left\{\sin\left[R\sqrt{\frac{\omega}{2k}}-\omega t+\varepsilon\right]\times\exp\left(-R\sqrt{\frac{\omega}{2k}}\right)+\sin(\omega t-\varepsilon)\right\}.$$

*) We assume here that $W(t)=W_0\cos(\omega t-\varepsilon)$.

Knowing the function Φ, we are able to determine the stress components from

$$\sigma_{ij} = 2\,G\left(\frac{\partial^2 \Phi}{\partial i\,\partial j} - \nabla^2 \Phi\,\delta_{ij}\right), \qquad i, j = x, y, z, \tag{1.14}$$

where δ_{ij} is Kronecker's delta.

We are concerned in our problem with a symmetry of the spherical type. In spherical co-ordinates we obtain the most simple expression for stress components. We have

$$\sigma_{rr} = 2\,G\left(\frac{d^2 \Phi}{d R^2} - \nabla^2 \Phi\right), \qquad \sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = 2\,G\left(\frac{1}{R}\,\frac{d\Phi}{dR} - \nabla^2 \Phi\right), \tag{1.15}$$

$$\sigma_{r\varphi} = 0, \qquad \sigma_{\varphi\vartheta} = 0, \qquad \sigma_{\vartheta\varphi} = 0, \qquad \nabla^2 \Phi = \frac{d^2 \Phi}{d R^2} + \frac{2}{R}\,\frac{d\Phi}{dR}.$$

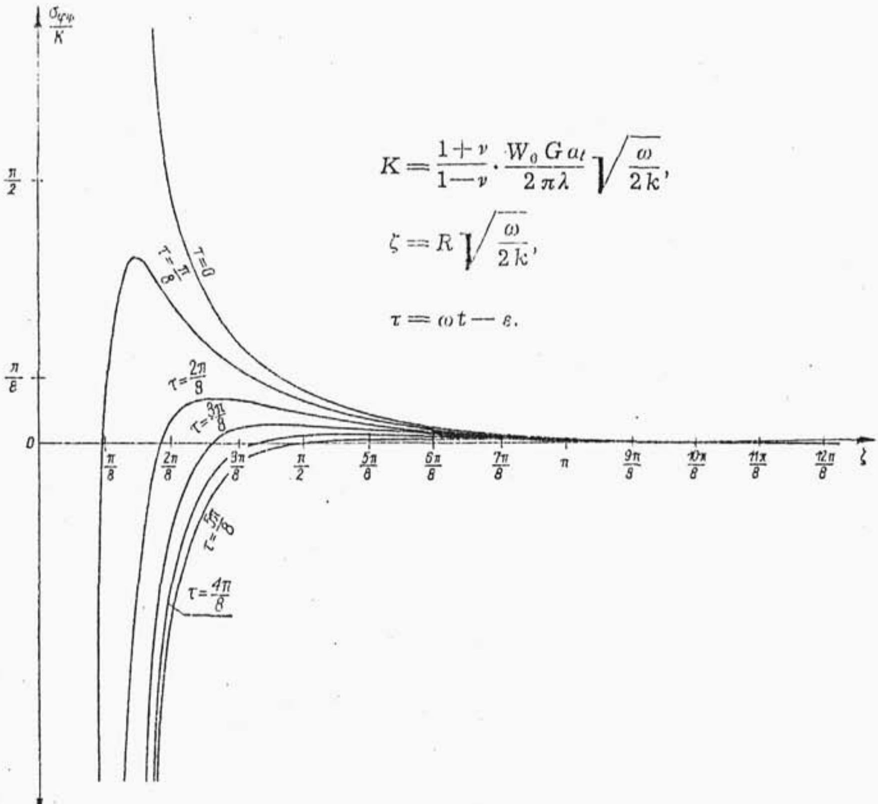


Fig. 1a

Using the Eq. (1.13), we find that

$$\sigma_{rr}=\frac{(1+\nu)\,\alpha_t\,k\,W_0\,G}{(1-\nu)\,\pi\,\lambda\,\omega}\left\{\exp\left(-R\sqrt{\frac{\omega}{2k}}\right)\left[\left(1+R\sqrt{\frac{\omega}{2k}}\right)\times\right.\right.$$

$$\left.\left.\times\sin\left(\omega t-\varepsilon-R\sqrt{\frac{\omega}{2k}}\right)+R\sqrt{\frac{\omega}{2k}}\right]-\sin(\omega t-\varepsilon)\right\}R^{-3},$$

