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APPLIED MECHANICS

The Stress Function in Three-Dimensional Problems Concerning an Elastic Body Characterized by Transverse Isotropy

by

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In the theory of elasticity of anisotropic bodies for three-dimensional problems there is but one stress function as yet known.

It is the function given by S. G. Lechnicki [1] for axially symmetrical problems concerning bodies characterized by the transverse isotropy and is a generalization of A. E. H. Love's [2] stress function for axially symmetrical problems in the case of isotropy.

In this paper we present a new stress function for elastic bodies characterized by the transverse isotropy. This function is not limited to the circularly symmetrical problems. In the particular case of an isotropic body it is identical with the stress function derived by B. G. Galerkin [3].

1. Let us consider an elastic, semi-infinite, transversely isotropic body. The system of coordinates is chosen in such a way, that its origin and x, y axes may lie in the plane limiting the semi-infinite body.

The components of stress are related to those of displacement as follows [4]:

$$\begin{aligned}
&\sigma_x = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{13}\frac{\partial w}{\partial z}, \qquad \tau_{yz} = A_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\\
(1.1)\qquad &\sigma_y = A_{21}\frac{\partial u}{\partial x} + A_{11}\frac{\partial v}{\partial y} + A_{13}\frac{\partial w}{\partial z}, \qquad \tau_{xz} = A_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\\
&\sigma_z = A_{31}\frac{\partial u}{\partial x} + A_{31}\frac{\partial v}{\partial y} + A_{33}\frac{\partial w}{\partial z}, \qquad \tau_{xy} = A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right);
\end{aligned}$$

where

$$A_{12} = A_{21}, \qquad A_{13} = A_{31}, \qquad A_{66} = \tfrac{1}{2}(A_{11} - A_{12}).$$

The equations of displacement in the theory of elasticity of transversely anisotropic bodies will take the form:

$$A_{11}\frac{\partial^2 u}{\partial x^2}+A_{66}\frac{\partial^2 u}{\partial y^2}+A_{44}\frac{\partial^2 u}{\partial z^2}+\frac{\partial}{\partial x}\left[(A_{12}+A_{66})\frac{\partial v}{\partial y}+(A_{13}+A_{44})\frac{\partial w}{\partial z}\right]=0,$$

$$(1.2)\quad A_{66}\frac{\partial^2 v}{\partial x^2}+A_{11}\frac{\partial^2 v}{\partial y^2}+A_{44}\frac{\partial^2 v}{\partial z^2}+\frac{\partial}{\partial y}\left[(A_{12}+A_{66})\frac{\partial u}{\partial x}+(A_{13}+A_{44})\frac{\partial w}{\partial z}\right]=0,$$

$$A_{44}\frac{\partial^2 w}{\partial x^2}+A_{44}\frac{\partial^2 w}{\partial y^2}+A_{33}\frac{\partial^2 w}{\partial z^2}+\frac{\partial}{\partial z}\left[(A_{13}+A_{44})\frac{\partial u}{\partial x}+(A_{13}+A_{44})\frac{\partial v}{\partial y}\right]=0.$$

We assume the stress function $\varphi(x, y, z)$ satisfying the following relations:

$$(1.3)\qquad u=-\frac{\partial^2\varphi}{\partial x\partial z},\quad v=-\frac{\partial^2\varphi}{\partial y\partial z},\quad w=a\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+b\frac{\partial^2\varphi}{\partial z^2},$$

where

$$a=\frac{A_{11}}{A_{13}+A_{44}},\qquad b=\frac{A_{44}}{A_{13}+A_{44}}.$$

The displacements u, v, w, expressed by equations (1.3) in terms of φ, transform the first two equations of the system (1.2) into identities; substituted into the last equation of the system (1.2) they result in the following differential equation for $f(\varphi)$:

$$(1.4)\quad A_{11}\left(\frac{\partial^4\varphi}{\partial x^4}+\frac{\partial^4\varphi}{\partial y^4}\right)+A_{33}\frac{\partial^4\varphi}{\partial z^4}+2A_{11}\frac{\partial^4\varphi}{\partial x^2\partial y^2}+2H\left(\frac{\partial^4\varphi}{\partial z^2\partial y^2}+\frac{\partial^4\varphi}{\partial z^2\partial x^2}\right)=0,$$

where

$$2H=\frac{A_{11}A_{33}-A_{13}^2-2A_{13}A_{44}}{A_{44}}.$$

Introducing

$$\varrho=\frac{H}{\sqrt{A_{11}A_{33}}},\qquad \varepsilon^{1}=\frac{A_{11}}{A_{33}}$$

we obtain

$$(1.5)\qquad \nabla_1^2\nabla_2^2\varphi(x,y,z)=0,$$

where

$$\nabla_{1,2}^2=\frac{\partial^2}{\partial z^2}+\mu_{1,2}^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

and

$$\mu_{1,2}=\varepsilon\sqrt{\varrho\pm\sqrt{\varrho^2-1}}\qquad \text{for}\quad \varrho>1$$

$$\mu_{1,2}=\varepsilon\qquad \text{for}\quad \varrho=1$$

$$\mu_{1,2}=\varepsilon\left[\sqrt{\frac{1+\varrho}{2}}\pm i\sqrt{\frac{1-\varrho}{2}}\right]\qquad \text{for}\quad \varrho<1.$$

The parameter ϱ is most important for solving equation (1.5). Depending on whether $\varrho > 1$ or $\varrho = 1$ or $\varrho < 1$ we obtain different types of solutions of eq. (1.5).

