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ON THE COMPLETENESS OF POTENTIALS IN MICROPOLAR ELASTICITY

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1. Introduction

Let us consider an elastic, homogeneous, isotropic and centrosymmetric body extending over the volume B and bounded by the surface A. Under influence of external forces this body will deform. Inside a body a field of displacements $\mathbf{u}(\mathbf{x}, t)$ and rotations $\boldsymbol{\omega}(\mathbf{x}, t)$ will appear and will change with the situation of the point $\mathbf{x}$ and time t.

The field of displacement $\mathbf{u}$ and rotations $\boldsymbol{\omega}$ should fulfill the following equations of motion [1]:

$$\Box_2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\omega}+\mathbf{X}=0\,, \tag{1.1}$$

$$\Box_4\boldsymbol{\omega}+(\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\omega}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=0\,, \tag{1.2}$$

where

$$\Box_2=(\mu+\alpha)\nabla^2-\varrho\partial_t^2,\qquad \Box_4=(\gamma+\varepsilon)\nabla^2-4\alpha-J\partial_t^2,$$

$$\nabla^2=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i},\qquad \partial_t^2=\frac{\partial^2}{\partial_t^2}.$$

In above equations $\mathbf{X}$ denotes the vector of body forces, $\mathbf{Y}$—vector of body couples, ϱ is a density and J—rotational inertia. μ, λ, α, β, γ, ε are material constants.

Equations (1.1)–(1.2) are the equations of elastokinetics of micropolar medium, leading out under assumptions of the adiabatic process.

The system of equations (1.1)–(1.2) can be separated in two different ways. First way, analogous to Lame's procedure applied in classic elastokinetics, consists of the decomposition of the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ on the potential and solenoidal parts, respectively [1]:

$$\mathbf{u}=\mathrm{grad}\,\Phi+\mathrm{rot}\,\boldsymbol{\Psi},\qquad \mathrm{div}\,\boldsymbol{\Psi}=0, \tag{1.3}$$

$$\boldsymbol{\omega}=\mathrm{grad}\,\Sigma+\mathrm{rot}\,\mathbf{H},\qquad \mathrm{div}\,\mathbf{H}=0. \tag{1.4}$$

Substituting Eqs. (1.3)–(1.4) to the equations of motion (1.1)–(1.2) and disregarding the body forces and body couples ($\mathbf{X}=0$, $\mathbf{Y}=0$) we obtain the following system of equations:

$$\Box_1\Phi=0, \tag{1.5}$$

$$\Box_3\Sigma=0, \tag{1.6}$$

$$\Box_2\boldsymbol{\Psi}+2\alpha\,\mathrm{rot}\mathbf{H}=0, \tag{1.7}$$

$$\Box_4\mathbf{H}+2\alpha\,\mathrm{rot}\boldsymbol{\Psi}=0. \tag{1.8}$$

From the last two equations, by means of elimination, we obtain

(1.9)
$$\Omega\Psi = 0, \quad \Omega\mathbf{H} = 0.$$

Here the following notations have been introduced

$$\Box_1 = (\lambda+2\mu)\nabla^2-\varrho\partial_t^2, \quad \Box_3 = (\beta+2\gamma)\nabla^2-4\alpha-J\partial_t^2,$$
$$\Omega = \Box_2\Box_4+4\alpha^2\nabla^2.$$

Solution of the combined system of hyperbolic Eqs. (1.1)–(1.2) has been reduced to the solution of the wave Eqs. (1.5)–(1.6) and Eqs. (1.9). Equation (1.5) represents longitudinal wave, equation (1.6)—rotational wave and equations $(1.9)_1$–$(1.9)_2$ represent transverse-torsional waves.

In Sec. 2 we shall present the proof of the completeness of introduced here potentials $\Phi, \Sigma, \Psi, \mathbf{H}$.

The second way of separation of the system of equations is analogous to that one which was used by GALERKIN [2] in relation to equations of classical elastostatics and by IACOVACHE [3] in reference to equations of classical elastokinetics. This type of functions have also been considered by N. SANDRU [4] in micropolar nonsymmetric elasticity. He applied an general algoritm introduced first by GR. C. MOISIL [5]. Below we shall present a different, and from our point of view the simplest way of obtaining of the stress functions, which permits to overcome the time consuming solutions of six order determinants.