(1.16)

$$\sigma_{\varphi\varphi}=\sigma_{\vartheta\vartheta}=-\sigma_{rr}-\frac{(1+\nu)\,\alpha_t\,W_0\,G k}{(1-\nu)\,2\,\lambda\pi\omega}R^{-1}\exp\left(-R\sqrt{\frac{\omega}{2k}}\right)\times$$

$$\times\cos\left(\omega t-\varepsilon-R\sqrt{\frac{\omega}{2k}}\right)=-\sigma_{rr}-\frac{1+\nu}{1-\nu}\alpha_t\,2\,G\,T, \qquad \sigma_{\varphi r}=\sigma_{\varphi\vartheta}=\sigma_{\vartheta r}=0.$$

In Fig. 1a the diagram of the function $\sigma_{\varphi\varphi}$ is given. Fig. 1b represents that of σ_{rr} for a few values of the parameters $\tau=\omega t-\varepsilon$.

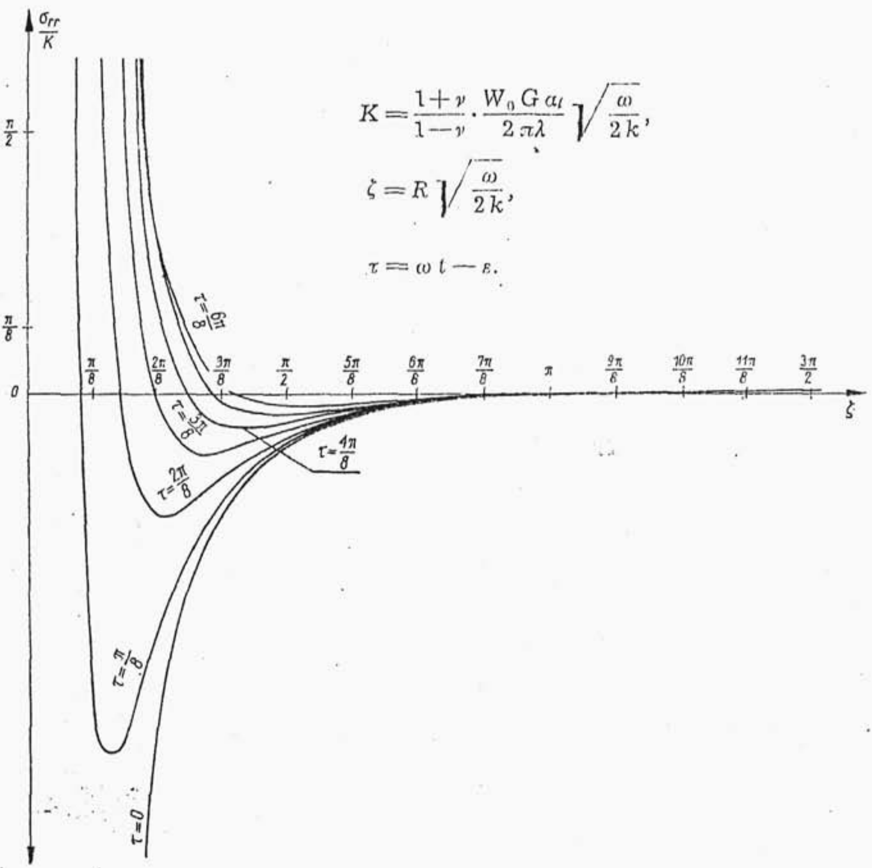


Fig. 1b

Let heat sources uniformly distributed along the z-axis, act in the elastic space. The problem is axially symmetric. The heat equation takes the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{k}\frac{\partial T}{\partial t} - \frac{W}{\lambda}\delta(r). \tag{1.17}$$

For a linear source of intensity W per unit length we assume that it changes with time in a harmonic manner, and

$$T(r,t) = U(r)\, e^{i(\omega t - \varepsilon)}, \qquad W = W_0\, e^{i(\omega t - \varepsilon)}. \tag{1.18}$$

The Eq. (1.17) takes the form

$$\frac{d^2 U}{d r^2} + \frac{1}{r}\frac{dU}{dr} - i\eta U = -\frac{W_0}{\lambda}\delta(r). \tag{1.19}$$

