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APPLIED MECHANICS

Influence Surfaces of Plates Shaped as Sectors of a Circular Ring

by

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The aim of the authors of the present work *) consists in determining an influence surface for deflection and other statical quantities like bending moments, twisting moments and the shear forces of a plate. This problem is identical with that of the determination of Green's function for the above-mentioned quantities.

As is well known, the statical quantities for a plate in polar coordinates can be expressed respectively by the following formulae:

a) bending moments

$$M_{r,j} = -N\left[\frac{\partial^2 w_j}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_j}{\partial \varphi^2}\right)\right];$$

$$M_{\varphi,j} = -N\left[\nu\frac{\partial^2 w_j}{\partial r^2} + \frac{1}{r}\frac{\partial w_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_j}{\partial \varphi^2}\right];$$

b) twisting moments

$$M_{r\varphi,j} = -N(1-\nu)\left[\frac{1}{r}\frac{\partial^2 w_j}{\partial r\,\partial \varphi} - \frac{1}{r^2}\frac{\partial w_j}{\partial \varphi}\right];$$

c) shearing forces

$$T_{r,j} = -N\frac{\partial \nabla^2 w_j}{\partial r}; \qquad T_{\varphi,j} = -N\frac{1}{r}\frac{\partial \nabla^2 w_j}{\partial \varphi}$$

where w_j is the deflection surface of the plate

$$N = \frac{Eh^3}{12(1-\nu^2)}$$

where E is the modulus of elasticity, ν — Poisson's ratio and h the thickness of the plate.

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The main task was to determine Green's function for a wedge-shaped plate of infinite radius resting freely on its edges — $\varphi = 0$, and $\varphi = \alpha$ (Fig. 1).

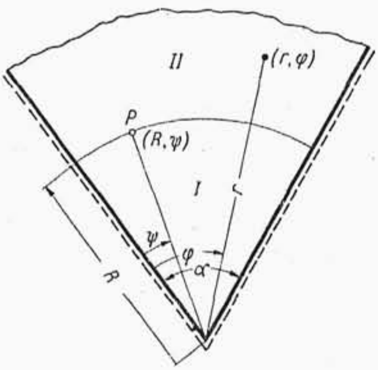

Fig. 1

The differential equation

$$\nabla^2 \nabla^2 w_j(r,\varphi) = 0 \qquad j = I, II$$

of plate deflection was replaced by a system of differential equations

$$\nabla^2 w_j(r,\varphi) = \Phi_j(r,\varphi)$$
$$\nabla^2 \Phi_j(r,\varphi) = 0$$

where the index $j = I$ is related to region I $(\varrho = \frac{r}{R} \leqslant 1)$ and the index $j = II$ to region II $(\varrho = \frac{r}{R} \geqslant 1)$.

The function $\Phi_j(r,\varphi)$ is proportional to the sum of the bending moments $(M_{r,j} + M_{\varphi j})$ and can be presented in the following form:

$$\text{(1a)} \qquad \Phi_I(r,\varphi) = \frac{P}{N\pi} \sum_{n=1,2\ldots}^{\infty} \frac{\varrho^{nk}}{n} \sin nk\psi \sin nk\varphi \qquad \text{for} \quad \varrho \leqslant 1,$$

$$\Phi_{II}(r,\varphi) = \frac{P}{N\pi} \sum \frac{\varrho^{-nk}}{n} \sin nk\psi \sin nk\varphi \qquad \text{for} \quad \varrho \geqslant 1,$$

where

$$k = \frac{\pi}{\alpha}, \quad \varrho = \frac{r}{R}$$

or in the closed form

$$\text{(1b)} \qquad \Phi_j(r,\varphi) = -\frac{P}{4N\pi} \ln \frac{\cosh(k \ln \varrho) - \cos k(\varphi - \psi)}{\cosh(k \ln \varrho) - \cos k(\varphi + \psi)} \qquad \text{for } \varrho \gtrless 1.$$

A simple calculation shows that the following relations exist between certain differential operations on function w_j and function Φ and its first derivatives:

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial w_j}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2 w_j}{\partial \varphi^2}\right) = \frac{1}{2}\Phi_j - \frac{1}{4}(\varrho - \varrho^{-1})\frac{\partial \Phi_j}{\partial \varrho},$$

$$\text{(2)} \qquad \frac{1}{R^2}\frac{\partial^2 w_j}{\partial \varrho^2} = \frac{1}{2}\Phi_j + \frac{1}{4}(\varrho - \varrho^{-1})\frac{\partial \Phi_j}{\partial \varrho},$$

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial^2 w_j}{\partial \varphi \partial \varrho} - \frac{1}{\varrho^2}\frac{\partial w_j}{\partial \varphi}\right) = (1 - \varrho^{-2})\frac{\partial \Phi_j}{\partial \varphi}.$$

Hence the statical quantities for a wedge-shaped plate of infinite radius charged with a concentrated force, and their influence surfaces can be represented by means of the function Φ and its derivatives.

And so we get:

$$M_{r,j} = -N\left[\frac{1+\nu}{2}\,\Phi + \frac{1-\nu}{4}(\varrho-\varrho^{-1})\frac{\partial \Phi_j}{\partial \varrho}\right],$$

$$M_{\varphi,j} = -N\left[\frac{1+\nu}{2}\,\Phi_j - \frac{1-\nu}{2}(\varrho-\varrho^{-1})\frac{\partial \Phi_j}{\partial \varrho}\right],$$

$$M_{r\varphi,j} = -N\left[\frac{1-\nu}{4}(1-\varrho^{-2})\frac{\partial \Phi_j}{\partial \varphi}\right],$$

$$T_{r,j} = -\frac{N}{R}\frac{\partial \Phi_j}{\partial \varrho},$$

$$T_{\varphi,j} = -\frac{N}{\varrho R}\frac{\partial \Phi_j}{\partial \varphi}.$$

Fig. 2 a represents the influence surface $8\pi M_r$; Fig. 2 b shows the influence surface $8\pi M_\varphi$ at a point $(R, \pi/8)$ for a wedge-shaped plate with central angle $\pi/4$.

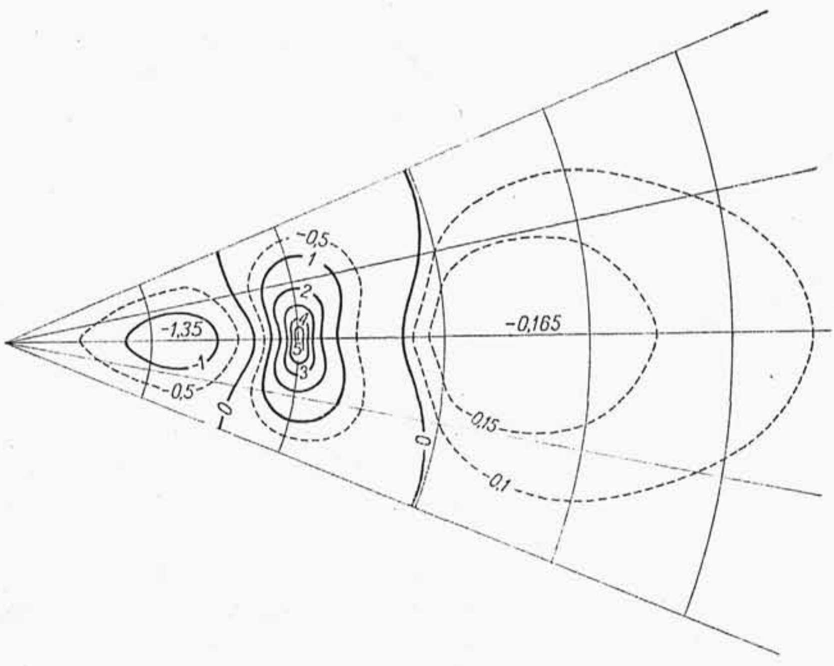

Fig. 2 a

The solution for a wedge of infinite radius contains, as a particular case, A. Nadai's [1] solution for a infinite plate band.