Eliminating from Eqs. (1.1)–(1.2) first, the function $\boldsymbol{\omega}$ and then the function $\mathbf{u}$, we obtain the following system of equations:

(1.10)
$$\Omega\mathbf{u}+\operatorname{grad}\operatorname{div}\Gamma\mathbf{u}+\Box_4\mathbf{X}-2\alpha\operatorname{rot}\mathbf{Y} = 0,$$

(1.11)
$$\Omega\boldsymbol{\omega}+\operatorname{grad}\operatorname{div}\Theta\boldsymbol{\omega}+\Box_2\mathbf{Y}-2\alpha\operatorname{rot}\mathbf{X} = 0,$$

$$\Omega = \Box_2\Box_4+4\alpha^2\nabla^2, \quad \Gamma = (\lambda+\mu-\alpha)\Box_4-4\alpha^2, \quad \Theta = (\beta+2\gamma-\varepsilon)\Box_2-4\alpha^2.$$

Let us consider first, the system of Eqs. (1.10) which can be rewritten in an operator form

(1.12)
$$L_{ij}(u_j)+\Box_4X_i-2\alpha\epsilon_{ijk}Y_{k,j} = 0, \quad i,j,k = 1,2,3,$$
$$L_{ij} = \Omega\delta_{ij}+\partial_i\partial_j\Gamma.$$

Let us introduce the vector function $\boldsymbol{\xi}$, connected with components of the displacements $\mathbf{u}$ by relations

(1.13)
$$u_1 = \begin{vmatrix} \xi_1 & L_{12} & L_{13} \\ \xi_2 & L_{22} & L_{23} \\ \xi_3 & L_{32} & L_{33} \end{vmatrix}, \quad u_2 = \begin{vmatrix} L_{11} & \xi_1 & L_{13} \\ L_{21} & \xi_2 & L_{23} \\ L_{31} & \xi_3 & L_{33} \end{vmatrix}, \quad u_3 = \begin{vmatrix} L_{11} & L_{12} & \xi_1 \\ L_{21} & L_{22} & \xi_2 \\ L_{31} & L_{32} & \xi_3 \end{vmatrix}.$$

After simple transformations we can confirm that the vector $\mathbf{u}$ is connected with vector $\boldsymbol{\xi}$ by the following relation:

(1.14)
$$\mathbf{u} = \Box_1\Box_4\boldsymbol{\xi}-\operatorname{grad}\operatorname{div}\Gamma\boldsymbol{\xi}.$$

Fully analogous procedure with Eq. (1.11) will give relation

(1.15)
$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\eta}-\operatorname{grad}\operatorname{div}\Theta\boldsymbol{\eta},$$

where $\boldsymbol{\eta}$ is the second stress vector function.

Putting Eqs. (1.14) into Eq. (1.10) and Eqs. (1.15) into Eq. (1.11) we obtain the following system of equations to determine the functions $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$:

$$\Omega\Box_1\Box_4\boldsymbol{\xi} = -\Box_4\mathbf{X}+2\alpha\,\mathrm{rot}\,\mathbf{Y}, \tag{1.16}$$

$$\Omega\Box_2\Box_3\boldsymbol{\eta} = -\Box_2\mathbf{Y}+2\alpha\,\mathrm{rot}\,\mathbf{X}. \tag{1.17}$$

We obtained two mutually independent systems of equations. However, these equations are not convenient to the further analysis, because; in their right-hand sides appear differential operations on the body forces and body couples. But, if instead of representations (1.14) and (1.15), we consider relations

$$\mathbf{u} = \Box_1\Box_4\boldsymbol{\varphi}-\mathrm{grad}\,\mathrm{div}\,\Gamma\boldsymbol{\varphi}-2\alpha\,\mathrm{rot}\,\Box_3\boldsymbol{\psi}, \tag{1.18}$$

$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\psi}-\mathrm{grad}\,\mathrm{div}\,\Theta\boldsymbol{\psi}-2\alpha\,\mathrm{rot}\,\Box_1\boldsymbol{\varphi}; \tag{1.19}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are the new stress functions, then the substitution of Eqs. (1.18)–(1.19) into the system of Eqs. (1.10) and (1.11) leads to the following relations

$$\Box_4(\Omega\Box_1\boldsymbol{\varphi}+\mathbf{X})-2\alpha\,\mathrm{rot}(\Omega\Box_3\boldsymbol{\psi}+\mathbf{Y}) = 0,$$

$$\Box_2(\Omega\Box_3\boldsymbol{\psi}+\mathbf{Y})-2\alpha\,\mathrm{rot}((\Omega\Box_1\boldsymbol{\varphi}+\mathbf{X}) = 0.$$

From the above relations result equations

$$\Box_1\Omega\boldsymbol{\varphi}+\mathbf{X} = 0, \quad \Box_3\Omega\boldsymbol{\psi}+\mathbf{Y} = 0, \tag{1.20}$$

which are used to determination of the stress functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$. Equations (1.20) become equivalent to the equations obtained in a different way by N. SANDRU [4].

In Sec. 4, we shall present a theorem on the completeness of the stress functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$.

