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THE PLANE PROBLEM OF MICROPOLAR THERMOELASTICITY

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1. Introduction

In this paper we shall be concerned with the plane states of strain and stress produced in an elastic micropolar (Cosserat) medium by the action of temperature.

We confine ourselves to the problem of stationary flow of heat.

However, prior to discussing the plane problem, we shall dwell briefly on the general state of stress in a micropolar body.

The action of temperature gives rise in the body to displacements $\mathbf{u}(\mathbf{x}, t)$ and rotations $\boldsymbol{\varphi}(\mathbf{x}, t)$. The state of strain of the body is described by two asymmetric tensors: the strain tensor γ_{ji} and the curvature-twist tensor $\varkappa_{ji}$. Both tensors are connected with the quantities $\mathbf{u}$ and $\boldsymbol{\varphi}$ by the relations [1-3]

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k, \quad \varkappa_{ji} = \varphi_{i,j}, \quad i, j, k = 1, 2, 3. \tag{1.1}$$

The state of stress is characterized by two asymmetric tensors. The tensor of force stresses σ_{ji} and of couple-stresses μ_{ji}. They are connected with the tensors γ_{ji}, $\varkappa_{ji}$ and σ_{ji}, μ_{ji} by the constitutive equations [4]

$$\begin{aligned} \sigma_{ji} &= (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + (\lambda\gamma_{kk} - \nu\theta)\delta_{ij} \\ \mu_{ji} &= (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ij}, \quad i, j, k = 1, 2, 3. \end{aligned} \tag{1.2}$$

The above equations should be regarded as Duhamel-Neumann equations extended to a micropolar body. In the relations (1.2) the symbols μ and λ are Lamé's costants, while α, β, γ, ε denote other material constants. We have $\nu = (3\lambda+2\mu)\alpha_t$ where α_t stands for the coefficient of thermal expansion. Substituting Eqs. (1.2) and (1.1) into the equations of equilibrium

$$\sigma_{ji,j} = 0, \quad \epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} = 0, \quad i, j, k = 1, 2, 3, \tag{1.3}$$

we obtain a system of equations in displacements and rotations, namely

$$\begin{aligned} &(\mu+\alpha)\nabla^2\mathbf{u} + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u} + 2\alpha\operatorname{rot}\boldsymbol{\varphi} = \nu\operatorname{grad}\theta, \\ &(\gamma+\varepsilon)\nabla^2\boldsymbol{\varphi} + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\varphi} - 4\alpha\boldsymbol{\varphi} + 2\alpha\operatorname{rot}\mathbf{u} = 0, \quad \nabla^2 = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}. \end{aligned} \tag{1.4}$$

The term θ representing the increase of temperature (the increase with respect to the temperature of the body in its natural state) may be determined from the equation of heat conduction

$$\nabla^2\theta = -\frac{W}{\lambda_0}. \tag{1.5}$$

Here the symbol W denotes the quantity of heat produced per unit time and volume while λ_0 is the coefficient of heat conduction. Equations (1.4) and (1.5) should be supplemented by boundary conditions. We write them in the form

$$\begin{gathered} p_i = \sigma_{ji}n_j = 0, \qquad m_i = \mu_{ji}n_j = 0, \\ \lambda_0\frac{\partial\theta}{\partial n} = \lambda_1(\theta_0-\theta), \qquad \mathbf{x}\in A. \end{gathered} \tag{1.6}$$

The first two conditions refer to the absence of loading (forces and moments) on the surface A bounding the body. The symbol θ_0 denotes here the temperature of the medium surrouding the considered body. λ_0 and λ_1 denote, respectively, the coefficients of internal and external heat conduction.

2. The Plane State of Strain

In the plane state of strain all causes and effects depend on two variables only. Assuming that the displacements and rotations do not depend on the variable x_3, we have

$$\mathbf{u} \equiv (u_1, u_2, 0), \qquad \boldsymbol{\varphi} \equiv (0, 0, \varphi_3), \tag{2.1}$$

where u_1, u_2, φ_3 are functions of the variables x_1, x_2.

In accordance with the definition (1.1) we obtain for the plane state of strain the following components of the tensors γ_{ji} and $\varkappa_{ji}$:

$$\begin{gathered} \gamma_{11} = \partial_1 u_1, \qquad \gamma_{22} = \partial_2 u_2, \qquad \gamma_{12} = \partial_1 u_2 - \varphi_3, \\ \gamma_{21} = \partial_2 u_1 + \varphi_3, \qquad \varkappa_{13} = \partial_1\varphi_3, \qquad \varkappa_{23} = \partial_2\varphi_3. \end{gathered} \tag{2.2}$$

The remaining values γ_{ji} and $\varkappa_{ji}$ are equal to zero. From the relations (1.2) we get

$$\begin{gathered} \sigma_{ji} = (\mu+\alpha)\gamma_{ji}+(\mu-\alpha)\gamma_{ij}+(\lambda\gamma_{kk}-\nu\theta)\delta_{ji}, \\ \sigma_{33} = \gamma_{kk}\lambda-\nu\theta, \qquad \mu_{j3} = (\gamma+\varepsilon)\varkappa_{j3}, \qquad \mu_{3j} = (\gamma-\varepsilon)\varkappa_{j3}, \qquad j = 1, 2. \end{gathered} \tag{2.3}$$

Here $\gamma_{kk} = \gamma_{11}+\gamma_{22}$. The state of stress σ_{ji} and the state of couple-stress μ_{ji} are characterized by the matrices

$$\boldsymbol{\sigma} \equiv \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{vmatrix}, \qquad \mathbf{u} \equiv \begin{vmatrix} 0 & 0 & \mu_{13} \\ 0 & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{vmatrix}. \tag{2.4}$$

The equations of equilibrium (1.3) for the plane state of strain are reduced to three equations, namely

$$\begin{gathered} \partial_1\sigma_{11}+\partial_2\sigma_{21} = 0, \\ \partial_1\sigma_{12}+\partial_2\sigma_{22} = 0, \\ \sigma_{12}-\sigma_{21}+\partial_1\mu_{13}+\partial_2\mu_{23} = 0. \end{gathered} \tag{2.5}$$

Eliminating the stresses from Eqs. (2.5) and taking into consideration Eqs. (2.2) and (2.3) we arrive at the following set of three equations:

$$
\begin{aligned}
&(\mu+\alpha)\nabla_1^2 u_1+(\mu+\lambda-\alpha)\partial_1 e+2\alpha\partial_2\varphi_3 = \nu\partial_1\theta\,,\\
&(\mu+\alpha)\nabla_1^2 u_2+(\mu+\lambda-\alpha)\partial_2 e-2\alpha\partial_1\varphi_3 = \nu\partial_2\theta\,,\\
&[(\gamma+\varepsilon)\nabla_1^2-4\alpha]\varphi_3+2\alpha(\partial_1 u_2-\partial_2 u_1) = 0\,.
\end{aligned}
\tag{2.6}
$$

We have

$$\partial_1 u_1+\partial_2 u_2 = e, \qquad \partial_1^2+\partial_2^2 = \nabla_1^2.$$

In the polar coordinate system we deal with the following vectors of displacements and rotations:

$$\mathbf{u}\equiv(u_r, u_\vartheta, 0), \qquad \boldsymbol{\varphi}\equiv(0, 0, \varphi_z). \tag{2.7}$$

In this system we have

$$
\begin{aligned}
&(\mu+\alpha)\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\vartheta}{\partial\vartheta}\right)+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}+\frac{2\alpha}{r}\frac{\partial\varphi_z}{\partial\vartheta} = \nu\frac{\partial\theta}{\partial r},\\
&(\mu+\alpha)\left(\nabla^2 u_\vartheta-\frac{u_\vartheta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\vartheta}\right)+(\lambda+\mu-\alpha)\frac{\partial e}{r\partial\vartheta}-2\alpha\frac{\partial\varphi_z}{\partial r} = \nu\frac{\partial\theta}{r\partial\vartheta},\\
&[(\gamma+\varepsilon)\nabla^2-4\alpha]\varphi_z+\frac{2\alpha}{r}\left(\frac{\partial}{\partial r}(ru_\vartheta)-\frac{\partial u_r}{\partial\vartheta}\right) = 0,
\end{aligned}
\tag{2.8}
$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}, \qquad e = \frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\vartheta}{\partial r}.$$

In the one-dimensional problem, for space, semi-space and an elastic layer, i.e. $u_1 = u_1(x_1)$, $u_2 = 0$, only one equation remains from the set (2.6)

$$(\lambda+2\mu)\partial_1^2 u_1 = \nu\partial_1\theta, \qquad u_2 = 0, \qquad \varphi_3 = 0\,. \tag{2.9}$$

In the case of axi-symmetric deformations, the system of Eqs. (2.8) reduces to the following one:

$$(\lambda+2\mu)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)u_r = \nu\frac{\partial\theta}{\partial r}. \tag{2.10}$$

Evidently, Eqs. (2.9) and (2.10) coincide with the equations of classical thermoelasticity for the case of one-dimensional problems. The stress tensor σ_{ji} is symmetric, the couple-stress tensor μ_{ji} is equal to zero.