The solutions obtained for a wedge-shaped plate were used to determine Green's function for a plate shaped as a flat ring sector. The solution for the deflection surface was represented as the sum of two functions:

$$w = w_j + w_1. \tag{3}$$

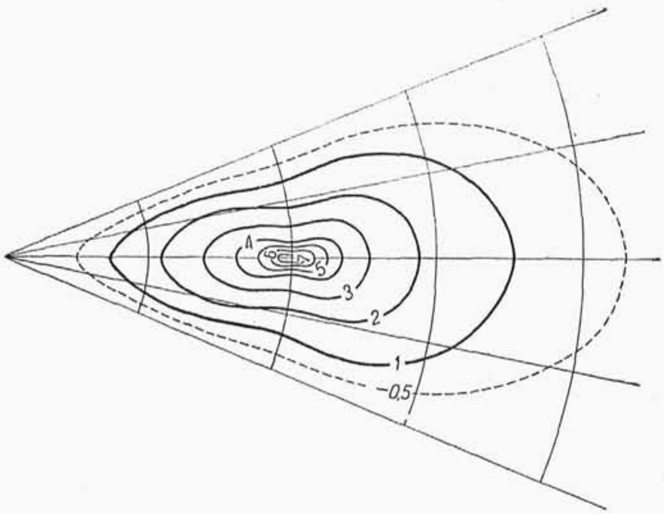

Fig. 2 b

Here w_j is Green's function for a wedge-shaped plate of infinite radius and function w_1 takes into account the boundary conditions at the edges. These particular cases where the section forces can be represented in closed form were elaborated in detail.

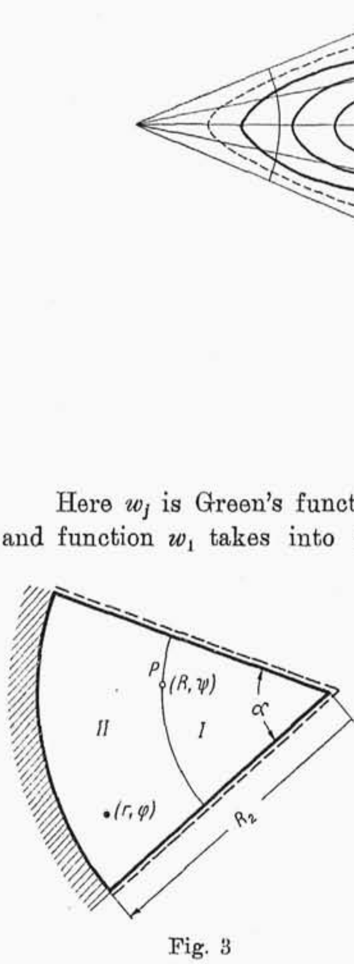

Fig. 3

a. Plate in the form of a circular sector and resting freely on its edges $\varphi = 0$ and $\varphi = \alpha$ and clamped on an arc $r = R_2$.

A solution was obtained using equation (3). All the statical quantities determined for function w_j are given when discussing the case of a infinite wedge-shaped plate. We give here only the derivatives of the supplementary function w_1 according to formula (3) by means of which we are able to determine the statical quantities of the plate.

Thus:

$$\frac{1}{R^2}\frac{\partial^2 w_1}{\partial \varrho^2} = -\frac{1}{2}\Psi - \frac{1}{4}\frac{\partial \Psi}{\partial \varrho}(4\varrho - \varrho_2^2\varrho^{-1} - 3\varrho\varrho_2^{-2}) - \frac{1}{4}\frac{\partial^2 \Psi}{\partial \varrho^2}(1 - \varrho_2^2 + \varrho^2 - \varrho^2\varrho_2^{-2}),$$

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial w_1}{\partial \varrho}+\frac{1}{\varrho^2}\frac{\partial^2 w_1}{\partial \varphi^2}\right)=-\frac{1}{2}\Psi-\frac{1}{4}\frac{\partial \Psi}{\partial \varrho}(\varrho_2^2\varrho^{-1}-\varrho\varrho_2^{-2})+$$
$$+\frac{1}{4}\frac{\partial^2 \Psi}{\partial \varrho^2}(1-\varrho_2^2+\varrho^2-\varrho^2\varrho_2^{-2}),$$