2. Completeness of the Potentials Φ, Σ, $\boldsymbol{\Psi}$, $\mathbf{H}$

THEOREM. *Let* $\mathbf{u}(\mathbf{x}, t)$ *and* $\boldsymbol{\omega}(\mathbf{x}, t)$ *be particular solutions of Eqs.* (1.1)–(1.2) *in a region of space* V *for* $t_1 < t < t_2$. *Then there exist such scalar functions* Φ, Σ *and vector functions* $\boldsymbol{\Psi}$, $\mathbf{H}$ *that the displacements* $\mathbf{u}(\mathbf{x}, t)$ *and rotations* $\boldsymbol{\omega}(\mathbf{x}, t)$ *can be expressed by the relations*

$$\mathbf{u} = \mathrm{grad}\,\Phi+\mathrm{rot}\,\boldsymbol{\Psi}, \quad \mathrm{div}\,\boldsymbol{\Psi} = 0, \tag{2.1}$$

$$\boldsymbol{\omega} = \mathrm{grad}\,\Sigma+\mathrm{rot}\,\mathbf{H}, \quad \mathrm{div}\,\mathbf{H} = 0. \tag{2.2}$$

Functions Φ, Σ, $\boldsymbol{\Psi}$, $\mathbf{H}$ *should satisfy the wave equations*

$$\Box_1\Phi = 0, \quad \Box_3\Sigma = 0, \quad \Omega\boldsymbol{\Psi} = 0, \quad \Omega\mathbf{H} = 0. \tag{2.3}$$

The proof of this theorem, presented below, is a generalisation of the analogous theorem of classical elastokinetics given by DUHAMEL [6] and repeated by E. STERNBERG [7].

An issue point in a proof of a theorem on the completeness of potentials Φ, Σ, $\boldsymbol{\Psi}$, $\mathbf{H}$ are two Newtonian vector potentials

$$\mathbf{W}(\mathbf{x}, t) = -\frac{1}{4\pi}\int\limits_V \frac{\mathbf{u}(\boldsymbol{\xi}, t)}{R(\mathbf{x}, \boldsymbol{\xi})}dV(\boldsymbol{\xi}), \quad \boldsymbol{\Omega}(\mathbf{x}, t) = -\frac{1}{4\pi}\int\limits_V \frac{\boldsymbol{\omega}(\boldsymbol{\xi}, t)}{R(\mathbf{x}, \boldsymbol{\xi})}dV(\boldsymbol{\xi}), \tag{2.4}$$

where $R(\mathbf{x}, \boldsymbol{\xi})$ is a distance between the points $\boldsymbol{\xi}$ and $\mathbf{x}$. Since

$$\nabla^2\mathbf{W} = \mathbf{u}, \quad \nabla^2\boldsymbol{\Omega} = \boldsymbol{\omega}$$

and

$$\nabla^2 \mathbf{W} = \operatorname{grad}\operatorname{div}\mathbf{W} - \operatorname{rot}\operatorname{rot}\mathbf{W}, \quad \nabla^2 \mathbf{\Omega} = \operatorname{grad}\operatorname{div}\mathbf{\Omega} - \operatorname{rot}\operatorname{rot}\mathbf{\Omega},$$

then functions $\mathbf{u}$ and $\boldsymbol{\omega}$ can be expressed in a form

$$\mathbf{u} = \operatorname{grad} U + \operatorname{rot}\mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0, \tag{2.5}$$

$$\boldsymbol{\omega} = \operatorname{grad} T + \operatorname{rot}\mathbf{S}, \quad \operatorname{div}\mathbf{S} = 0, \tag{2.6}$$

where

$$U = \operatorname{div}\mathbf{W}, \quad \mathbf{V} = -\operatorname{rot}\mathbf{W},$$

and

$$T = \operatorname{div}\mathbf{\Omega}, \quad \mathbf{S} = -\operatorname{rot}\mathbf{\Omega}.$$

Inserting Eqs. (2.5) and (2.6) into a basic system of equations (1.1) and (1.2) (with $\mathbf{X} = \mathbf{Y} = 0$), we obtain

$$\operatorname{grad}(\Box_1 U) + \operatorname{rot}(\Box_2 \mathbf{V} + 2\alpha \operatorname{rot}\mathbf{S}) = 0, \tag{2.7}$$

$$\operatorname{grad}(\Box_3 T) + \operatorname{rot}(\Box_4 \mathbf{S} + 2\alpha \operatorname{rot}\mathbf{V}) = 0. \tag{2.8}$$

Let us perform on the above equations the operation of divergence. As a result we shall obtain the equations

$$\nabla^2 \Box_1 U = 0, \quad \nabla^2 \Box_3 T = 0. \tag{2.9}$$

After an operation of rotation on Eqs. (2.7)–(2.8), we have

$$\nabla^2(\Box_2 \mathbf{V} + 2\alpha \operatorname{rot}\mathbf{S}) = 0, \quad \nabla^2(\Box_4 \mathbf{S} + 2\alpha \operatorname{rot}\mathbf{V}) = 0. \tag{2.10}$$

Let us consider now Eq. (2.9) . It will be satisfied if

$$\Box_1 U = a(\mathbf{x}, t), \quad \nabla^2 a = 0. \tag{2.11}$$

Let us determine a new function $A(\mathbf{x}, t)$ as

$$\varrho A(\mathbf{x}, t) = \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\lambda\, a(\mathbf{x}, \lambda). \tag{2.12}$$