Let us introduce in Eqs. (2.6) the vector

$$\boldsymbol{\zeta} = \frac{1}{2}\operatorname{rot}\mathbf{u}-\boldsymbol{\varphi}\,. \tag{2.11}$$

or

$$\zeta_1 = 0, \qquad \zeta_2 = 0, \qquad \zeta_3 = \frac{1}{2}(\partial_1 u_2-\partial_2 u_1)-\varphi_3\,.$$

Equations (2.6) now take the form

$$\begin{aligned} &\mu\nabla_1^2 u_1+(\lambda+\mu)\partial_1 e-2\alpha\partial_2\zeta_3 = \nu\partial_1\theta\,,\\ &\mu\nabla_1^2 u_2+(\lambda+\mu)\partial_2 e+2\alpha\partial_1\zeta_3 = \nu\partial_2\theta\,,\\ &[(\gamma+\varepsilon)\nabla_1^2-4\alpha]\zeta_3-\frac{1}{2}(\gamma+\varepsilon)\nabla_1^2(\partial_1 u_2-\partial_2 u_1) = 0\,. \end{aligned} \tag{2.12}$$

The solution of this system of equations will be composed of two parts

$$\begin{aligned} &u_1 = u_1'+u_1'', \quad u_2 = u_2'+u_2'',\\ &\zeta_3 = \zeta_3'+\zeta_3'', \quad \zeta_3' = 0\,. \end{aligned} \tag{2.13}$$

The primed functions are particular solutions of non-homogeneous Eqs. (2.12) while the functions with a double prime stand for the general solutions of homogeneous Eqs. (2.12). Introducing (2.13) into (2.12) we obtain

$$\begin{aligned} &\mu\nabla_1^2 u_1'+(\lambda+\mu)\partial_1 e' = \nu\partial_1\theta\,,\\ &\mu\nabla_1^2 u_2'+(\lambda+\mu)\partial_2 e' = \nu\partial_2\theta\,,\\ &\nabla_1^2(\partial_1 u_2'-\partial_2 u_1') = 0, \quad \zeta_3' = 0\,, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\mu\nabla_1^2 u_1''+(\lambda+\mu)\partial_1 e''-2\alpha\partial_2\zeta_3'' = 0\,,\\ &\mu\nabla_1^2 u_2''+(\lambda+\mu)\partial_2 e''+2\alpha\partial_1\zeta_3'' = 0\,,\\ &[(\gamma+\varepsilon)\nabla_1^2-4\alpha]\zeta_3''-\frac{1}{2}(\gamma+\varepsilon)\nabla_1^2(\partial_1 u_2'-\partial_2 u_1'') = 0\,. \end{aligned} \tag{2.15}$$

Thus we arrived at Eqs. (2.14) identical with equations of classical thermoelasticity [5]. The condition $\xi_3' = 0$ leads to the relation $\varphi_3' = \frac{1}{2}(\partial_1 u_2-\partial_2 u_1')$ which holds in the classical theory of thermoelasticity. The condition $(2.14)_3$ will be satisfied if we assume the displacements u_1', u_2' in the form

$$u_1' = \partial_1\Phi, \quad u_2' = \partial_2\Phi\,. \tag{2.16}$$

Substituting (2.16) into Eqs. $(2.14)_{1,2}$ we obtain after integration the Poisson equation for the function Φ:

$$\nabla_1^2\Phi = m\theta, \quad m = \frac{\nu}{\lambda+2\mu}. \tag{2.17}$$

The function Φ is the particular integral of the system of Eqs. (2.14), and by the same, is the particular integral of differential Eqs. (2.12)

The primed stresses and strains can be expressed with the help of function Φ in the following way:

$$\begin{aligned} &\gamma_{ji}' = \Phi_{,ij}, \quad \varkappa_{ji}' = 0\,,\\ &\sigma_{ji}' = 2\mu(\Phi_{,ij}-\delta_{ji}\nabla_1^2\Phi), \quad \mu_{ji}' = 0\,. \end{aligned} \tag{2.18}$$

In the case of an infinite region the function Φ is given by the formula [5]

$$\Phi(\xi_1, \xi_2) = -\frac{m}{4\pi} \int_A \frac{\theta(x_1, x_2)\, dx_1 dx_2}{R(x_1, x_2, \xi_1, \xi_2)}, \tag{2.19}$$

where

$$R = [(x_1-\xi_1)^2+(x_2-\xi_2)^2]^{1/2}.$$

For a bounded region we have to solve Eq. (2.17) with the boundary condition $\Phi = 0$.

Thus we have to solve only the system of Eqs. (2.15) which refers to the isothermal problem ($\theta = 0$). This is a typical boundary problem of the theory of elasticity of micropolar medium. If we assume the boundary to be free of loading, we may write the boundary conditions for the system of equations in the form

$$(\sigma'_{ji}+\sigma''_{ji})n_j = 0, \quad \mu''_{ji} n_j = 0, \quad i, j = 1, 2. \tag{2.20}$$

Thus, the displacements u_1'', u_2'' and the rotation φ_3'' being known, we can determine the stresses σ'_{ji} and μ_j'' from Eqs. (2.3) (obviously we have to set there $\theta = 0$).

Let us return to Eqs. (2.15). Differentiating the second with respect to x_1, the first with respect to x_2 and subtracting the results we obtain

$$\mu\nabla_1^2(\partial_1 u_2''-\partial_2 u_1'') = -2\alpha\nabla_1^2\xi_3. \tag{2.21}$$

With the help of this relation we reduce the system of Eqs. (2.15) to the form

$$\begin{gathered}
\mu\nabla_1^2 u_1''+(\lambda+\mu)\partial_1 e''-2\alpha\partial_2\xi_3'' = 0,\\
\mu\nabla_1^2 u_2''+(\lambda+\mu)\partial_2 e''+2\alpha\partial_1\xi_3'' = 0,\\
(1-l^2\nabla_1^2)\zeta_3'' = 0, \qquad l^2 = \frac{(\gamma+\varepsilon)(\mu+\alpha)}{4\mu\alpha}.
\end{gathered} \tag{2.22}$$

or, in the operator form

$$\begin{gathered}
L_{11}u_1''+L_{12}u_2''+L_{13}\zeta_3'' = 0,\\
L_{21}u_1''+L_{22}u_2''+L_{23}\zeta_3'' = 0,\\
L_{31}u_1''+L_{32}u_2''+L_{33}\zeta_3'' = 0,
\end{gathered} \tag{2.23}$$

where

$$\begin{aligned}
&L_{11} = \mu\nabla_1^2+(\lambda+\mu)\partial_1^2, && L_{12} = (\lambda+\mu)\partial_1\partial_2 && L_{13} = -2\alpha\partial_2,\\
&L_{21} = (\lambda+\mu)\partial_2\partial_1, && L_{22} = \mu\nabla_1^2+(\lambda+\mu)\partial_2^2, && L_{23} = 2\alpha\partial_1,\\
&L_{31} = 0, && L_{32} = 0, && L_{33} = (l^2\nabla_1^2-1).
\end{aligned}$$

We now introduce the functions $\Omega_1, \Omega_2, \Omega_3$ connected with the displacements u_1'', u_2' and the rotation ζ_3'' by the relations

$$u_1'' = \begin{vmatrix} \Omega_1 & L_{12} & L_{13}\\ \Omega_2 & L_{22} & L_{23}\\ \Omega_3 & L_{32} & L_{33}\end{vmatrix}, \quad u_2'' = \begin{vmatrix} L_{11} & \Omega_1 & L_{13}\\ L_{21} & \Omega_2 & L_{23}\\ L_{31} & \Omega_3 & L_{33}\end{vmatrix}, \quad \zeta_3'' = \begin{vmatrix} L_{11} & L_{12} & \Omega_1\\ L_{21} & L_{22} & \Omega_2\\ L_{31} & L_{32} & \Omega_3\end{vmatrix}. \tag{2.24}$$

Performing the operations in (2.24) and introducing the new notation

$$\psi_1 = (l^2\nabla_1^2-1)\Omega_1, \quad \psi_2 = (l^2\nabla^2-1)\Omega_2, \quad \psi_3 = \nabla_1^2\Omega_3,$$

we arrive at the relations

$$
\begin{aligned}
u_1'' &= (\lambda+2\mu)\left[\nabla_1^2\psi_1 - \frac{\lambda+\mu}{\lambda+2\mu}\partial_1(\partial_1\psi_1+\partial_2\psi_2)+2\alpha\partial_2\psi_3\right],\\
u_2'' &= (\lambda+2\mu)\left[\nabla_1^2\psi_2 - \frac{\lambda+\mu}{\lambda+2\mu}\partial_2(\partial_1\psi_1+\partial_2\psi_2)-2\alpha\partial_1\psi_3\right],\\
\zeta_3'' &= (\lambda+2\mu)\mu\nabla_1^2\psi_3\,.
\end{aligned}
\tag{2.25}
$$

Introducing (2.25) into Eqs. (2.23), we obtain the following differential equations for the functions ψ_1, ψ_2, ψ_3:

$$
\nabla_1^2\nabla_1^2\psi_1 = 0, \quad \nabla_1^2\nabla_1^2\psi_2 = 0, \quad \nabla_1^2(l^2\nabla_1^2-1)\psi_3 = 0\,. \tag{2.26}
$$

The derived representation (2.25) for the displacements u_1'', u_2'' and the rotation ζ_3'' in terms of the stress functions ψ_1, ψ_2, ψ_3, can be regarded as a generalization of the Galerkin functions to the two-dimensional micropolar elasticity.

