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APPLIED MECHANICS

The Problem of Rectangular Plates with Mixed Boundary Conditions

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The object of this note*) is to find the deflection surface of a rectangular plate arbitrarily loaded, the boundary being divided into sections of different boundary conditions.

Let us consider the simple case of a rectangular plate (Fig. 1) arbitrarily loaded and simply supported at the edges $x=0$, $x=a$, and $x=b$.

Let us now assume two pairs of different boundary conditions existing at c_1 and c_2. For c_1 let us put

$$l_1 w(x, 0)=0, \qquad l_2 w(x, 0)=0 \tag{1}$$

and for c_2

$$g_1 w(x, 0)=0 \qquad g_2 w(x, 0)=0, \tag{2}$$

where l_1, l_2, g_1, g_2 denote linear differential operators, and $w(x, y)$, any ordinate of the deflection surface of the plate.

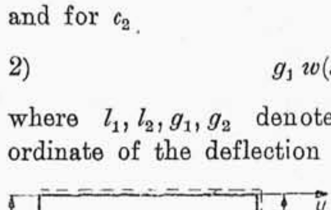

Fig. 1

The problem is to solve the differential equations of the deflection surface

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p(x, y)}{N}$$

with boundary conditions (1) and (2) at the edges $y=0$, the conditions for other edges being $\nabla^2_w=0$, $w=0$.

Let us note that for c the operators $g_1 w(x, 0)$ and $g w(x, 0)$ are unknown functions of x; the same may be said of the operators $l_1 w(x, 0)$ and $l_2 w(x, 0)$ at c_2.

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*) This note will be published in extenso in "Archiwum Mechaniki Stosowanej" **5** (1953), 2.

Let us assume as basic — a system with $g_1 w(x,0)=0$ and $g_2 w(x,0)=0$ as boundary conditions at the edge $y=0$ (for the portions c_1 and c_2 respectively), the conditions for the other edges being $\nabla^2_w = 0$, $w=0$.

Let us denote by $w_0(x,y)$ the deflection surface corresponding to the load $p(x,y)$ and $G_1(x,y;\xi,0)$, $G_2(x,y;\xi,0)$, the functions of Green for the conditions $g_1 w(\xi)=1$ and $g_2 w(\xi)=1$ in the basic system defined above. The deflection surface $w(x,y)$ of the plate may be expressed as the sum of the two components $w(x,y)$ and $w_1(x,y)$, the latter taking into account the unknown boundary functions $g_1 w(\xi)$ and $g_2 w(\xi)$ at c_1:

$$w(x,y) = w_0(x,y) + w_1(x,y) \tag{3}$$

where

$$w_1(x,y) = \int_{c_1} g_1 w(\xi)\cdot G_1(x,y;\xi,0)\,d\xi + \int_{c_2} g_2 w(\xi)\cdot G_2(x,y;\xi,0)\,d\xi\,;\ 0 \leqq \xi \leqq c_1. \tag{4}$$

The unknown functions $g_1 w(\xi)$ and $g_2 w(\xi)$ are obtained from the boundary conditions (1):

$$l_1 w(x,0) = 0, \qquad l_2 w(x,0) = 0.$$

Thus we arrive at a system of two Fredholm integral equations of the first type:

$$\begin{aligned} l_1 w_0(x,0) + \int_{c_1} g_1 w(\xi)\cdot l_1 G_1(x,0;\xi,0)\,d\xi + \int_{c_2} g_2 w(\xi)\cdot l_2 G_2(x,0;\xi,0)\,d\xi = 0 \\ l_2 w_0(x,0) + \int_{c_1} g_1 w(\xi)\cdot l_2 G_1(x,0;\xi,0)\,d\xi + \int_{c_2} g_2 w(\xi)\cdot l_2 G_2(x,0;\xi,0)\,d\xi = 0 \end{aligned} \tag{5}$$

where, according to Betti's reciprocal theorem,

$$l_2 G_1(x,0;\xi,0) = l_1 G_2(x,0;\xi,0). \tag{6}$$

From the system of equations (5) we find the unknown functions $g_1 w(\xi)$, $g_2 w(\xi)$ and from the equation (3, 4) the deflection surface of the plate. Of course, a system with $l_1 w(x,0)$ and $l_2 w(x,0) = 0$ as boundary conditions for the edge $y=0$ (c_1 and c_2 respectively) may also be assumed as basic.