Thus, taking into consideration Eqs. (2.12) and $(2.11)_2$, we have

$$a(\mathbf{x}, t) = \varrho \frac{\partial^2 A}{\partial t^2}, \quad \nabla^2 A = 0. \tag{2.13}$$

From Eq. $(2.9)_2$ one obtains

$$\Box_3 T = c(\mathbf{x}, t), \quad \nabla^2 c = 0. \tag{2.14}$$

Let us introduce a new function $C(\mathbf{x}, t)$, such that

$$\begin{aligned} c(\mathbf{x}, t) &= (J\partial_t^2 + 4\alpha) C(\mathbf{x}, t), \\ C(\mathbf{x}, t) &= \frac{1}{J\varkappa} \int_0^t c(\mathbf{x}, t) \sin\varkappa(t-\tau)\, d\tau, \quad \varkappa = \left(\frac{4\alpha}{J}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.15}$$

Notice, that in view of Eq. $(2.14)_2$, occurs $\nabla^2 C = 0$.

Let us introduce two new functions $U_1 = U+A$ and $T_1 = T+C$. Substituting them into Eqs. (2.11) and (2.14) and, taking into account the relations (2.13) and (2.15), we obtain

(2.16) $$\Box_1 U_1 = 0, \quad \Box_3 T_1 = 0.$$

Let us consider by turn Eqs. (2.7) and (2.8). It is easily seen that

(2.17) $$\Box_2 \mathbf{V} + 2\alpha \operatorname{rot} \mathbf{S} = \mathbf{b}(\mathbf{x}, t), \quad \nabla^2 \mathbf{b} = 0, \quad \operatorname{div} \mathbf{b} = 0,$$

(2.18) $$\Box_4 \mathbf{S} + 2\alpha \operatorname{rot} \mathbf{V} = \mathbf{d}(\mathbf{x}, t), \quad \nabla^2 \mathbf{d} = 0, \quad \operatorname{div} \mathbf{d} = 0.$$

Let us introduce the functions **B** and **D** defined in a following way:

(2.19) $$\varrho \mathbf{B} = \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\lambda \mathbf{b}(\mathbf{x}, \lambda), \quad \mathbf{D} = \frac{1}{J\varkappa} \int_0^t \mathbf{d}(\mathbf{x}, t) \sin \varkappa (t-\tau) d\tau,$$

whence

$$\mathbf{b} = \varrho \frac{\partial^2 \mathbf{B}}{\partial t^2}, \quad \mathbf{d} = (J\partial_t^2 + 4\alpha)\mathbf{D}.$$

From relations $\operatorname{div}\mathbf{b} = 0$, $\operatorname{div}\mathbf{d} = 0$ it results also that

(2.20) $$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{div} \mathbf{D} = 0$$

and from relations $(2.17)_2$ and $(2.18)_2$, we have

(2.21) $$\nabla^2 \mathbf{B} = 0, \quad \nabla^2 \mathbf{D} = 0.$$

Let us introduce new functions $\boldsymbol{\Psi}$, **H**, such that

(2.22) $$\boldsymbol{\Psi} = \mathbf{V} + \mathbf{B}, \quad \mathbf{H} = \mathbf{S} + \mathbf{D}.$$

By substitution above functions into Eqs. (2.17)–(2.18) one gets

(2.23) $$\begin{aligned} &\Box_2 (\boldsymbol{\Psi} - \mathbf{B}) + 2\alpha \operatorname{rot} (\mathbf{H} - \mathbf{D}) = \mathbf{b}, \\ &\Box_4 (\mathbf{H} - \mathbf{D}) + 2\alpha \operatorname{rot} (\boldsymbol{\Psi} - \mathbf{B}) = \mathbf{d}. \end{aligned}$$

In view of a definition of the functions **B** and **D**, [formulae $(2.19)_1$–$(2.19)_2$], we obtain from Eq. (2.23) the following relations:

(2.24) $$\Box_2 \boldsymbol{\Psi} + 2\alpha \operatorname{rot} \mathbf{H} = 2\alpha \operatorname{rot} \mathbf{D},$$

(2.25) $$\Box_4 \mathbf{H} + 2\alpha \operatorname{rot} \boldsymbol{\Psi} = 2\alpha \operatorname{rot} \mathbf{B}.$$

After these preparations let us come back to Eqs. (2.5)–(2.6). The relation (2.5) now becomes

(2.26) $$\mathbf{u} = \operatorname{grad} U + \operatorname{rot} \mathbf{V} = \operatorname{grad} (U_1 - A) + \operatorname{rot} (\boldsymbol{\Psi} - \mathbf{B}) = \operatorname{grad} U_1 + \operatorname{rot} \boldsymbol{\Psi} + \mathbf{u}^*,$$

where

(2.27) $$\mathbf{u}^* = -(\operatorname{grad} A + \operatorname{rot} \mathbf{B}).$$