The procedure leading to the solution of the thermoelastic problem by means of the stress functions ψ_1, ψ_2, ψ_3 is the following. We assume the particular solution in the form (2.16), determine the thermoelastic displacement potential Φ and then the stresses σ_{ji}', μ_{ji}' from the formulae (2.18). In general this solution satisfies only a part of the boundary conditions. The additional solutions u_1'', u_2'', ζ_3'' are taken in the form (2.25). Then we solve the system of Eqs. (2.26) taking into account the boundary conditions. Finally the stresses are obtained by superposition

$$
\sigma_{ji} = \sigma_{ji}'+\sigma_{ji}'', \quad \mu_{ji} = \mu_{ji}'+\mu_{ji}''\,. \tag{2.27}
$$

3. The Stress Functions for the Thermoelastic Problem

Let us return to the formulae (2.2). It is readily observed that the quantities appearing in these formulae are connected by means of the relations

$$
\begin{gathered}
\partial_1\gamma_{21}-\partial_2\gamma_{11}-\varkappa_{13} = 0, \quad \partial_1\gamma_{22}-\partial_2\gamma_{12}-\varkappa_{23} = 0,\\
\partial_1\varkappa_{23}-\partial_2\varkappa_{13} = 0
\end{gathered}
\tag{3.1}
$$

which can also be written in the form

$$
\begin{aligned}
&\partial_1^2\gamma_{22}+\partial_2^2\gamma_{11} = \partial_1\partial_2(\gamma_{12}+\gamma_{21}),\\
&\partial_2^2\gamma_{12}-\partial_1^2\gamma_{21} = \partial_1\partial_2(\gamma_{22}-\gamma_{11})-(\partial_1\varkappa_{13}+\partial_2\varkappa_{23}),\\
&\partial_1\varkappa_{23}-\partial_2\varkappa_{13} = 0\,.
\end{aligned}
\tag{3.2}
$$

These are the compatibility equations for the two-dimensional problem of micropolar medium.

Solving Eqs. (2.3) for the strains γ_{ji} and $\varkappa_{j3}$ $(i, j = 1, 2, 3)$, we have

$$
\gamma_{11} = \frac{1}{2\mu}\left[\sigma_{11} - \frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22})\right] + \frac{\nu\theta}{2(\lambda+\mu)}
$$

$$\gamma_{22}=\frac{1}{2\mu}\left[\sigma_{22}-\frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22})\right]+\frac{\nu\theta}{2(\lambda+\mu)},$$

$$\gamma_{12}=\frac{1}{4\mu}(\sigma_{12}+\sigma_{21})+\frac{1}{4\alpha}(\sigma_{12}-\sigma_{21}),$$

$$\gamma_{21}=\frac{1}{4\mu}(\sigma_{21}+\sigma_{12})+\frac{1}{4\alpha}(\sigma_{21}-\sigma_{12}).$$

Introducing the above relations into the compatibility equations we arrive at the following three equations in stresses:

$$\partial_2^2\sigma_{11}+\partial_1^2\sigma_{22}-\frac{\lambda}{2(\lambda+\mu)}\nabla_1^2(\sigma_{11}+\sigma_{22})+\frac{\mu\nu}{\lambda+\mu}\nabla_1^2\theta=\partial_1\partial_2(\sigma_{12}+\sigma_{21}),$$

$$(3.3)\quad (\partial_2^2-\partial_1^2)(\sigma_{12}+\sigma_{21})+\frac{\mu}{\alpha}(\nabla_1^2(\sigma_{12}-\sigma_{21})=2\partial_1\partial_2(\sigma_{22}-\sigma_{11})-\frac{4\mu}{\gamma+\varepsilon}(\partial_1\mu_{13}+\partial_2\mu_{23}),$$

$$\partial_1\mu_{23}-\partial_2\mu_{13}=0.$$

In the case of the isothermal problem ($\theta=0$) the above equations are reduced to those deduced by H. SCHAEFER [6].

We now introduce the stress functions F and Ψ and connect them with the stresses by the relations [7]

$$(3.4)\quad \begin{aligned} &\sigma_{11}=\partial_2^2F-\partial_1\partial_2\Psi, && \sigma_{22}=\partial_1^2F+\partial_1\partial_2\Psi,\\ &\sigma_{12}=-\partial_1\partial_2F-\partial_2^2\Psi, && \sigma_{21}=-\partial_1\partial_2F+\partial_1^2\Psi,\\ &\mu_{13}=\partial_1\Psi, && \mu_{23}=\partial_2\Psi. \end{aligned}$$

Substituting relations (3.4) into Eqs. (2.5) we find that they are identically satisfied. Substituting, in turn, (3.4) into the compatibility equations $(3.3)_{1,2}$ we obtain

$$(3.5)\quad \begin{aligned} &\nabla_1^2\nabla_1^2F+2\mu m\nabla_1^2\theta=0,\\ &\nabla_1^2(1-l^2\nabla_1^2)\Psi=0, \end{aligned}$$

where

$$l^2=\frac{(\gamma+\varepsilon)(\mu+\alpha)}{4\mu\alpha},\qquad m=\frac{(3\lambda+2\mu)\alpha_t}{\lambda+2\mu}.$$

The functions F and Ψ are not independent. They are connected by the relations $(3.2)_{1,2}$. Consequently, we obtain

$$(3.6)\quad \begin{aligned} &-\partial_1(1-l^2\nabla_1^2)\Psi=A\partial_2\nabla_1^2F+B\partial_2\theta,\\ &\partial_2(1-l^2\nabla_1^2)\Psi=A\partial_1\nabla_1^2F+B\partial_1\theta,\\ &A=\frac{(\lambda+2\mu)(\gamma+\varepsilon)}{4\mu(\lambda+\mu)},\qquad B=\frac{\nu(\gamma+\varepsilon)}{2(\lambda+\mu)}. \end{aligned}$$

We have still to give the boundary conditions for Eqs. (3.5). We assume the boundary s to be free of loading. This condition in expressed by the equations

$$(3.7)\quad \sigma_{ji}n_j=0,\qquad \mu_{j3}n_j=0,\qquad j=1,2,\qquad \mathbf{x}\in s,$$

which, if expressed in Ψ and F, lead to the following ones:

$$\frac{d}{ds}(\partial_2 F-\partial_1\Psi)=0,\quad \frac{d}{ds}(\partial_1 F+\partial_2\Psi)=0,\quad \frac{\partial\Psi}{\partial n}=0. \tag{3.8'}$$

The quantities d/ds and d/dn are the derivatives along the boundary s and along the normal to this boundary.

The boundary equations may be written in the form

$$\frac{\partial F}{\partial s}=0,\quad \frac{\partial F}{\partial n}+\frac{\partial\Psi}{\partial s}=0,\quad \frac{\partial\Psi}{\partial n}=0. \tag{3.8''}$$

We take the solutions of Eqs. (3.5) in the form

$$F=F'+F'',\ \Psi=\Psi'+\Psi''. \tag{3.9}$$

Here F' is the particular solution of the equation

$$\nabla_1^2\nabla_1^2 F'+2\mu m\nabla_1^2\theta=0,\quad \Psi'=0. \tag{3.10}$$

It may easily be verified that the functions F' and Ψ' lead to the symmetric stress tensor

$$\sigma'_{ji}=\sigma'_{ij}=-\partial_i\partial_j F'+\delta_{ji}\nabla_1^2 F',\quad \mu'_{ji}=0. \tag{3.11}$$

The stress function F' is the particular solution for Hooke's medium and, at the same time, the particular solution of the equations of classical thermoelasticity.

The particular integral F' can also be assumed in such a way that the Poisson equation is satisfied, with an arbitrary boundary condition, e.g. $F'=0$.

$$\nabla_1^2 F'+2\mu m\theta=0 \tag{3.10'}$$

Assuming $\Psi'=0$ we satisfy relations (3.6), for

$$\partial_2(A\nabla_1^2 F'+B\theta)=0,$$
$$\partial_1(A\nabla_1^2 F'+B\theta)=0.$$

It is further readily observed that there exists a relation between the function F' and the thermoelastic displacement potential Φ, namely $F'=-2\mu\Phi$.

The functions F'', Ψ'' should satisfy the equations

$$\nabla_1^2\nabla_1^2 F''=0, \tag{3.12}$$

$$\nabla_1^2(1-l^2\nabla_1^2)\Psi''=0 \tag{3.13}$$

with the boundary conditions

$$(\sigma'_{ji}+\sigma''_{ji})n_j=0,\quad \mu''_{j3}n_j=0,\quad i,j=1,2. \tag{3.14}$$

Furthermore, the following relations should hold:

$$\begin{aligned}-\partial_1(1-l^2\nabla_1^2)\Psi''&=A\partial_2\nabla_1^2 F'',\\ \partial_2(1-l^2\nabla_1^2)\Psi''&=A\partial_1\nabla_1^2 F''.\end{aligned} \tag{3.15}$$

Thus we succeeded in proving, in general, that for the plane state of strain the solution of the thermoelastic problem consists of two parts

a) the particular solution of non-homogeneous equations of classical thermoelasticity and

b) the general solution of homogeneous differential equations of micropolar elasticity (with $\theta = 0$)

In the particular case of one-dimensional problem ($u_1 \neq 0$, $u_2 = 0$) or the two-dimensional axisymmetric problem ($u_r \neq 0$, $u_\vartheta = 0$) it is sufficient to have only the solution a).

Let us return once more to the system (3.5) with the homogeneous boundary conditions (3.8″). Consider the state of stress in an infinitely long cylinder heated on the side surface. Assuming the absence of sources, the temperature satisfies the Laplace equation. In this case Eqs. (3.5) and the boundary conditions are homogeneous. The solution of the system is trivial, i.e.

$$F \equiv 0, \qquad \Psi \equiv 0$$

only when the temperature θ is constant. Only under this assumption the additional relations (3.6) are satisfied.