The solution of the Eq. (1.19) is the integral

$$U = \frac{W_0}{2\pi\lambda}\int_0^\infty a J_0(ar)(a^2 + i\eta)^{-1}\, da = \frac{W_0}{2\pi\lambda} K_0(r\sqrt{i\eta}), \tag{1.20}$$

where $K_0(r\sqrt{i\eta})$ is a modified Bessel function of the third kind otherwise called a Basset function. Thus,*)

$$T = Re\left[\frac{W_0}{2\pi\lambda} e^{i(\omega t - \varepsilon)} K_0(r\sqrt{i\eta})\right]. \tag{1.21}$$

Bearing in mind that

$$e^{-\frac{i\nu\pi}{2}} K_0(r\sqrt{i\eta}) = \mathrm{ker}_\nu(r\sqrt{\eta}) + i\,\mathrm{kei}_\nu(r\sqrt{\eta}),$$

where the functions $\mathrm{ker}_\nu(z)$, $\mathrm{kei}_\nu(z)$ are Kelvin functions, we can express the real part of the function (1.21) by

$$T = \frac{W_0}{2\pi\lambda}\left[\mathrm{ker}_0\left(r\sqrt{\frac{\omega}{2k}}\right)\cos(\omega t - \varepsilon) - \mathrm{kei}_0\left(r\sqrt{\frac{\omega}{2k}}\right)\sin(\omega t - \varepsilon)\right]. \tag{1.22}$$

From the equation

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} = \frac{1+\nu}{1-\nu}\alpha_t T$$

we determine the function Φ in the form

$$\Phi = -Re\left[\frac{W_0}{2\pi\lambda}\int_0^\infty a^{-1}(a^2 + i\eta)^{-1} J_0(a, r)\, da\right] = \tag{1.23}$$

$$= \frac{1+\nu}{1-\nu}\,\frac{W_0}{2\pi\lambda}\,\frac{\alpha_t}{i\eta}\left[\ln\frac{a}{r} - K_0(r\sqrt{i\eta})\right].$$

* For $W = W_0 \cos(\omega t - \varepsilon)$.

Knowing the function Φ, we can determine the complex stress from the following equation:

$$\sigma_{rr}^{*} = -2G\frac{1}{r}\frac{\partial \Phi}{\partial r}, \qquad \sigma_{\varphi\varphi}^{*} = -2G\frac{\partial^2 \Phi}{\partial r^2}, \qquad \sigma_{r\varphi}^{*} = 0. \tag{1.24}$$

We obtain

$$\begin{cases} \sigma_{rr}^{*} = \dfrac{1+\nu}{1-\nu}\alpha_t \dfrac{W_0 G}{\pi\lambda}\dfrac{e^{i(\omega t-\varepsilon)}}{i\eta}\left[\sqrt{i\eta}\,K_1(r\sqrt{i\eta}) - \dfrac{1}{r^2}\right], \\ \sigma_{\varphi\varphi}^{*} = -\dfrac{1+\nu}{1-\nu}\alpha_t \dfrac{W_0 G}{\pi\lambda}\dfrac{e^{i(\omega t-\varepsilon)}}{i\eta}\left[\sqrt{i\eta}K_1(r\sqrt{i\eta}) + \eta i K_0(r\sqrt{i\eta}) - \dfrac{1}{r^2}\right] = \\ \qquad\qquad = -\sigma_{rr}^{*} - \dfrac{1+\nu}{1-\nu}\alpha_t\, 2GT, \\ \sigma_{r\varphi}^{*} = 0. \end{cases} \tag{1.25}$$

The stresses σ_{rr}, $\sigma_{\varphi\varphi}$ will be obtained as the real parts of the functions σ_{rr}^{*}, $\sigma_{\varphi\varphi}^{*}$.