In a completely analogous way we shall derive the relation (2.6)

(2.28) $$\boldsymbol{\omega} = \operatorname{grad} T + \operatorname{rot} \mathbf{S} = \operatorname{grad} (T_1 - C) + \operatorname{rot} (\mathbf{H} - \mathbf{D}) = \operatorname{grad} T_1 + \operatorname{rot} \mathbf{H} + \boldsymbol{\omega}^*,$$

where

(2.29) $$\omega^* = -(\operatorname{grad} C + \operatorname{rot} \mathbf{D}).$$

Notice that performation of the divergence and rotations operations on the expressions (2.27) and (2.29), respectively, gives

$$\begin{aligned} \operatorname{div}\mathbf{u}^* &= -\nabla^2 A = 0, \quad & \operatorname{rot}\mathbf{u}^* &= \nabla^2\mathbf{B} = 0, \\ \operatorname{div}\boldsymbol{\omega}^* &= -\nabla^2 C = 0, \quad & \operatorname{rot}\boldsymbol{\omega}^* &= \nabla^2\mathbf{D} = 0. \end{aligned} \tag{2.30}$$

From above relations it results that there exist such functions U_2, T_2, that

$$\mathbf{u}^* = \operatorname{grad} U_2, \quad \nabla^2 U_2 = 0, \quad \boldsymbol{\omega}^* = \operatorname{grad} T_2, \quad \nabla^2 T_2 = 0. \tag{2.31}$$

Inserting last equations into Eqs. (2.26) and (2.28) we have

$$\mathbf{u} = \operatorname{grad}(U_1+U_2)+\operatorname{rot}\boldsymbol{\Psi}, \quad \boldsymbol{\omega} = \operatorname{grad}(T_1+T_2)+\operatorname{rot}\mathbf{H}. \tag{2.32}$$

We substitute these relations into the equations of motion (1.1) and (1.2). In this manner we shall obtain the system of equations

$$\begin{aligned} &\operatorname{grad}\big(\Box_1(U_1+U_2)\big)+\operatorname{rot}(\Box_2\boldsymbol{\Psi}+2\alpha\operatorname{rot}\mathbf{H}) = 0, \\ &\operatorname{grad}\big(\Box_3(T_1+T_2)\big)+\operatorname{rot}(\Box_4\mathbf{H}+2\alpha\operatorname{rot}\boldsymbol{\Psi}) = 0. \end{aligned} \tag{2.33}$$

Taking into account the relations (2.16) and $(2.31)_2$, together with $(2.31)_4$, and considering the system of Eqs. (2.24)–(2.25), we shall transform Eq. (2.33) to a form

$$\begin{aligned} &-\operatorname{grad}\left(\varrho\frac{\partial^2 U_2}{\partial t^2}\right)+2\alpha\operatorname{rot}\operatorname{rot}\mathbf{D} = 0, \\ &-\operatorname{grad}(J\partial_t^2+4\alpha)\,T_2+2\alpha\operatorname{rot}\operatorname{rot}\mathbf{B} = 0. \end{aligned} \tag{2.34}$$

But, if we notice that $\operatorname{rot}\operatorname{rot}\mathbf{D} = \operatorname{grad}\operatorname{div}\mathbf{D}-\nabla^2\mathbf{D}$ and relations $\operatorname{div}\mathbf{D} = 0$, $\nabla^2\mathbf{D} = 0$ occur, we shall obtain $\operatorname{rot}\operatorname{rot}\mathbf{D} = 0$ and similarly $\operatorname{rot}\operatorname{rot}\mathbf{B} = 0$.

Thus, the system of Eqs. (2.34), becomes

$$\operatorname{grad}\left(\frac{\partial^2 U_2}{\partial t^2}\right) = 0, \quad \operatorname{grad}\left(J\frac{\partial^2 T_2}{\partial t^2}+4\alpha T_2\right) = 0. \tag{2.35}$$

Integrating Eq. $(2.35)_1$ one gets

$$U_2 = \alpha(t)+t\beta(\mathbf{x})+\gamma(\mathbf{x}). \tag{2.36}$$

Since $\nabla^2 U_2 = 0$, what results from Eq. $(2.31)_2$, then

$$\nabla^2\beta(\mathbf{x}) = 0, \quad \nabla^2\gamma(\mathbf{x}) = 0. \tag{2.37}$$

Integration of Eq. $(2.35)_2$ leads to an ordinary differential equation

$$\frac{d^2 T_2}{dt^2}+\varkappa^2 T_2 = f(t),$$

with a solution

$$T_2 = \sigma(\mathbf{x})\cos\varkappa t+\tau(\mathbf{x})\sin\varkappa t+g(t), \tag{2.38}$$

where

$$g(t) = \int_0^t f(\tau)\sin\varkappa(t-\tau)\,dt.$$

In virtue of Eq. $(2.31)_4$ one can obtain

$$\nabla^2\sigma(\mathbf{x}) = 0, \quad \nabla^2\tau(\mathbf{x}) = 0. \tag{2.39}$$