Thus, we find that for $\theta = \text{const}$ the stress functions F and Ψ in a simply-connected cylinder vanish, which leads to the zero values of $\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}, \mu_{13}, \mu_{31}, \mu_{23}, \mu_{32}$. The only non-vanishing stress is σ_{33}, given by the formula

$$\sigma_{33} = \lambda\gamma_{kk} - \nu\theta = -\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\alpha_t\theta. \tag{3.16}$$

All other temperature fields satisfying the Laplace equation lead to non-vanishing stresses.

4. Elastic Semi-space under the Action of Temperature Distributed over the Boundary

Consider the elastic semi-space $x_1 \geqslant 0$ heated in the plane $x_1 = 0$ to the temperature $f(x_2)$. We assume that this plane is free of stress and hence, for $x_1 = 0$ we have

$$\sigma_{11} = 0, \quad \sigma_{12} = 0, \quad \mu_{13} = 0. \tag{4.1}$$

To determine the stresses we apply the exponential Fourier transform

$$\begin{aligned} \tilde{g}(x_1, \zeta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x_1, x_2) e^{i\zeta x_2} dx_2, \\ g(x_1, x_2) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(x_1, \zeta) e^{-i\zeta x_2} d\zeta. \end{aligned} \tag{4.2}$$

First we determine the temperature $\theta(x_1, x_2)$ solving the Laplace equation

$$\nabla_1^2 \theta = 0 \tag{4.3}$$

with the boundary condition

$$\theta(0, x_2) = f(x_2) \tag{4.4}$$

and the regularity condition at infinity. Making use of the Fourier transform we express the temperature in the form of the integral

$$\theta(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\zeta) e^{-\zeta x_1 - i x_2 \zeta} d\zeta, \tag{4.5}$$

where

$$\tilde{f}(\zeta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x_2)\,e^{i\zeta x_2}dx_2\,.$$

To determine the state of stress we introduce the functions F, Ψ considered in the preceding section. Set

$$F=F'+F'',\qquad \Psi=\Psi'+\Psi'' \tag{4.6}$$

where F', Ψ' are particular solutions of the system of Eqs. (3.5).

Let us assume that $\Psi'=0$ and that the function F' satisfies the differential equation

$$\nabla_1^2F'+2\mu m\theta=0 \tag{4.7}$$

with the boundary condition

$$F'=0\qquad \text{for}\qquad x_1=0\,, \tag{4.8}$$

and the regularity condition for $|x_1^2+x_2|\to\infty$.

Applying the exponential Fourier transform we have

$$F'(x_1,x_2)=\frac{\mu m x_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{f}(\zeta)}{\zeta}\,e^{-\zeta x_1-i\zeta x_2}d\zeta\,. \tag{4.9}$$

From the formulae (3.11) we determine the stresses connected with the function F':

$$\begin{aligned}
\sigma'_{11}&=-\frac{\mu m x_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\zeta\tilde{f}(\zeta)\,e^{-\zeta x_1-i\zeta x_2}d\zeta\,,\\
\sigma'_{22}&=-\frac{\mu m}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\zeta)\,(2-x_1\zeta)\,e^{-\zeta x_1-i\zeta x_2}d\zeta\,,\\
\sigma'_{12}=\sigma'_{21}&=\frac{i\mu m}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\zeta)\,(1-x_1\zeta)\,e^{-\zeta x_1-i\zeta x_2}d\zeta\,,\\
\mu'_{13}=\mu'_{23}&=0\,.
\end{aligned} \tag{4.10}$$

The functions F'', Ψ'' should satisfy the equations

$$\nabla_1^2\nabla_1^2F''=0,\qquad \nabla_1^2(1-l^2\nabla_1^2)\,\Psi''=0\,, \tag{4.11}$$

with the boundary conditions

$$\sigma'_{11}+\sigma''_{11}=0,\qquad \sigma'_{12}+\sigma''_{12}=0,\qquad \mu''_{13}=0\qquad \text{for}\qquad x_1=0\,. \tag{4.12}$$

Applying the exponential Fourier transform to Eqs. (4.11) and taking into account the regularity conditions at infinity we arrive at the integrals

$$F''(x_1,x_2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(M+N\zeta x_1)\,e^{-\zeta x_1-i\zeta x_2}d\zeta\,, \tag{4.12'}$$

$$\Psi''(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{-\zeta x_1} + De^{-\varrho x_1}) e^{-i\zeta x_2} d\zeta, \tag{4.13}$$

$$\varrho = \left(\zeta^2 + \frac{1}{l^2}\right)^{1/2},$$

The quantities M, N, C and D are to be determined by means of the boundary conditions (4.12) and the relations (3.6). The boundary conditions (4.12) in terms of the functions F'', Ψ'' take the form

$$\begin{gathered} |\sigma'_{11} + \partial_2^2 F'' - \partial_1 \partial_2 \Psi''|_{x_1=0} = 0, \qquad |\sigma'_{12} - \partial_1 \partial_2 F'' - \partial_2^2 \Psi''|_{x_1=0} = 0, \\ |\partial_1 \Psi''|_{x_1=0} = 0. \end{gathered} \tag{4.14}$$

The first boundary condition leads to the result $M = 0$. The last yields the relation

$$\zeta C + \varrho D = 0.$$

Finally the condition (4.11) for the Fourier transform has the form

$$|\tilde{\sigma}'_{12} + \tilde{\sigma}''_{12}|_{x_1=0} = |i\mu m \tilde{f}(1-\zeta x_1) e^{-\zeta x_1} + i\zeta \tilde{F}'' + \zeta^2 \tilde{\Psi}''|_{x_1=0} = 0,$$

and leads to the relation

$$\mu i m \tilde{f} + iN\zeta^2 + C\zeta^2 \left(1 - \frac{\zeta}{\varrho}\right) = 0. \tag{4.15}$$

We still have to satisfy Eqs. (3.15). In the transformed form they are

$$\begin{aligned} l^2 \partial_1 (\partial_1^2 - \varrho^2) \tilde{\Psi}'' &= -i\zeta A (\partial_1^2 - \zeta^2) \tilde{F}'', \\ -i\zeta l^2 (\partial_1^2 - \varrho^2) \tilde{\Psi}'' &= A \partial_1 (\partial_1^2 - \zeta^2) \tilde{F}''. \end{aligned} \tag{4.16}$$

Introducing $\tilde{\Psi}''$, $\tilde{F}''$ into these equations we obtain the relation

$$C = 2iN\zeta^2 A.$$

Thus, we determine the quantities C, D, M and N:

$$N = -\frac{\mu m \tilde{f}}{\zeta^2 \Delta_0}, \quad C = -\frac{2Ai\mu m \tilde{f}}{\Delta_0}, \quad D = -\frac{\zeta C}{\varrho}, \quad M = 0.$$

Here

$$\Delta_0 = 1 + 2A\zeta^2 \left(1 - \frac{\zeta}{\varrho}\right).$$

Hence, we obtain the stress functions F'', Ψ'' in terms of the Fourier integrals

$$F'' = -\frac{\mu m x_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{f}(\zeta)}{\zeta \Delta_0} e^{-\zeta x_1 - i x_2 \zeta} d\zeta, \tag{4.17}$$

$$\Psi'' = -\frac{2i\mu m A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{f}(\zeta)}{\Delta_0} \left(e^{-\zeta x_1} - \frac{\zeta}{\varrho} e^{-\varrho x_1}\right) e^{-i x_2 \zeta} d\zeta. \tag{4.18}$$

We still have to find the total stresses by means of the formulae (3.4). We have here

$$\sigma_{11} = -\frac{\mu m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\zeta)\left[x_1\zeta\left(1-\frac{1}{\Delta_0}\right)e^{-\zeta x_1}+\frac{2A\zeta^2}{\Delta_0}(e^{-\zeta x_1}-e^{-\varrho x_1})\right]e^{-i\zeta x_2}d\zeta,$$

$$\sigma_{22} = -\frac{\mu m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\zeta)\left[(2-x_1\zeta)\left(1-\frac{1}{\Delta_0}\right)e^{-\zeta x_1}-\frac{2A\zeta^2}{\Delta_0}(e^{-\zeta x_1}-e^{-\varrho x_1})\right]e^{-i\zeta x_2}d\zeta,$$

$$\sigma_{12} = \frac{i\mu m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\zeta)\left[(1-\zeta x_1)\left(1-\frac{1}{\Delta_0}\right)e^{-\zeta x_1}-\right.$$

$$(4.19) \qquad \left.-\frac{2A\zeta^2}{\Delta_0}\left(e^{-\zeta x_1}-\frac{\zeta}{\varrho}e^{-\varrho x_1}\right)\right]e^{-i\zeta x_2}d\zeta,$$

$$\sigma_{21} = \frac{i\mu m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\zeta)\left[(1-\zeta x_1)\left(1-\frac{1}{\Delta_0}\right)e^{-\zeta x_1}+\right.$$

$$\left.+\frac{2A\zeta^2}{\Delta_0}(\zeta e^{-\zeta x_1}-\varrho e^{-\varrho x_1})\right]e^{-i\zeta x_2}d\zeta,$$

$$\mu_{13} = \frac{2iA\mu m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{f}(\zeta)}{\Delta_0}\zeta(e^{-\zeta x_1}-e^{-\varrho x_1})e^{-i\zeta x_2}d\zeta,$$

$$\mu_{23} = -\frac{2\mu m A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{f}(\zeta)}{\Delta_0}\left(e^{-\zeta x_1}-\frac{\zeta}{\varrho}e^{-\varrho x_1}\right)e^{-i\zeta x_2}d\zeta.$$

Consider now the normal stresses

$$\sigma_{33} = \gamma_{kk}\lambda-\nu\theta = \frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22})-\frac{\mu\nu\theta}{\lambda+\mu},$$

We have

$$(4.20) \qquad \sigma_{33} = -\frac{\mu m\lambda}{(\lambda+\mu)\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\zeta)\left(1-\frac{1}{\Delta_0}\right)e^{-\zeta x_1-i\zeta x_2}d\zeta-\frac{\nu\mu\theta}{\lambda+\mu}.$$

In the particular case of the Hookean body we set in the stresses $\alpha = 0$ (whence $l^2 \to 0$, $\varrho \to \zeta$, $\Delta_0 \to 1$). It is readily observed that in this case the stresses $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \mu_{13}, \mu_{23}, \mu_{31}, \mu_{32}$ vanish. Only the stress σ_{33} remains different from zero. These results agree with the well known. I. N. Muskhelishvili theorem [8].