(4) $$\nabla^2 w_1=-\Psi-\frac{\partial \Psi}{\partial \varrho}(\varrho-\varrho\varrho_2^{-2}),$$

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial^2 w_1}{\partial \varrho\partial\varphi}-\frac{1}{\varrho^2}\frac{\partial w_1}{\partial \varphi}\right)=-\frac{1}{4}\frac{\partial \Psi}{\partial \varphi}(1-\varrho_2^{-2})-\frac{1}{4}\frac{\partial^2 \Psi}{\partial \varrho\partial\varphi}(1-\varrho_2^{-2})(\varrho-\varrho_2^2\varrho^{-1}),$$

where

$$\varrho_2=\frac{R_2}{R},$$

but

(5) $$\Psi(\varrho,\varphi)=-\frac{P}{4N\pi}\ln\frac{\cosh(k\ln\varrho\varrho_2^{-2})-\cos k(\varphi-\psi)}{\cosh(k\ln\varrho\varrho_2^{-2})-\cos k(\varphi+\psi)}.$$

b. For the Fig. 4 plate we introduce a new function

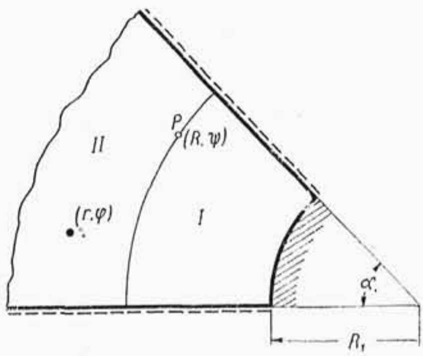

Fig. 4

(6) $$\Theta(\varrho,\varphi)=-\frac{P}{4N\pi}\ln\frac{\cosh(k\ln\varrho\varrho_1^{-2})-\cos k(\varphi-\psi)}{\cosh(k\ln\varrho\varrho_1^{-2})-\cos k(\varphi+\psi)}$$

where

$$\varrho_1=\frac{R_1}{R}.$$

In this case a solution was also obtained by using equation (3). The following relations exist between function $\Theta(\varrho,\varphi)$ and the derivatives of function w_1:

$$\frac{1}{R^2}\frac{\partial^2 w_1}{\partial \varrho^2}=-\frac{1}{2}\Theta-\frac{1}{4}\frac{\partial \Theta}{\partial \varrho}(4\varrho-\varrho_1^2\varrho^{-1}-3\varrho\varrho_1^{-2})-\frac{1}{4}\frac{\partial^2 \Theta}{\partial \varrho^2}(1-\varrho_1^2+\varrho^2-\varrho^2\varrho_1^{-2}),$$

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial w_1}{\partial \varrho}+\frac{1}{\varrho^2}\frac{\partial^2 w_1}{\partial \varphi^2}\right)=-\frac{1}{2}\Theta-\frac{1}{4}\frac{\partial \Theta}{\partial \varrho}(\varrho_1^2\varrho^{-1}-\varrho\varrho_1^{-2})+$$
$$+\frac{1}{4}\frac{\partial^2 \Theta}{\partial \varrho^2}(1-\varrho_1^2+\varrho^2-\varrho^2\varrho_1^{-2}),$$

$$\nabla^2 w_1 = -\Theta - \frac{\partial \Theta}{\partial \varrho}(\varrho - \varrho\varrho_1^{-2}), \tag{7}$$

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial^2 w_1}{\partial\varrho\,\partial\varphi} - \frac{1}{\varrho^2}\frac{\partial w_1}{\partial\varphi}\right) = -\frac{1}{4}\frac{\partial\Theta}{\partial\varphi}(1-\varrho_1^{-2}) - \frac{1}{4}\frac{\partial^2\Theta}{\partial\varrho\,\partial\varphi}(1-\varrho_1^{-2})(\varrho - \varrho_1^2\varrho^{-1}).$$

c. Finally, let us consider the case of a wedge-shaped plate (Fig. 5) charged with a force P and additionally supported on a curvilinear support