Let us introduce now the new scalar functions Φ and Σ, determined in a following way:

$$\Phi = U_1+U_2-\alpha(t), \quad \Sigma = T_1+T_2-g(t). \tag{2.40}$$

By means of them the relations (2.32) may be written as

$$\mathbf{u} = \operatorname{grad}\Phi+\operatorname{rot}\mathbf{\Psi}, \quad \boldsymbol{\omega} = \operatorname{grad}\Sigma+\operatorname{rot}\mathbf{H}. \tag{2.41}$$

In this way we have obtained the sought representation of the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ by potential part and solenoidal part, respectively. It is still necessary to demonstrate that the conditions $\operatorname{div}\mathbf{\Psi} = 0$, $\operatorname{div}\mathbf{H} = 0$ are satisfied.

From a definition of the function $\mathbf{\Psi}$, we have

$$\operatorname{div}\mathbf{\Psi} = \operatorname{div}(\mathbf{V}-\mathbf{B}).$$

Since $\operatorname{div}\mathbf{V} = 0$, $\operatorname{div}\mathbf{B} = 0$, then also $\operatorname{div}\mathbf{\Psi} = 0$. Analogically from formulae $(2.22)_2$, $(2.5)_2$, $(2.6)_2$ and $(2.20)_2$ it results that $\operatorname{div}\mathbf{H} = 0$. It still remains to show that the functions Φ, Σ, $\mathbf{\Psi}$, $\mathbf{H}$ satisfy the wave equation (2.3).

Let us use first the Eq. $(2.16)_1$, taking into account the relation $(2.40)_1$. Here we have

$$\Box_1 U_1 = 0 \quad \text{or} \quad \Box_1(\Phi-U_2+\alpha(t)) = 0. \tag{2.42}$$

In view of Eq. (2.36) one can write

$$\Box_1\Phi-\Box_1(t\beta(\mathbf{x})+\gamma(\mathbf{x})) = 0.$$

Since $\nabla^2\beta = 0$, $\nabla^2\gamma = 0$, then it remains us the wave equation

$$\Box_1\Phi = 0. \tag{2.43}$$

To derive the wave Eq. $(2.3)_2$ we start with Eq. $(2.16)_2$

$$\Box_3 T_1 = 0, \quad \text{or} \quad \Box_3(\Sigma-T_2-g(t)) = 0. \tag{2.44}$$

Taking into consideration (2.38), we have

$$\Box_3\Sigma-\Box_3(\sigma(\mathbf{x})\cos\varkappa t+\tau(\mathbf{x})\sin\varkappa t) = 0.$$

The second term in this equation, in virtue of Eq. (2.39), is equal to zero. Thus it remains the required wave equation

$$\Box_3\Sigma = 0. \tag{2.45}$$

It is still necessary to obtain the wave Eqs. $(2.3)_3$, and $(2.3)_4$. Let us introduce the notations

$$\mathbf{M} = \Box_2\mathbf{\Psi}+2\alpha\operatorname{rot}\mathbf{H}, \quad \mathbf{N} = \Box_4\mathbf{H}+2\alpha\operatorname{rot}\mathbf{\Psi}. \tag{2.46}$$

Then from Eqs. (2.24)–(2.25) one obtains

$$\operatorname{rot}\mathbf{M} = 0, \quad \operatorname{rot}\mathbf{N} = 0, \tag{2.47}$$

because $\operatorname{rot}\operatorname{rot}\mathbf{D} = -\nabla^2\mathbf{D} = 0$ and $\operatorname{rot}\operatorname{rot}\mathbf{B} = -\nabla^2\mathbf{B} = 0$.

Performing on Eq. (2.46) the divergence operation we have

$$\operatorname{div}\mathbf{M} = 0, \quad \operatorname{div}\mathbf{N} = 0. \tag{2.48}$$

If divergence and rotation of a vector disappear in a considered region, then this vector is identically equal to zero. In this manner we obtain the system of homogeneous equations,

$$\Box_2\mathbf{\Psi}+2\alpha\operatorname{rot}\mathbf{H} = 0, \quad \Box_4\mathbf{H}+2\alpha\operatorname{rot}\mathbf{\Psi} = 0, \tag{2.49}$$

whence, by elimination—of the quantities $\mathbf{\Psi}$ or $\mathbf{H}$., we get the wave Eqs. $(2.3)_3$ and $(2.3)_4$. This ends the proof of a theorem on the completeness of potentials Φ, Σ, $\mathbf{\Psi}$, $\mathbf{H}$.