For the stress σ_{33} we have

$$(4.21) \qquad \sigma_{33} = -\frac{\nu\mu}{\lambda+\mu}\theta(x_1, x_2).$$

In the case of the micropolar body the stresses are in principle different from zero, except for the case of constant temperature.

Let us now calculate the stress σ_{22} on the boundary of the semi-space. We have

$$\sigma_{22}(0, x_2) = -\frac{2\mu m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\zeta)\left(1-\frac{1}{\Delta_0}\right) e^{-i\zeta x_2} d\zeta . \tag{4.22}$$

Assuming that the quantity $1/l^2$ is very small as compared with unity we expand ϱ into an infinite series. Taking only two terms of this expansion we obtain

$$\varrho = \zeta\left(1+\frac{1}{\zeta^2 l^2}\right)^{1/2} \approx \zeta\left(1+\frac{1}{2l^2\zeta^2}\right).$$

Thus, we have

$$1-\frac{1}{\Delta_0} = \sigma \frac{\zeta^2}{\zeta^2+k^2},$$

where

$$\sigma = \frac{(\lambda+2\mu)\alpha}{\alpha(2\lambda+3\mu)+\mu(\lambda+\mu)}, \quad k^2 = \frac{2\mu\alpha(\lambda+\mu)}{(\gamma+\varepsilon)\left[\alpha(2\lambda+3\mu)+\mu(\lambda+\mu)\right]}.$$

Assume that the distribution of temperature on the boundary $x_1 = 0$ is given by the expression

$$\theta(0, x_2) = \theta_0[H(x_2-c)-H(x_2+c)], \tag{4.23}$$

where $H(z)$ is the Heaviside function. Over the infinite strip $|x_2| \leqslant c$, $-\infty < x_3 < \infty$ there acts the temperature θ_0, while for $|x_2| > c$, $-\infty < x_3 < \infty$ we have $\theta = 0$. The formula (4.22) takes the form

$$\sigma_{22}(0, x_2) = -\frac{4\mu m\sigma\theta_0}{\pi} \int_0^{\infty} \frac{\zeta \sin\zeta c}{\zeta^2+k^2} \cos\zeta x_2 d\zeta .$$

This integral can be solved, namely we obtain

$$\sigma_{22}(0, x_2) = 2\mu m\sigma\theta_0 \begin{cases} -e^{-kc}\operatorname{ch}(kx_2), & x_2 < c, \\ e^{-k|x_2|}\operatorname{sh}(kc), & x_2 > c, \end{cases} \tag{4.24}$$

5. Action of Heat Sources in the Elastic Semi-space

Suppose that at the point $(x_1', 0)$ of the elastic semi-space $x_1 \geqslant 0$ there acts a stationary heat source with intensity $W(x_1, x_2) = W_0\delta(x_1-x_1')\delta(x_2)$. We assume that on the plane $x_1 = 0$ bounding the semispace, the temperature is zero and the stresses vanish. The boundary conditions of our problem are the following:

$$\theta = 0, \quad \sigma_{11} = 0, \quad \sigma_{12} = 0, \quad \mu_{13} = 0. \tag{5.1}$$

The solution of the heat-conduction equation

$$\nabla_1^2\theta = -\frac{W}{\lambda_\theta} \tag{5.2}$$

with the boundary condition (5.1) and the regularity condition at $|x_1^2+x_2^2|\to\infty$, yields the expression

$$\theta=\frac{W_0}{2\pi\lambda}\ln\frac{r_2}{r_1}, \tag{5.3}$$

where

$$r_{1,2}=[(x_1\mp x_1')^2+x_2^2]^{1/2}.$$

The particular solution F' is taken to be the solution of the equation

$$\nabla_1^2\nabla_1^2F'+2\mu m\nabla_1^2\theta=0 \tag{5.4}$$

with the boundary conditions

$$F'=0,\qquad \nabla_1^2F'=0$$

and the regularity condition for $|x_1^2+x_2^2|\to\infty$. It can be represented in the form of a double Fourier integral [5]

$$F'=\frac{4K}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\eta x_1'}{(\zeta^2+\eta^2)^2}\sin\eta x_1\cos\zeta x_2 d\zeta d\eta \tag{5.5}$$

where

$$K=\frac{\mu m W_0}{\lambda_0}.$$

Knowing the function F' we determine the stresses from the formula (3.11). Thus we obtain

$$\begin{aligned}\sigma_{11}'=\partial_2^2F'&=-\frac{4K}{\pi^2}\int_0^\infty\int_0^\infty\frac{\zeta^2\sin\eta x_1'}{(\zeta^2+\eta^2)^2}\sin\eta x_1\cos\zeta x_2 d\eta d\zeta=\\&=-\frac{K}{2\pi}\left[\ln\frac{r_2}{r_1}-x_2^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\right],\\ \sigma_{22}'=\partial_1^2F'&=-\frac{K}{2\pi}\left[\ln\frac{r_2}{r_1}+x_2^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\right],\\ \sigma_{12}'=\sigma_{21}'=-\partial_1\partial_2F'&=-\frac{K}{2\pi}x_2\left[\frac{(x_1-x_1')^2}{r_1^2}-\frac{(x_1+x_1')^2}{r_2^2}\right].\end{aligned} \tag{5.6}$$

On the boundary $x_1=0$ we have $\sigma_{11}'=\sigma_{22}'=0$ but $\sigma_{12}'\neq0$. The tangential stress σ_{12}' on the boundary $x_1=0$ can be represented in the form of the integral

$$\sigma_{12}'(0,x_2)=\frac{Kx_1'}{\pi}\int_0^\infty e^{-\zeta x_1'}\sin\zeta x_2 d\zeta. \tag{5.7}$$

To remove the stresses on the plane $x_1=0$ we have to add to the state σ_{ji}', μ_{ji}' the state of stress σ_{ji}'', μ_{ji}'' determined by the functions F'', Ψ''. Thus we have to solve the system of homogeneous equations

$$\nabla_1^2\nabla_1^2F''=0,\qquad \nabla_1^2(1-l^2\nabla_1^2)\Psi''=0 \tag{5.8}$$

with the boundary conditions

$$(5.9) \qquad \sigma'_{11}+\sigma''_{11}=0, \quad \sigma'_{12}+\sigma''_{12}=0, \quad \mu''_{13}=0 \quad \text{for} \quad x_1=0,$$

and the regularity conditions at infinity.

We take the solution of the differential Eqs. (5.8) in the form of Fourier integrals

$$(5.10) \qquad F''=\int_0^\infty (M+N\zeta x_1)e^{-\zeta x_1}\cos\zeta x_2 d\zeta,$$

$$(5.11) \qquad \Psi''=\int_0^\infty (Ce^{-\zeta x_1}+De^{-\varrho x_1})\sin\zeta x_2 d\zeta,$$

where

$$\varrho=\left(\zeta^2+\frac{1}{l^2}\right)^{1/2}.$$

It is evident that the boundary condition $\mu''_{13}=0$ for $x_1=0$ leads to the relation

$$(5.12) \qquad \zeta C+\varrho D=0.$$

The condition $(5.9)_1$ is satisfied when $M=0$. The second boundary condition in terms of the functions F'', Ψ'',

$$|\sigma'_{12}-\partial_1\partial_2 F''-\partial_2^2\Psi''|_{x_1=0}=0,$$

yields the relation

$$(5.13) \qquad \zeta^2 N+\zeta^2(C+D)=-\frac{Kx'_1 e^{-x'_1\zeta}}{\pi}.$$

We now make use of the conditions (3.15). Substituting into them the functions F'', Ψ'' we arrive at only one relation, namely

$$(5.14) \qquad C=2\zeta^2 AN.$$

Thus, we obtain

$$(5.15) \qquad C=-\frac{2AK}{\pi\Delta_0}x'_1 e^{-x'_1\zeta}, \quad N=-\frac{Kx'_1}{\pi\zeta^2\Delta_0}e^{-x'_1\zeta}, \quad D=-\frac{\zeta}{\varrho}C,$$

where

$$\Delta_0=1+2A\zeta^2\left(1-\frac{\zeta}{\varrho}\right).$$

We have therefore determined all quantities required for the calculation of the stresses σ''_{ji}, μ''_{ji}. From the formulae (3.4) we find

$$\sigma''_{11}=\frac{Kx'_1}{\pi}\int_0^\infty \frac{e^{-x'_1\zeta}}{\Delta_0}[x_1\zeta e^{-x_1\zeta}-2A\zeta^2(e^{-\zeta x_1}-e^{-\varrho x_1})]\cos\zeta x_2 d\zeta,$$

$$\sigma''_{22}=\frac{Kx'_1}{\pi}\int_0^\infty \frac{e^{-x'_1\zeta}}{\Delta_0}[(2-x_1\zeta)e^{-x_1\zeta}+2A\zeta^2(e^{-\zeta x_1}-e^{-\varrho x_1})]\cos\zeta x_2 d\zeta,$$

$$\sigma''_{12} = -\frac{Kx'_1}{\pi}\int_0^\infty \frac{e^{-x'_1\zeta}}{\Delta_0}\left[(1-\zeta x_1)e^{-\zeta x_1}+2A\zeta^2\left(e^{-\zeta x_1}-\frac{\zeta}{\varrho}e^{-\varrho x_1}\right)\right]\sin\zeta x_2 d\zeta,$$