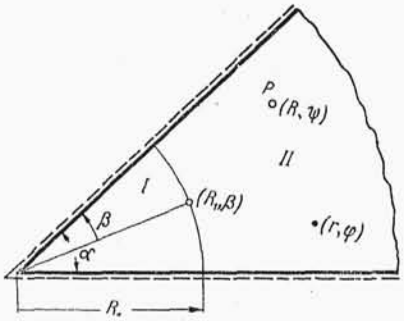

Fig. 5

$r = R_1$. The solution was obtained in this case by using the method of integral equations [2] for plates with mixed boundary conditions. If $R > R_1$, i. e. if a fonce P is placed in region *II*, the plate deflection is expressed in the closed form by:

$$w(r,\varphi) = -\frac{PR_1^2}{32\,N\pi}\left(1 - \frac{Tr^2}{R_1^2}\right)\left(\frac{R^2}{R_1^2} - 1\right)\ln\frac{\cosh(k\ln\varrho) - \cos k(\varphi-\psi)}{\cosh(k\ln\varrho) - \cos k(\varphi+\psi)}. \tag{8}$$

The deflection in region *II* is expressed by different series for $r > R$ and $r < R$, but their derivatives can be added up and presented in the closed form:

$$\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial\varrho^2}\right) = \frac{1}{2}\Phi + \frac{1}{4}(\varrho - \varrho^{-1})\frac{\partial\Phi}{\partial\varrho} - \frac{1}{4}\left[(1+\varrho_1^{-2})\Theta + (2\varrho - \varrho_1^2\varrho^{-1} - \varrho\varrho_1^{-2})\frac{\partial\Theta}{\partial\varrho}\right] - $$
$$ - \frac{1}{8}(1 + \varrho^2 - \varrho^2\varrho_1^{-2} - \varrho_1^2)\frac{\partial^2\Theta}{\partial\varrho^2}.$$

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial w}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2 w}{\partial\varrho^2}\right) = \frac{1}{2}\Phi - \frac{1}{4}(\varrho - \varrho^{-1})\frac{\partial\Phi}{\partial\varrho} - \frac{1}{4}\left[(1+\varrho_1^{-2})\Theta + (\varrho_1^2\varrho^{-1} - \varrho\varrho_1^{-2})\frac{\partial\Theta}{\partial\varrho}\right] + $$
$$ + \frac{1}{8}(1 + \varrho^2 - \varrho^2\varrho_1^{-2} - \varrho_1^2)\frac{\partial^2\Theta}{\partial\varrho^2},$$

$$\nabla^2 w = \Phi - \frac{1}{2}\left[(1+\varrho_1^2)\Theta + (\varrho - \varrho\varrho_1^{-2})\frac{\partial\Theta}{\partial\varrho}\right], \tag{9}$$

$$\frac{1}{R^2}\left(\frac{1}{\varrho}\frac{\partial^2 w}{\partial\varrho\,\partial\varphi} - \frac{1}{\varrho^2}\frac{\partial w}{\partial\varphi}\right) =$$
$$= \frac{1}{4}(1-\varrho^{-2})\frac{\partial\Phi}{\partial\varphi} - \frac{1}{8}\left[(1 - \varrho^{-2} + \varrho_1^{-2} - \varrho_1^2\varrho^{-2})\frac{\partial\Theta}{\partial\varphi} + (\varrho + \varrho^{-1} - \varrho\varrho_1^{-2} - \varrho_1^2\varrho^{-1})\frac{\partial^2\Theta}{\partial\varrho\,\partial\varphi}\right].$$

The function Φ is expressed by formula (1) and Θ by formula (6).

If in the given cases (a to c) we pass for α to $\rightarrow 0$ and for R_1 to $\rightarrow \infty$, we shall obtain solutions for the semi-infinite strip or the well-known solution of S. Wojnowsky-Krieger [3] for a strip with additional rectilinear support.

A knowledge of influence surfaces in the plate systems we have been discussing makes possible to determine deflections and statical quantities for arbitrary plate charges.

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