Then

$$w(x,y) = \overline{w}_0(x,y) + \int_{c_2} l_1 w(\xi)\cdot L_1(x,y;\xi,0)\,d\xi + \int_{c_2} l_2 w(\xi)\cdot L_2(x,y;\xi,0)\,d\xi \tag{7}$$

where $\overline{w}_0(x,y)$ denotes the deflection surface corresponding to the load $p(x,y)$ in the basic system assumed. $L_1(x,y;\xi,0)$ and $L_2(x,y;\xi,0)$ are the functions of Green for $l_1 w = 1$, $l_2 w = 1$ in the basic system. The unknown functions $l_1 w(\xi)$ and $l_2 w(\xi)$ can be obtained from the system of equations resulting from the operation (2) performed on the equation (7).

Let us consider the particular case of the edge built in at c_1 and free at c_2.

We have for c_1

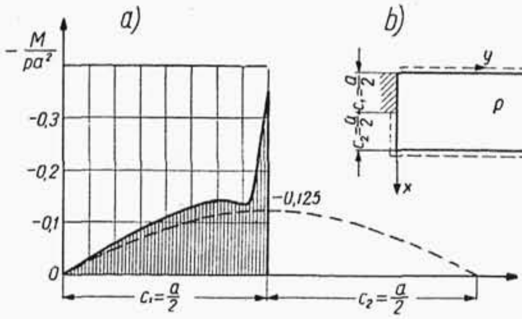
$$l_1 w(x,0) = w(x,0) = 0, \qquad l_2 w(x,0) = \frac{\partial w(x,0)}{\partial y} = 0.$$

and for c_2

$$g_1 w(x,0) = -N\left(\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right)_{y=0} = \dot{\bar{q}}_y(x,0) = 0$$

$$g_2 w(x,0) = -N\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)_{y=0} = m_y(x,0) = 0.$$

Here the basic system can be the plate under consideration, free at the edge $y=0$ and simply supported at the remaining edges. The unknown functions for c_1 are: the reaction at the support $g_1 w(\xi) = \bar{q}_y(\xi)$ and the bending moment at the built-in portion of the edge $g_2 w(\xi) = m_y(\xi)$



2Fig.

The basic system may also consist of the same plate built-in at the edge $y=0$.

The unknown functions for c_2 will be: the deflection $l_1 w(\xi) = w(\xi)$ and the angle of the tangent to the deflection surface $l_2 w(\xi) = \left.\frac{\partial w}{\partial y}\right|_{y=0} = \varphi(\xi)$.

The method described above of solving the problem of the bending of a rectangular plate with different boundary conditions for each of the two sections c_1 and c_2 of one edge can be generalised for any number of sections of one edge as well as for the case where all four edges are divided into any number of sections of different boundary conditions.

This method of solving problems concerning rectangular plates with mixed boundary conditions will be illustrated by several examples.

1. A narrow plate whose breadth is a is built-in at section c_1 of the short edge, c_2 and the remaining edges being simply supported. The plate is loaded by a uniform load p. Let us assume the system consisting of the plate simply supported at the edges $x=0$, $x=a$, and $y=0$ as basic. The unknown function is the bending moment $M(x) = -N\frac{\partial^2 w(x,0)}{\partial y}$ at the built-in section $c_1 = \frac{a}{2}$ of the edge. From the second equation of the

system (5) taking

$$l_1 w(x,0) = w(x,0) = 0, \qquad l_2 w(x,0) = \frac{\partial w(x,0)}{\partial y} = 0 \qquad \text{for} \quad c_1$$

and

$$g_1 w(x,0) = w(x,0) = 0, \qquad g_2 w(x,0) = -N \frac{\partial^2 w(x,0)}{\partial y} = 0 \quad \text{for} \quad c_2,$$

we obtain the following integral equation:

$$l_2 w_0(x,0) + \int_0^{a/2} g_2(\xi) \cdot l_2 G_2(x,0;\xi,0)\, d\xi = 0 \tag{8}$$

or

$$\frac{\partial w_0(x,0)}{\partial y} + \int_0^{a/2} M(\xi) \frac{\partial G_2(x,0;\xi,0)}{\partial y}\, d\xi = 0.$$

The function of Green G_2 should be satisfied by the differential equation $\nabla^2 \nabla^2 G_2 = 0$ and the boundary conditions $G_2 = 0$, $\nabla^2 G_2 = 0$, at the edges $x = 0$, $x = a$, $y = \infty$ and

$$G_2(x,0;\xi,0) = 0, \qquad -N \frac{\partial^2 G_2(x,0;\xi,0)}{\partial y^2} = \frac{2}{a} \sum_{n=1,2,\dots}^{\infty} \sin \frac{n\pi\xi}{a} \sin \frac{n\pi x}{a}.$$