3. Theorem on the Completeness of Stress Functions

THEOREM. *Let* $\mathbf{u}(\mathbf{x}, t)$ *and* $\boldsymbol{\omega}(\mathbf{x}, t)$ *be particular solutions of the system of equations* (1.1)–(1.2) *in a region of space* V *for* $t_1 < t < t_2$. *Then, there exist such vector functions* $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, *that displacements* $\mathbf{u}$ *and rotations* $\boldsymbol{\omega}$ *are expressed by the following representation:*

$$\mathbf{u} = \Box_1\Box_4\boldsymbol{\xi}-\operatorname{grad}\operatorname{div}\Gamma\boldsymbol{\xi}, \tag{3.1}$$

$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\eta}-\operatorname{grad}\operatorname{div}\Theta\boldsymbol{\eta}, \tag{3.2}$$

and functions $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ *satisfy the wave equations*

$$\Omega\Box_1\Box_4\boldsymbol{\xi} = -\Box_4\mathbf{X}+2\alpha\operatorname{rot}\mathbf{Y}, \tag{3.3}$$

$$\Omega\Box_2\Box_3\boldsymbol{\eta} = -\Box_2\mathbf{Y}+2\alpha\operatorname{rot}\mathbf{X}. \tag{3.4}$$

In a proof of the above theorem we start with the representation of the vectors $\mathbf{u}$, $\boldsymbol{\omega}$ in a form of solution

$$\mathbf{u} = \operatorname{grad}\Phi+\operatorname{rot}\mathbf{\Psi}, \quad \operatorname{div}\mathbf{\Psi} = 0, \tag{3.5}$$

$$\boldsymbol{\omega} = \operatorname{grad}\Sigma+\operatorname{rot}\mathbf{H}, \quad \operatorname{div}\mathbf{H} = 0. \tag{3.6}$$

Substituting Eqs. (3.5)–(3.6) into the motion equations, Eqs. (1.1) and (1.2), and taking into account

$$\Omega = \Box_2\Box_4+4\alpha^2\nabla^2 = \Box_1\Box_4-\nabla^2\Gamma = \Box_2\Box_3-\Theta\nabla^2, \tag{3.7}$$

we obtain the following system of equations:

$$\operatorname{grad}(\Box_1\Box_4\Phi)+\operatorname{rot}(\Omega\mathbf{\Psi})+\Box_4\mathbf{X}-2\alpha\operatorname{rot}\mathbf{Y} = 0, \tag{3.8}$$

$$\operatorname{grad}(\Box_2\Box_3\Sigma)+\operatorname{rot}(\Omega\mathbf{H})+\Box_2\mathbf{Y}-2\alpha\operatorname{rot}\mathbf{X} = 0. \tag{3.9}$$

Let us apply Eqs. (3.3)–(3.4) and express the vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ by their potential part and solenoidal part

$$\boldsymbol{\xi} = \operatorname{grad}\vartheta+\operatorname{rot}\boldsymbol{\lambda}, \quad \operatorname{div}\boldsymbol{\lambda} = 0, \tag{3.10}$$

$$\boldsymbol{\eta} = \operatorname{grad}\sigma+\operatorname{rot}\boldsymbol{\chi}, \quad \operatorname{div}\boldsymbol{\chi} = 0. \tag{3.11}$$

Inserting Eq. (3.10) into Eq. (3.3), and Eq. (3.11) into Eq. (3.4), one gets the following system of equations:

$$\operatorname{grad}(\Omega\Box_1\Box_4\vartheta)+\operatorname{rot}(\Omega\Box_1\Box_4\boldsymbol{\lambda})+\Box_4\mathbf{X}-2\alpha\operatorname{rot}\mathbf{Y} = 0, \tag{3.12}$$

$$\operatorname{grad}(\Omega\Box_2\Box_3\sigma)+\operatorname{rot}(\Omega\Box_2\Box_3\boldsymbol{\chi})+\Box_2\mathbf{Y}-2\alpha\operatorname{rot}\mathbf{X} = 0. \tag{3.13}$$

A comparison of Eq. (3.8) with Eq. (3.12) and Eq. (3.9) with Eq. (3.13) leads to

$$\Phi = \Omega\vartheta, \qquad \Sigma = \Omega\sigma, \tag{3.14}$$

$$\boldsymbol{\Psi} = \Box_1\Box_4\boldsymbol{\lambda}, \quad \mathbf{H} = \Box_2\Box_3\boldsymbol{\chi}. \tag{3.15}$$

Performing operation $\Box_1\Box_4$ on Eq. (3.10), and taking into account Eq. $(3.15)_1$, we shall find that

$$\Box_1\Box_4\boldsymbol{\xi} = \operatorname{grad}(\Box_1\Box_4\vartheta) + \operatorname{rot}\boldsymbol{\Psi}. \tag{3.16}$$

Elimination of the expression $\operatorname{rot}\boldsymbol{\Psi}$ from Eqs. (3.16) and (3.5) leads to the relation

$$\mathbf{u} = \operatorname{grad}\Phi + \Box_1\Box_4\boldsymbol{\xi} - \operatorname{grad}\Box_1\Box_4\vartheta. \tag{3.17}$$