We have

$$\begin{aligned} \sigma_{rr} = -\frac{1+\nu}{1-\nu}\alpha_t \frac{W_0 G k}{\pi\lambda\omega}\Bigg\{\sqrt{\frac{\omega}{2k}}\Bigg[\mathrm{kei}_1\left(r\sqrt{\frac{\omega}{2k}}\right) - \\ - \mathrm{ker}_1\left(r\sqrt{\frac{\omega}{2k}}\right)\Bigg]\cos(\omega t-\varepsilon) + \Bigg[\mathrm{ker}_1\left(r\sqrt{\frac{\omega}{2k}}\right) + \\ + \mathrm{kei}_1\left(r\sqrt{\frac{\omega}{2k}}\right)\Bigg]\sin(\omega t-\varepsilon)\Bigg] + \frac{1}{r^2}\sin(\omega t-\varepsilon)\Bigg\}, \\ \sigma_{\varphi\varphi} = -\sigma_{rr} - \frac{1+\nu}{1-\nu}\alpha_t\frac{W_0 G}{\pi\lambda}\Bigg[\mathrm{ker}_0\left(r\sqrt{\frac{\omega}{2k}}\right)\cos(\omega t-\varepsilon) - \mathrm{kei}_0 \times \\ \times\left(r\sqrt{\frac{\omega}{2k}}\right)\sin(\omega t-\varepsilon)\Bigg]. \end{aligned} \tag{1.26}$$

Let us now consider the following problem. Let uniformly distributed heat sources act in the plane $x=\xi$ of an elastic space. The intensity of these stresses per unit area of the plane $x=\xi$ will be denoted by $W = W_0\, e^{i(\omega t-\varepsilon)}$.

The heat equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t} - \frac{W}{\lambda}\delta(x-\xi). \tag{1.27}$$

Introducing the relation $T(x,t) = U(x)\,e^{i(\omega t-\varepsilon)}$, we reduce the partial differential equation (1.27) to the ordinary equation

$$\frac{d^2 U}{dx^2} - i\eta U = -\frac{W_0}{\lambda}\delta(x-\xi). \tag{1.28}$$

The solution of this equation is

$$(1.29)\qquad U = \frac{W_0}{\lambda\pi}\int_0^{\infty}(\alpha^2+i\eta)^{-1}\cos\alpha(x-\xi)\,d\alpha = \frac{W_0}{2\lambda}(i\eta)^{-1/2}\exp\left[-(x-\xi)\sqrt{i\eta}\right].$$

Hence,

$$(1.30)\qquad T = Re\left\{\frac{W_0}{2\lambda}(i\eta)^{-1/2}\exp\left[i(\omega t-\varepsilon)-(x-\xi)\sqrt{i\eta}\right]\right\}.$$

The real part of this function is the temperature field

$$(1.31)\qquad T = \frac{W_0}{4\lambda}(\eta)^{-1/2}\exp\left[-(x-\xi)\sqrt{\eta}\right]\cos\left[\omega t-\varepsilon-(x-\xi)\sqrt{\eta}\right]$$

which is sought.

The Eq. (1.8) reduces to the form

$$(1.32)\qquad \frac{\partial^2\Phi}{\partial x^2} = \frac{1+\nu}{1-\nu}\alpha_t T.$$

From (1.14) it is seen that

$$\sigma_{xx} = 0,\quad \sigma_{zx} = 0,\quad \sigma_{zy} = 0,\quad \sigma_{xy} = 0,$$

and

$$(1.33)\qquad \sigma_{yy} = \sigma_{zz} = -2G\frac{\partial^2\Phi}{\partial x^2} = -2G\frac{1+\nu}{1-\nu}\alpha_t T.$$

Thus,

$$(1.34)\qquad \sigma_{yy} = \sigma_{zz} = \frac{GW_0\alpha_t(1+\nu)}{2\lambda(1-\nu)}\left(\frac{2k}{\omega}\right)^{1/2}\exp\left[-(x-\xi)\sqrt{\frac{\omega}{2k}}\right]\times \cos\left[\omega t-\varepsilon-(x-\xi)\sqrt{\frac{\omega}{2k}}\right].$$

In Figure 2 the function σ_{yy} is represented for various values of the parameters $\tau = \omega t - \varepsilon$.