(5.16)

$$\sigma''_{21} = -\frac{Kx'_1}{\pi}\int_0^\infty \frac{e^{-x'_1\zeta}}{\Delta_0}[(1-\zeta x_1)e^{-\zeta x_1}+2A\zeta(\zeta e^{-\zeta x_1}-\varrho e^{-\varrho x_1})]\sin\zeta x_2 d\zeta,$$

$$\mu''_{13} = \frac{2AKx'_1}{\pi}\int_0^\infty \frac{e^{-x'_1\zeta}\zeta}{\Delta_0}(e^{-\zeta x_1}-e^{-\varrho x_1})\sin\zeta x_2 d\zeta,$$

$$\mu''_{23} = -\frac{2AKx'_1}{\pi}\int_0^\infty \frac{e^{-x'_1\zeta}}{\Delta_0}\zeta\left(e^{-\zeta x_1}-\frac{\zeta}{\varrho}e^{-\varrho x_1}\right)\cos\zeta x_2 d\zeta.$$

Observe that the singularities appear only in the stresses σ'_{ij}. The stresses σ''_{ij}, μ''_{ji} are regular functions in the considered semi-space. The singularities of the stresses σ'_{ji} are identical with those in the Hookean body.

We obtain the latter by setting $\alpha = 0$ and $\varrho = \zeta$, $\Delta_0 = 1$. Introducing these values into the integrals (5.16) we can represent the stresses σ''_{ji} for the Hookean body in the closed form

$$\sigma''_{11} = \frac{Kx_1x'_1}{\pi r_2^4}[(x_1+x'_1)^2-x_2^2],$$

$$\sigma''_{22} = \frac{K}{2\pi r_2^4}[(x_1+2x'_1)r_2^2+2x_1x_2^2],$$

(5.17)

$$\sigma''_{12} = \sigma''_{21} = -\frac{Kx'_1x_2}{\pi r_2^4}(x_2^2+x_1'^2-x_1^2),$$

$$\mu''_{13} = \mu''_{23} = 0.$$

For the micropolar body the asymmetry of the stress tensor appears in the additional solution needed to remove the stresses in the plane $x_1 = 0$. Observe finally that the stresses $\mu_{13}, \mu_{23}, \mu_{31}, \mu_{32}$ are regular functions.

6. Plane State of Stress

Consider a cylinder with generators parallel to the x_3-axis and the bases in the planes $x_3 = \pm h$. This cylinder will be called plate if its height $2h$ is small as compared with the linear dimensions of the cross-section. Assume that the side surface of the cylinder is loaded by the forces $\mathbf{p}$ and moments $\mathbf{m}$, where

$$\mathbf{p} \equiv (p_1, p_2, 0), \qquad \mathbf{m} \equiv (0, 0, m_3). \tag{6.1}$$

We further asssume that the loadings p_1, p_2 and the moment m_3 are distributed symmetrically with respect to the middle plane $x_3 = 0$.

Suppose that the plate is also acted on by the body forces $\mathbf{X}$ and the moments body $\mathbf{Y}$, i.e.

$$\mathbf{X} \equiv (X_1, X_2\, 0), \qquad \mathbf{Y} \equiv (0, 0, Y_3), \tag{6.2}$$

also symmetric with respect to the middle plane. Furthermore, we assume that there acts in the plate the temperature $\theta(x_1, x_2, x_3)$ distributed symmetrically with respect to the plane $x_3 = 0$.

Under the action of these loadings there exists in the plate a state of stress, in general spatial. There occur all components of the state of stress σ_{ji}, μ_{ji} as functions of x_1, x_2, x_3. We assume that the planes $x_3 = \pm h$ are free of stress, i.e.

$$\sigma_{33} = \sigma_{31} = \sigma_{32} = 0, \quad \mu_{33} = \mu_{31} = \mu_{32} = 0 \quad \text{for} \quad x_3 = \pm h. \tag{6.3}$$

Consider the first three equations of equilibrium

$$\begin{aligned} &\partial_1\sigma_{11}+\partial_2\sigma_{21}+\partial_3\sigma_{31}+X_1 = 0, \\ &\partial_1\sigma_{12}+\partial_2\sigma_{22}+\partial_3\sigma_{32}+X_2 = 0, \\ &\partial_1\sigma_{13}+\partial_2\sigma_{23}+\partial_3\sigma_{33} \qquad = 0. \end{aligned} \tag{6.4}$$

In view of the symmetric distribution of the body forces X_1, X_2 and the forces p_1, p_2 with respect to the middle plane, the stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$ and σ_{21} are symmetric and σ_{31}, σ_{32} antisymmetric with respect to this plane. The stress σ_{33} is a symmetric function of x_3 and therefore σ_{13} and σ_{23} are antisymmetric with respect to the variable x_3. On the basis of the remaining equilibrium equations

$$\begin{aligned} \sigma_{23}-\sigma_{32}+\partial_1\mu_{11}+\partial_2\mu_{21}+\partial_3\mu_{31} &= 0, \\ \sigma_{31}-\sigma_{13}+\partial_1\mu_{12}+\partial_2\mu_{22}+\partial_3\mu_{32} &= 0, \\ \sigma_{12}-\sigma_{21}+\partial_1\mu_{13}+\partial_2\mu_{23}+\partial_3\mu_{33}+Y_3 &= 0 \end{aligned} \tag{6.5}$$

we find that in view of the antisymmetry of the stresses σ_{23}, σ_{32}, σ_{31}, σ_{13} with respect to the plane $x_3 = 0$ the stresses μ_{11}, μ_{21}, μ_{12} and μ_{22} are antisymmetric and the stresses μ_{31}, μ_{32} symmetric with respect to this plane. In view of the symmetry of the stresses σ_{12}, σ_{21} and the symmetry of the body moment Y_3 with respect to the middle plane $x_3 = 0$, it follows from the last equation (6.5) that μ_{13}, μ_{23} are symmetric and the stress μ_{33} antisymmetric with respect to the plane $x_3 = 0$.

Let us integrate over the thickness of the plate the first two equations of the group (6.4) and the last Eq. (6.5)

$$\begin{aligned} &\int_{-h}^{h} \partial_1\sigma_{11}+\partial_2\sigma_{21}+\partial_3\sigma_{31}+X_1)\,dx_3 = 0, \\ &\int_{-h}^{h} (\partial_1\sigma_{12}+\partial_2\sigma_{22}+\partial_3\sigma_{23}+X_2)\,dx_3 = 0. \\ &\int_{-h}^{h} (\sigma_{12}-\sigma_{21}+\partial_1\mu_{13}+\partial_2\mu_{23}+\partial_3\mu_{33}+Y_3)\,dx_3 = 0. \end{aligned} \tag{6.6}$$

Observe that

$$\int_{-h}^{h} \partial_3 \sigma_{3\nu} dx_3 = \sigma_{3\nu}(x_1, x_2, \pm h) = 0, \qquad \nu = 1, 2,$$

$$\int_{-h}^{h} \partial_3 \mu_{33} dx_3 = \mu_{33}(x_1, x_2, \pm h) = 0.$$

The above integrals vanish in view of the boundary conditions (6.3). Equation (6.6) can be represented in the form

$$\begin{gathered} \partial_1 \sigma_{11}^* + \partial_2 \sigma_{21}^* + X_1^* = 0, \\ \partial_1 \sigma_{12}^* + \partial_2 \sigma_{22}^* + X_2^* = 0, \\ \sigma_{12}^* - \sigma_{21}^* + \partial_1 \mu_{13}^* + \partial_2 \mu_{23}^* + Y_3^* = 0. \end{gathered} \tag{6.7}$$

Here, the quantities

$$\sigma_{\nu\mu}^*(x_1, x_2) = \frac{1}{2h} \int_{-h}^{h} \sigma_{\nu\mu}^*(x_1, x_2, x_3) dx_3, \qquad X_\nu^*(x_1, x_2) = \frac{1}{2h} \int_{-h}^{h} X_\nu(x_1, x_2, x_3) dx_3,$$

$$\mu_{\nu 3}^* = \frac{1}{2h} \int_{-h}^{h} \mu_{\nu 3} dx_3, \qquad Y_3^* = \frac{1}{2h} \int_{-h}^{h} Y_3 dx_3, \qquad \nu, \mu = 1, 2$$

are mean values of the stresses $\sigma_{\mu\nu}$, μ_{ν_3}, ν, $\mu = 1, 2$, the body forces X_ν and the body moments Y_3 along the thickness of the plate. We arrived at a system of three equilibrium equations, in which the mean values of the stresses depend on the variables x_1 and x_2 only.