Thus we obtain

$$G_2(x,y;\xi,0) = \frac{y}{N\pi} \sum_{n=1,2,\dots}^{\infty} \frac{e^{-\frac{n\pi y}{a}}}{n} \sin \frac{n\pi\xi}{a} \sin \frac{n\pi x}{a}$$

or

$$G_2(x,y;\xi,0) = -\frac{y}{4N\pi} \ln \frac{\cosh \frac{\pi y}{a} - \cos \frac{\pi}{a}(x-\xi)}{\cosh \frac{\pi y}{a} - \cos \frac{\pi}{a}(x+\xi)}$$

For the given load we have in the basic system:

$$w_0(x,y) = \frac{4pa^4}{N\pi^5} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^5} \left[1 - \left(1 + \frac{n\pi y}{2a}\right) e^{-\frac{n\pi y}{a}}\right] \sin \frac{n\pi x}{a}$$

Equation (8) thus appears as:

$$\int_0^{a/2} M(\xi) \ln \left| \frac{\sin \frac{\pi}{2a}(x-\xi)}{\sin \frac{\pi}{2a}(x+\xi)} \right| d\xi = \frac{4pa^3}{\pi^3} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^4} \sin \frac{n\pi x}{a} \tag{9}$$

Fig. 2 represents a diagram of the function $M(\xi)$ obtained as a result of an approximate solution of the integral equation (9).

For $c_1 = a$, $c_2 = 0$ we obtain the solution of equation (9) in the form of

$$M(x) = -\frac{px}{2}(a - x).$$

2. A narrow plate infinitely long with a slot coinciding with the x-axis, uniformly loaded over the whole area (Fig. 3). The shearing forces at the x-axis are equal to zero. Assuming the bending moment $m_y = -N \frac{\partial^2 w(x,0)}{\partial y^2} = M(x)$ as unknown we obtain the function $M(x)$ as a solution of the integral equation

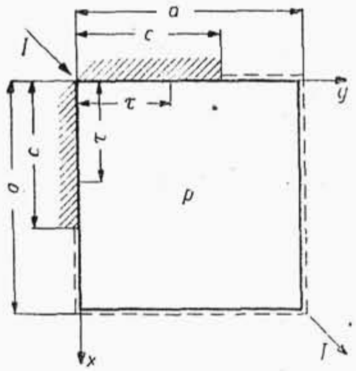


Fig. 3

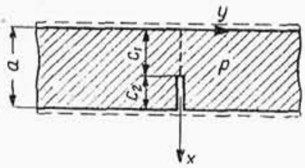


Fig. 4

$$\int_0^{c_1} M(\xi) \ln \left| \frac{\sin \frac{\pi}{2a}(x-\xi)}{\sin \frac{\pi}{2a}(x+\xi)} \right| d\xi = -\frac{4pa^2\nu}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^4} \sin \frac{n\pi x}{a}$$

(ν-being the inverse of Poisson's ratio).

For $c_2 = a$ we obtain

$$M(x) = \frac{\nu p x}{2}(a-x).$$

3. A rectangular plate is shown in Fig. 4. In order to find the unknown moment $M(\tau)$ at the built-in sections AC and AD of the edges we obtain a single integral equation

$$\int_0^c M(\tau) \left\{ \sum_{n=1,2,\ldots}^{\infty} \left[\frac{1}{n} \left(\operatorname{ctgh} \varrho_n - \frac{\varrho_n}{\sinh^2 \varrho_n} \right) \sin \alpha_n \tau + \right. \right.$$

$$\left. \left. + \frac{4}{\pi} \sum_{m=1,2\ldots}^{\infty} \frac{\sin \alpha_m \tau}{mn\left(\frac{n}{m} + \frac{m}{n}\right)^2} \right] \sin \alpha_n x \right\} d\tau =$$

$$= \frac{-2pa^3}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^4} \left(\operatorname{tgh} \frac{\varrho_n}{2} - \frac{\varrho_n}{2\cosh^2 \frac{\varrho_n}{2}} \right) \sin \alpha_n x.$$

$$\varrho_n = n\pi, \quad \alpha_n = \frac{n\pi}{a}, \quad \alpha_m = \frac{m\pi}{a}.$$

the system being symmetrical with respect to one diagonal.

Numerous examples of this type can be given. The main difficulty of the method described above is that of solving a system of integral equations. The method can be applied to plates of areas composed of rectangles as well as to continuous plates.