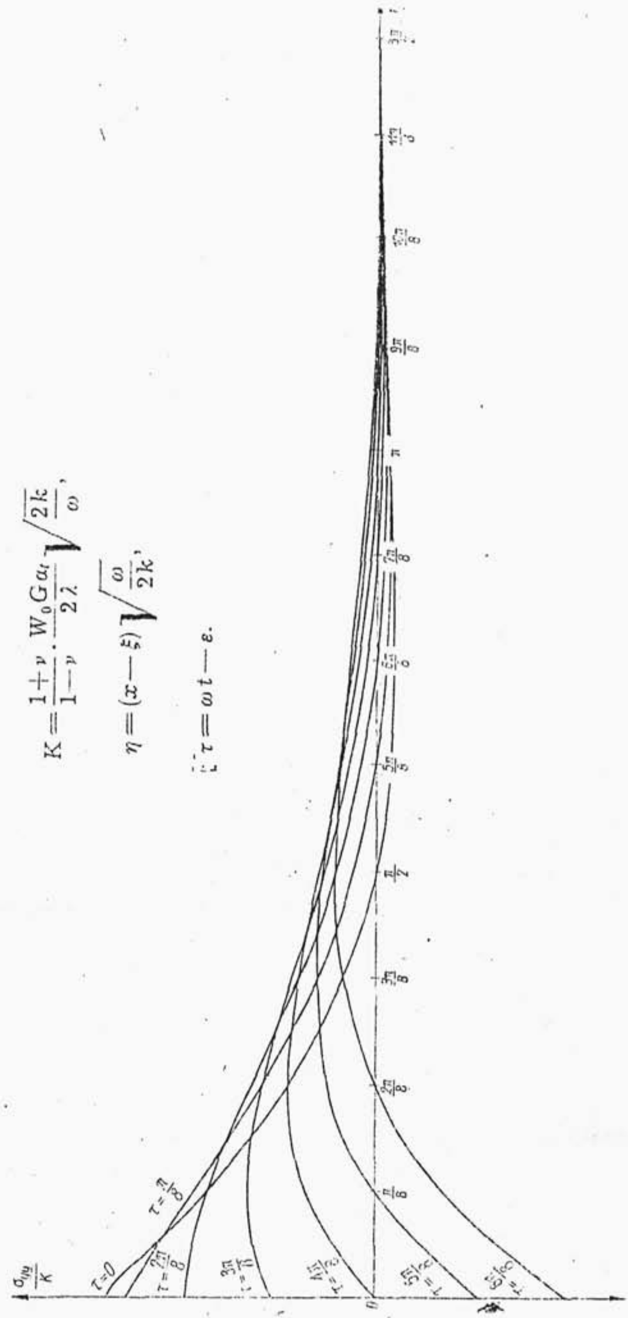
Let a positive plane heat source act in the plane $x = \xi$ and a negative plane source in the plane $x = -\xi$. In this case we have $T = 0$ and $\sigma_{yy} = 0$, $\sigma_{xx} = 0$ in the plane $x = 0$.

We are concerned with the case of an elastic semi-space ($x > 0$) in which a plane heat source acts in the plane $x = \xi$.

The stresses σ_{yy}, σ_{zz} will be obtained from the equations

$$(1.35)\qquad \sigma_{yy} = \sigma_{zz} = \frac{GW_0\alpha_t(1+\nu)}{2\lambda(1-\nu)}\left(\frac{2k}{\omega}\right)^{1/2}\left\{\exp\left[-(x-\xi)\sqrt{\frac{\omega}{2k}}\right]\times \cos\left[\omega t-\varepsilon-(x-\xi)\sqrt{\frac{\omega}{2k}}\right]-\exp\left[-(x+\xi)\sqrt{\frac{\omega}{2k}}\right]\times \cos\left[\omega t-\varepsilon-(x+\xi)\sqrt{\frac{\omega}{2k}}\right]\right\}\quad \text{for}\quad x>\xi.$$

For $x < \xi$ we should replace $(x-\xi)$ by $(\xi - x)$.



$$K=\frac{1+\nu}{1-\nu}\cdot\frac{W_0 G\alpha_t}{2\lambda}\sqrt{\frac{2k}{\omega}},$$

$$\eta=(x-\xi)\sqrt{\frac{\omega}{2k}},$$

$$\tau=\omega t-\varepsilon.$$

Fig. 2

$$K = \frac{1+\nu}{1-\nu} \cdot \frac{W_0 G \alpha_t}{2\lambda} \sqrt{\frac{2k}{\omega}},$$