Let us integrate the third Eq. (6.4) and the first Eqs. (6.5) along the thickness of the plate. We obtain

$$\begin{gathered} \partial_1 \sigma_{13}^* + \partial_2 \sigma_{23}^* = 0, \\ \sigma_{23}^* - \sigma_{32}^* + \partial_1 \mu_{11}^* + \partial_2 \mu_{21}^* = 0, \\ \sigma_{31}^* - \sigma_{32}^* + \partial_1 \mu_{12}^* + \partial_2 \mu_{22}^* = 0. \end{gathered} \tag{6.8}$$

We have used here the boundary conditions (6.3), for

$$\int_{-h}^{h} \partial_3 \mu_{3\nu}(x_1, x_2, x_3) dx_3 = \mu_{3\nu}(x_1 \cdot x_2, \pm h) = 0,$$

$$\int_{-h}^{h} \partial_3 \sigma_{33} dx_3 = \sigma_{33}(x_1, x_2, \pm h) = 0.$$

Equations (6.8) are identically satisfied in view of the antisymmetry of the functions $\sigma_{13}, \sigma_{31}, \sigma_{23}, \sigma_{32}, \mu_{11}, \mu_{21}, \mu_{22}, \mu_{12}$. Thus

$$\sigma_{13}^* = \sigma_{23}^* = \sigma_{32}^* = \sigma_{31}^* = \mu_{11}^* = \mu_{21}^* = \mu_{12}^* = \mu_{22}^* = 0.$$

We do not make an appreciable error by assuming that the stresses σ_{13}, σ_{31}, σ_{32}, σ_{23}, σ_{33} and μ_{11}, μ_{22}, μ_{12}, μ_{21} are very small as compared with σ_{11}, σ_{22}, σ_{12}, σ_{21}, μ_{13}, μ_{32}, μ_{31}, μ_{23}.

This assumption is the better, the smaller the height of the plate as compared with the other linear dimensions of the plate.

Thus, in the thin plate the state of stress is approximately described by the tensors

$$\boldsymbol{\sigma}^* = \begin{vmatrix} \sigma_{11}^* & \sigma_{12}^* & 0 \\ \sigma_{21}^* & \sigma_{22}^* & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad \boldsymbol{\mu}^* = \begin{vmatrix} 0 & 0 & \mu_{13}^* \\ 0 & 0 & \mu_{23}^* \\ \mu_{31}^* & \mu_{32}^* & 0 \end{vmatrix}. \tag{6.9}$$

This state of stress will be called the generalized plane state of stress of the Cosserat medium. The state of displacements and rotations in the plate can be described by the mean values of the vectors

$$\mathbf{u}^* \equiv (u_1^*, u_2^*, 0), \qquad \boldsymbol{\varphi}^* \equiv (0, 0, \varphi_3^*). \tag{6.10}$$

where

$$u_j^*(x_1, x_2) = \frac{1}{2h} \int_{-h}^{h} u_j(x_1, x_2, x_3)\, dx_3,$$

$$\varphi_j^*(x_1, x_2) = \frac{1}{2h} \int_{-h}^{h} \varphi_j(x_1, x_2, x_3)\, dx_3, \qquad j = 1, 2, 3.$$

Assuming the symmetry of the functions p_1, p_2, m_3 and the functions X_1, X_2, Y_3 with respect to the middle plane $x_3 = 0$ the displacement u_3 and the rotations φ_1, φ_2 vanish on this plane and are antisymmetric with respect to it. Hence $u_3^* = \varphi_1^* = \varphi_2^* = 0$. Observe, however, that the quantity $\partial_3 u_3$ is symmetric with respect to this plane and the quantity

$$\gamma_{33}^* = \frac{1}{2h} \int_{-h}^{h} \partial_3 u_3\, dx_3$$

is different from zero.

Thus, we obtain a state of strain of the plate, described by the tensors

$$\boldsymbol{\gamma}^* = \begin{vmatrix} \gamma_{11}^* & \gamma_{12}^* & 0 \\ \gamma_{21}^* & \gamma_{22}^* & 0 \\ 0 & 0 & \gamma_{33}^* \end{vmatrix}, \qquad \boldsymbol{\varkappa}^* = \begin{vmatrix} 0 & 0 & \varkappa_{13}^* \\ 0 & 0 & \varkappa_{23}^* \\ 0 & 0 & 0 \end{vmatrix}, \tag{6.11}$$

We now proceed to the constitutive relations (1.2). Averaging the stresses over the thickness $2h$ yields

$$\sigma_{ji}^* = (\mu+\alpha)\gamma_{ji}^* + (\mu-\alpha)\gamma_{ij}^* + (\lambda\gamma_{kk}^* - \nu\theta^*)\delta_{ji} \tag{6.12}$$

$$\mu_{ji}^* = (\gamma+\varepsilon)\varkappa_{ji}^* + (\gamma-\varepsilon)\varkappa_{ij}^* + \beta\varkappa_{kk}^*\delta_{ji}, \qquad i, j, k = 1, 2, 3, \tag{6.13}$$

where

$$\theta^*(x_1, x_2) = \frac{1}{2h} \int_{-h}^{h} \theta(x_1, x_2, x_3)\, dx_3. \tag{6.14}$$

Let us first determine the quantity γ_{33}^* from the condition $\sigma_{33}^* = 0$,

$$\gamma_{33}^* = -\frac{\lambda}{\lambda+2\mu}(\gamma_{11}^*+\gamma_{22}^*)+\frac{\nu\theta^*}{\lambda+2\mu}. \tag{6.15}$$

Taking into account (6.15) and the matrices (6.9) and (6.11) we obtain the constitutive relations

$$\begin{aligned}
&\sigma_{11}^* = 2\mu\left[\gamma_{11}^*+\frac{\lambda}{\lambda+2\mu}(\gamma_{11}^*+\gamma_{22}^*)-\frac{\nu\theta^*}{\lambda+2\mu}\right],\\
&\sigma_{22}^* = 2\mu\left[\gamma_{22}^*+\frac{\lambda}{\lambda+2\mu}(\gamma_{11}^*+\gamma_{22}^*)-\frac{\nu\theta^*}{\lambda+2\mu}\right],\\
&\sigma_{12}^* = (\mu+\alpha)\gamma_{12}^*+(\mu-\alpha)\gamma_{21}^*, \qquad \sigma_{21}^* = (\mu+\alpha)\gamma_{21}^*+(\mu-\alpha)\gamma_{12}^*,\\
&\mu_{13}^* = (\gamma+\varepsilon)\varkappa_{13}^*, \qquad \mu_{23}^* = (\gamma+\varepsilon)\varkappa_{23}^*,\\
&\mu_{31}^* = (\gamma-\varepsilon)\varkappa_{13}^*, \qquad \mu_{32}^* = (\gamma-\varepsilon)\varkappa_{23}^*.
\end{aligned} \tag{6.16}$$

Let us introduce the relations (6.16) into the equilibrium equations (6.7), making use of the formulae

$$\begin{aligned}
&\gamma_{11}^* = \partial_1 u_1^*, \quad \gamma_{22}^* = \partial_2 u_2^*, \quad \gamma_{12}^* = \partial_1 u_2^*-\varphi_3^*,\\
&\gamma_{21}^* = \partial_2 u_1^*+\varphi_3^*, \quad \varkappa_{13}^* = \partial_1\varphi_3^*, \quad \varkappa_{23}^* = \partial_2\varphi_3^*.
\end{aligned} \tag{6.17}$$

Then we arrive at the system of differential equations in displacements and rotations

$$\begin{aligned}
&(\mu+\alpha)\nabla_1^2 u_1^*+\beta_0\partial_1 e^*+2\alpha\partial_2\varphi_3^*+X_1^* = 2\mu m\partial_1\theta,\\
&(\mu+\alpha)\nabla_1^2 u_2^*+\beta_0\partial_2 e^*-2\alpha\partial_1\varphi_3^*+X_2^* = 2\mu m\partial_2\theta,\\
&[(\gamma+\varepsilon)\nabla_1^2-4\alpha]\varphi_3^*+2\alpha(\partial_1 u_2^*-\partial_2 u_1^*)+Y_3^* = 0,
\end{aligned} \tag{6.18}$$

where

$$e^* = \partial_1 u_1^*+\partial_2 u_2^*, \quad \beta_0 = \frac{\mu(3\lambda+2\mu)-\alpha(\lambda+2\mu)}{\lambda+2\mu}, \quad m = \frac{\nu}{\lambda+2\mu}. \tag{6.19}$$

Here, also, the particular solution of the system of equations can be taken in the form

$$u_1'^* = \partial_1\Phi, \quad u_2'^* = \partial_2\Phi, \quad \varphi_3'^* = 0. \tag{6.20}$$

We obtain a solution identical with that of classical thermoelasticity.

In the plane state of stress there exists also a representation of stresses by the functions F, Ψ. As in Sec. 3 of this paper we have to introduce the stresses (6.16) into the compatibility Eqs. (3.2).

Below we shall present a different procedure for deriving the differential equations for the functions F and Ψ, analogous to Eqs. (3.5) of the plane state of strain.