In the particular case of $\varrho = 1$; $\varepsilon > 0$, equation (1.5) can be reduced to the biharmonic equation

$$\nabla^4 \varphi(\xi, \eta, \zeta) = 0,$$

where

$$\xi = x, \qquad \eta = y, \qquad \zeta = \varepsilon z.$$

The function φ is chosen to satisfy the equation (1.5) and the boundary conditions. The knowledge of that function allows us to determine the components of the stress tensor from eqs. (1.1) and the displacements from eqs. (1.3).

2. The application of the stress function φ will be explained on two examples.

a. Let us consider a plane limiting a semi-infinite body loaded by a vertical load $p(x, y)$, symmetrical with respect to the axes x and y.

The stress function is assumed in the form of a symmetrical Fourier integral

$$\varphi = \int_0^\infty \int_0^\infty Z(z) \cos \alpha x \cos \beta y \, d\alpha \, d\beta. \tag{2.1}$$

In the case of $z \to \infty$ the stresses vanish, and we assume for $\varrho > 1$

$$Z = Ae^{-\gamma\mu_1 z} + Be^{-\gamma\mu_2 z},$$

where

$$\mu_{1,2} = \varepsilon \sqrt{\varrho \pm \sqrt{\varrho^2 - 1}}, \qquad \gamma = \sqrt{\alpha^2 + \beta^2}.$$

For $z = 0$ we should have

$$\sigma_z(x, y, o) = -p(x, y), \qquad \tau_{xz}(x, y, o) = \tau_{yz}(x, y, o) = 0.$$

The first condition leads to the relation

$$\frac{\partial}{\partial z}\left[(A_{33}a - A_{13})\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) + A_{33} b \frac{\partial^2 \varphi}{\partial z^2}\right]_{z=0} = -p(x, y). \tag{2.2}$$

The other two conditions can be reduced to a single one

$$\left[(b-1)\frac{\partial^2 \varphi}{\partial z^2} + a\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)\right]_{z=0} = 0. \tag{2.3}$$

We express the external load by means of Fourier's integral

$$p(x, y) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty p(\alpha, \beta) \cos \alpha x \cos \beta y \, d\alpha \, d\beta, \tag{2.4}$$

which is possible if the value of integral $\int_0^\infty \int_0^\infty p(\alpha, \beta) \, d\alpha \, d\beta$ is finite.

Substituting (2.1) and (2.4) into the boundary condition we obtain the constants A, B. Then

$$(2.5)\qquad \varphi = \frac{4}{\pi^2}\frac{\sqrt{A_{11}A_{33}}}{(A_{11}A_{33}-A_{13}^2)(\mu_1-\mu_2)}\int_0^\infty\int_0^\infty \frac{p(\alpha,\beta)}{\gamma^3}\left[(1+\mu_2^2\eta)e^{-\gamma\mu_1 z} - (1+\mu_1^2\eta)e^{-\gamma\mu_2 z}\right]\cos\alpha x\cos\beta y\,d\alpha\,d\beta,$$

where $\eta = A_{13}/A_{11}$.

The knowledge of the function φ allows us to determine the components of stress and displacement and the problem can be regarded as solved.

For example, we have

$$w(x,y,z) = \frac{4\sqrt{A_{11}A_{33}}}{\pi^2(A_{11}A_{33}-A_{13}^2)(\mu_1-\mu_2)}\int_0^\infty\int_0^\infty \frac{p(\alpha,\beta)}{\gamma}\left[(\mu_2^2\eta+1)(b\mu_1^2-a)e^{-\gamma\mu_1 z} - (\mu_1^2\eta+1)(b\mu_2^2-a)e^{-\gamma\mu_2 z}\right]\cos\alpha x\cos\beta y\,d\alpha\,d\beta.$$

b. Let us consider a thin isotropic infinite plate resting on an elastic semi-infinite body of transverse isotropy. We assume that there exists no friction between the plate and the foundation.

The differential equation of deflection will take the form

$$(2.6)\qquad N\nabla^4\,\overline{w}(x,y) = q(x,y) - p(x,y),$$

where $\overline{w}(x,y)$ denotes the deflection of the plate, N — its flexural rigidity, $q(x,y)$ — the load, and $p(x,y)$ — the reaction of the elastic foundation. We assume that the plate cannot be separated from the base. Then we have

$$\overline{w}(x,y) = w(x,y,0),$$

where $w(x,y,0)$ is a vertical displacement of the plane limiting the semi-infinite elastic body.

Substituting φ from equation (2.5) in the expression for w (1.3), we have

$$(2.7)\qquad w(x,y,0) = C\int_0^\infty\int_0^\infty \frac{p(\alpha,\beta)}{\gamma}\cos\alpha x\cos\beta y\,d\alpha\,d\beta,$$

where

$$C = \frac{4\sqrt{A_{11}A_{33}}(\mu_1+\mu_2)}{\pi^2(A_{11}A_{33}-A_{13}^2)}.$$

Expressing the right side of equation (2.6) in terms of the double Fouriers's integral we have

$$NC\gamma^3 p(\alpha,\beta) = \frac{4}{\pi^2}\left[q(\alpha,\beta) - p(\alpha,\beta)\right].$$

Hence the deflection of the plate, resting on elastic semi-infinite body, will take the form

$$\overline{w}(x,y) = C \int_0^\infty \int_0^\infty \frac{q(\alpha,\beta)}{\gamma + \frac{NC\gamma^4\pi^2}{4}} \cos \alpha x \cos \beta y \, d\alpha \, d\beta. \tag{2.8}$$

Knowing the deflection surface of the plate, we can determine any statical quantity (bending and twisting moments, shearing forces etc.) of the plate.

The applications of the function in question are somewhat limited, however, and it cannot be applied in the case of problems with three boundary conditions instead of two.

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