$$\eta = x \sqrt{\frac{\omega}{2k}},$$

$$\xi \sqrt{\frac{\omega}{2k}} = \frac{\pi}{2},$$

$$\tau = \omega t - \varepsilon.$$

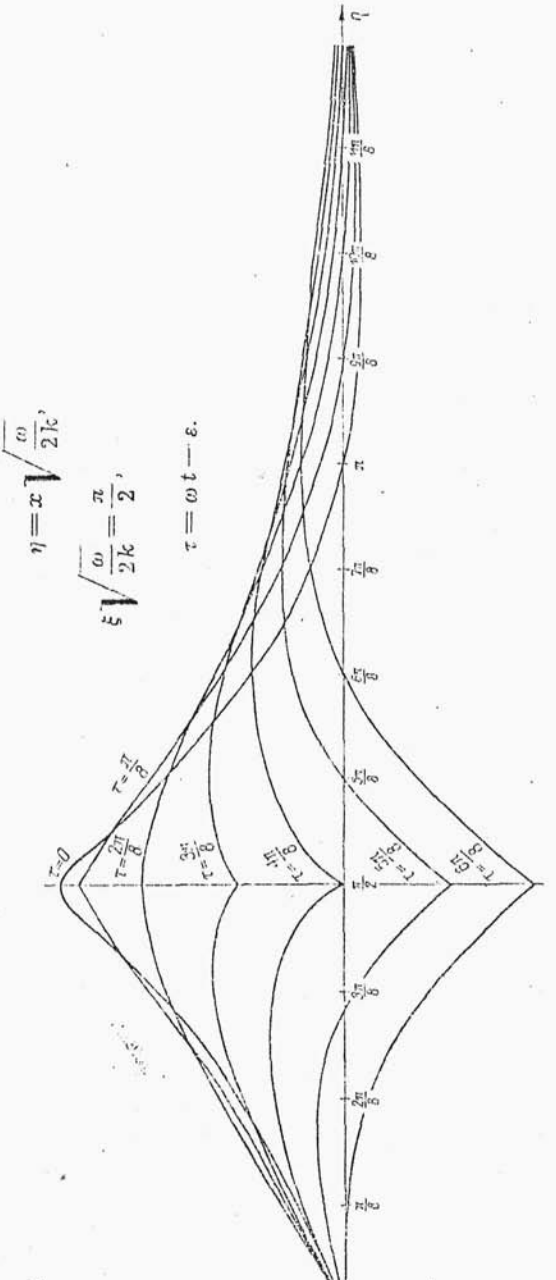


Fig. 3.

In Figure 3 the function σ_{yy} is represented for various values of the parameter τ.

The solutions obtained for a heat source varying in a harmonic manner can be used for constructing solutions changing in a periodic manner.

Expanding the function $W(t)$ in a Fourier series

$$W(t) = \sum_{n=0}^{\infty} A_n \cos(n \omega t - \varepsilon_n), \tag{1.36}$$

we obtain the temperature and stress field as a result of superposition of harmonic elements.

Thus, in the case of a source of intensity $W(t)$ acting in an infinite elastic space and varying with time in a periodic manner, we obtain for the temperature field the following expression:

$$T = \frac{1}{4\pi\lambda R} \sum_{n=0}^{\infty} A_n \exp\left(- R \sqrt{\frac{\omega n}{2k}}\right) \cos\left(n\omega t - \varepsilon_n - R\sqrt{\frac{\omega n}{2k}}\right). \tag{1.37}$$

Moreover, the solutions obtained can be used to determine the temperature and stress fields in the case of heat sources distributed over any region Γ of the elastic space. If in the region Γ there acts a heat source, constituting a harmonic function of time and any function of the coordinate, the temperature field will be expressed as

$$T(x, y, z, t) = \frac{1}{4\pi\lambda} \iiint_{(\Gamma)} \frac{W_0(\xi, \eta, \zeta)}{R} \exp\left(-R\sqrt{\frac{\omega}{2k}}\right) \times \\ \times \cos\left(\omega t - \varepsilon - R\sqrt{\frac{\omega}{2k}}\right) d\xi\, d\eta\, d\zeta,$$

where

$$R = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}.$$

This investigation was communicated at the meeting of the Department of Mechanics of Continuous Media of the Institute of Basic Technical Problems, on January 28, 1957.

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[1] E. Melan, H. Parcus, *Wärmespannungen stationärer Temperaturfelder*, Vienna, 1953.