Let us contract the first two Eqs. (6.18). We obtain

$$\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\nabla_1^2 e^*+\partial_1 X_1^*+\partial_2 X_2^* = 2\mu m\nabla_1^2\theta. \tag{6.21}$$

In the relation (6.16) we have

$$e^* = \frac{\lambda+2\mu}{2\mu(3\lambda+2\mu)}(\sigma_{11}^*+\sigma_{22}^*)+2\alpha_t\theta^* .$$

Introducing this expression into (6.21) and making use of the representation by the functions F and Ψ,

$$(6.22)\qquad \begin{aligned} &\sigma_{11}^* = \partial_2^2 F-\partial_1\partial_2\Psi, && \sigma_{22}^* = \partial_1^2 F+\partial_1\partial_2\Psi ,\\ &\sigma_{12}^* = -\partial_1\partial_2 F-\partial_2^2\Psi, && \sigma_{21}^* = -\partial_1\partial_2 F+\partial_1^2\Psi ,\\ &\mu_{13}^* = \partial_1\Psi, && \mu_{23}^* = \partial_2\Psi , \end{aligned}$$

we reduce Eq. (6.21) to the form

$$(6.23)\qquad \nabla_1^2\nabla_1^2 F+\frac{3\lambda+2\mu}{2(\lambda+\mu)}(\partial_1 X_1^*+\partial_2 X_2^*)+\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\alpha_t\nabla^2\theta = 0 .$$

Let us now differentiate Eq. $(6.18)_2$ with respect to x_1 and Eq. $(6.18)_1$ with respect to x_2 and subtract the results; then

$$(6.24)\qquad \nabla_1^2(\partial_1 u_2^*-\partial_2 u_1^*) = \frac{2\alpha}{\mu+\alpha}\nabla_1^2\varphi_3^*-\frac{1}{\mu+\alpha}(\partial_1 X_2^*-\partial_2 X_1^*) .$$

Applying the operator ∇_1^2 to Eq. $(6.18)_3$ and making use of Eq. (6.24), we obtain

$$(6.25)\qquad \nabla_1^2(1-l^2\nabla_1^2)\Psi = -\frac{1}{2\mu}(\partial_1 X_2^*-\partial_2 X_1^*)+\frac{1}{\gamma+\varepsilon}l^2\nabla_1^2 Y_3^* ,$$

where

$$l^2 = \frac{(\mu+\alpha)(\gamma+\varepsilon)}{4\mu\alpha} .$$

The functions F and Ψ are not independent. The equations

$$(6.26)\qquad \partial_1\gamma_{21}^*-\partial_2\gamma_{11}^*-\varkappa_{13}^* = 0, \quad \partial_1\gamma_{22}^*-\partial_2\gamma_{12}^*-\varkappa_{23}^* = 0 ,$$

following from the relations (6.17) yield after having expressed strains by stresses, the following relations connecting the functions F and Ψ:

$$(6.27)\qquad \begin{aligned} -\partial_1(1-l^2\nabla_1^2)\Psi &= A\partial_2\nabla_1^2 F+B\partial_2\theta^* ,\\ \partial_2(1-l^2\nabla_1^2)\Psi &= A\partial_1\nabla_1^2 F+B\partial_2\theta^* . \end{aligned}$$

Here

$$A = \frac{(\lambda+\mu)(\gamma+\varepsilon)}{\mu(3\lambda+2\mu)}, \qquad B = \frac{\nu(\gamma+\varepsilon)}{3\lambda+2\mu} .$$

The differential equations for the function F and relations of the type (6.27) differ in the two states, the plane state of strain and the plane state of stress, only in the values of the coefficients. It can also easily be proved that in the case of the plane state of stress we obtain the boundary conditions analogous to (3.8). Namely we have

$$(6.28)\qquad \frac{\partial F}{\partial n}+\frac{\partial\Psi}{\partial s} = f_1 n_1+f_2 n_2, \quad \frac{\partial F}{\partial s}-\frac{\partial\Psi}{\partial n} = f_2 n_1-f_1 n_2, \quad \frac{\partial\Psi}{\partial n} = m_3^* ,$$

where

$$f_1 = -\int\limits_{s_0}^{s} p_2^*(s)\,ds\,, \quad f_2^* = \int\limits_{s_0}^{s} p_1^*(s)\,ds\,,$$

$$p_\alpha^* = \frac{1}{2h}\int\limits_{-h}^{h} p_\alpha(x_1, x_2, x_3)\,dx_3\,, \quad \alpha = 1, 2\,.$$

In the particular case of the action of the temperature field only $\theta^*(x_1, x_2)$, we have to solve the system of equations

$$\nabla_1^2\nabla_1^2 F + \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\alpha_t\nabla_1^2\theta^* = 0\,, \tag{6.29}$$

$$\nabla_1^2(1-l^2\nabla_1^2)\Psi = 0 \tag{6.30}$$

with the homogeneous boundary conditions

$$\frac{\partial F}{\partial n} + \frac{\partial \Psi}{\partial s} = 0\,, \quad \frac{\partial F}{\partial s} = 0\,, \quad \frac{\partial \Psi}{\partial n} = 0\,. \tag{6.31}$$

Further, the relations (6.27) have to be satisfied.

The temperature θ^* appearing in the constitutive equations and in Eq. (6.29) is determined from the heat conduction equation

$$(\partial_1^2+\partial_2^2+\partial_3^2)\theta(x_1, x_2, x_3) = -\frac{W(x_1, x_2, x_3)}{\lambda_0} \tag{6.32}$$

Let us now integrate the above equation over the thickness of the plate. We obtain

$$(\partial_1^2+\partial_1^2)\frac{1}{2h}\int\limits_{-h}^{h}\theta\,dx_3 + \frac{1}{2h}\left|\frac{\partial\theta}{\partial x_3}\right|_{-h}^{h} = -\frac{1}{2h}\int\limits_{-h}^{h} W\,dx_3\,, \tag{6.33}$$

Introducing the functions

$$\theta^*(x_1, x_2) = \frac{1}{2h}\int\limits_{-h}^{h}\theta\,dx_3\,, \quad W^* = \frac{1}{2h}\int\limits_{-h}^{h} W\,dx_3\,,$$

and bearing in mind the conditions of heat exchange on the planes $x_3 = \pm h$,

$$\lambda_0\left|\frac{\partial\theta}{\partial x_3}\right|_{x_3=h} = \lambda_1\big(\vartheta-\theta(x_1, x_2, h)\big)\,, \quad \lambda_0\left|\frac{\partial\theta}{\partial x_3}\right|_{x_3=-h} = -\lambda_1\big(\vartheta-\theta(x_1, x_2, -h)\big)\,,$$

where λ_1 is the external heat conduction coefficient and ϑ the temperature of the surroundings, we reduce Eq. (6.33) to the form

$$(\partial_1^2+\partial_2^2)\theta^* + \varepsilon_0[\vartheta-\theta(x_1, x_2, h)] = -\frac{W^*}{\lambda_0}\,. \tag{6.34}$$

where

$$\varepsilon_0 = \frac{2\lambda_1}{h\lambda_0}\,.$$

If the planes $x_3 = \pm h$ are thermally insulated, $\lambda_1 = 0$, $\varepsilon_0 = 0$ and for θ^* we obtain the Poisson equation

$$\nabla_1^2 \theta^* = -\frac{W^*}{\lambda_0}. \tag{6.35}$$

Consider the particular case of the simply-connected plate, free of external loadings ($p_1 = p_2 = m_3 = 0$, $X_1 = X_2 = Y_3 = 0$) but heated symmetrically with respect to the plane $x_3 = 0$. Further, assume that there are no heat sources in the plate and the planes $x_3 = \pm h$ are thermally insulated.

The trivial solution of Eqs. (6.29), (6.30), the boundary conditions (6.31) and the relations (6.27)

$$F \equiv 0 \qquad \Psi \equiv 0 \tag{6.36}$$

is possible only when $\theta^* = \text{const}$. The solution (6.36) implies zero values of all stresses. In the case of other solutions of the homogeneous Eq. (6.34) than $\theta^* = \text{const}$, the values of stresses are different from zero.

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Streszczenie

PŁASKI PROBLEM W MIKROPOLARNEJ TERMOSPRĘŻYSTOŚCI

W niniejszej pracy podano podstawowe związki i równania różniczkowe dla płaskiego zagadnienia termosprężystości w ośrodku mikropolarnym. Podano równania różniczkowe tak w przemieszczeniach i obrotach jak i w funkcjach naprężeń.

Z równań wynika, że każde rozwiązanie może być złożone z rozwiązania szczególnego, identycznego w swej postaci z rozwiązaniem w ośrodku Hooke'a, oraz z rozwiązania ogólnego jednorodnego układu równań różniczkowych, wpisanych dla ośrodka mikropolarnego, rozpatrywanego w stanie izotermicznym.

Podano wreszcie dwa przykłady szczególne, obrazujące naprężenia termiczne, występujące w półprzestrzeni sprężystej wskutek ogrzania jej brzegu i działania źródła ciepła w jej obszarze.

Резюме

ПЛОСКАЯ ЗАДАЧА В МИКРОПОЛЯРНОЙ ТЕРМОУПРУГОСТИ

В настоящей работе даются основные соотношения и дифференциальные уравнения для плоской задачи термоупругости в микрополярной среде. Даются дифференциальные уравнения, так в перемещениях и вращениях, как и в функциях напряжений.

Из рассуждений следует, что всякое решение может быть составлено из частного решения идентичного вида, как частное решение в среде Гука, и из общего решения однородной системы дифференциальных уравнений, записанных для микрополярной среды рассматриваемой в изотермическом состоянии.

Даются наконец два частных примера, представляющие термические напряжения, выступающие в упругом полупространстве вследствие нагрева его границы и действия теплоисточников в его области.

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