Taking into account Eq. $(3.14)_1$, relation (3.7), and formula $\operatorname{div}\boldsymbol{\xi} = \nabla^2\vartheta$ resulting from Eq. (3.10), we reduce Eq. (3.17) to the equation

$$\mathbf{u} = \Box_1\Box_4\boldsymbol{\xi} - \operatorname{grad}\operatorname{div}\Gamma\boldsymbol{\xi}, \tag{3.18}$$

we have obtained then the formula identical to that derived previously, Eq. (1.14). Performing operation $\Box_2\Box_3$ on Eq. (3.11), and taking into consideration formula $(3.15)_2$, one can obtain

$$\Box_2\Box_2\boldsymbol{\eta} = \operatorname{grad}(\Box_2\Box_3\sigma) + \operatorname{rot}\mathbf{H}. \tag{3.19}$$

Elimination $\operatorname{rot}\mathbf{H}$ from Eqs. (3.19), and (3.6), and applying Eqs. $(3.14)_2$ and (3.7) together with relation $\operatorname{div}\boldsymbol{\eta} = \nabla^2\sigma$, leads to the representation

$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\eta} - \operatorname{grad}\operatorname{div}\Theta\boldsymbol{\eta}, \tag{3.20}$$

consistent with formula (1.15). In this way a theorem on the completeness of solutions, when the stress functions $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are used, is proved. In an analogical way one can show that there exist such vector functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ that displacements $\mathbf{u}$ and rotations $\boldsymbol{\omega}$ are determined by Eqs. (1.18)–(1.19), and functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ satisfy the wave Eqs. (1.20).

4. Relations Between Potentials Φ, Σ, $\boldsymbol{\Psi}$, $\mathbf{H}$ and Stress Functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$

Let us consider the homogeneous wave Eq. (1.20) body forces and body couples are disregarded):

$$\Box_1\Omega\boldsymbol{\varphi} = 0, \quad \Box_3\Omega\boldsymbol{\psi} = 0. \tag{4.1}$$

A solution of the above equations in a view of T. Boggio theorem may be composed of two parts

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}' + \boldsymbol{\varphi}'', \quad \boldsymbol{\psi} = \boldsymbol{\psi}' + \boldsymbol{\psi}''. \tag{4.2}$$

Functions $\boldsymbol{\varphi}'$, $\boldsymbol{\varphi}''$, $\boldsymbol{\psi}'$, $\boldsymbol{\psi}''$ satisfy the following equations:

$$\Box_1\boldsymbol{\varphi}' = 0, \quad \Omega\boldsymbol{\varphi}'' = 0, \tag{4.3}$$

$$\Box_3\boldsymbol{\psi}' = 0, \quad \Omega\boldsymbol{\psi}'' = 0. \tag{4.4}$$

Putting Eqs. (4.2) into Eqs. (1.18) and (1.19), and applying Eqs. (4.3)-(4.4), we obtain the following representation:

$$\mathbf{u} = \Box_1\Box_4\boldsymbol{\varphi}''-\operatorname{grad}\operatorname{div}\Gamma(\boldsymbol{\varphi}'+\boldsymbol{\varphi}'')-2\alpha\operatorname{rot}\Box_3\boldsymbol{\psi}'', \tag{4.5}$$

$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\psi}''-\operatorname{grad}\operatorname{div}\Theta(\boldsymbol{\psi}'+\boldsymbol{\psi}'')-2\alpha\operatorname{rot}\Box_1\boldsymbol{\varphi}''. \tag{4.6}$$

Using the well known relation

$$\operatorname{rot}\operatorname{rot}\mathbf{U} = \operatorname{grad}\operatorname{div}\mathbf{U}-\nabla^2\mathbf{U},$$

and taking into account relations (3.7), we shall reduce the representations (4.5) and (4.6) to a form

$$\mathbf{u} = -\operatorname{grad}\operatorname{div}\Gamma\boldsymbol{\varphi}'-2\alpha\operatorname{rot}\Box_3\boldsymbol{\psi}'', \tag{4.7}$$

$$\boldsymbol{\omega} = -\operatorname{grad}\operatorname{div}\Theta\boldsymbol{\psi}'-2\alpha\operatorname{rot}\Box_1\boldsymbol{\varphi}''. \tag{4.8}$$

Comparing the Stokes-Helmholtz representation (1.3), (1.4) with a representation (4.7), (5.8), one obtains

$$\Phi = -\operatorname{div}\Gamma\boldsymbol{\varphi}', \quad \boldsymbol{\Psi} = -2\alpha\Box_3\boldsymbol{\psi}'', \tag{4.9}$$

$$\Sigma = -\operatorname{div}\Theta\boldsymbol{\psi}', \quad \mathbf{H} = -2\alpha\Box_1\boldsymbol{\varphi}'', \tag{4.10}$$

The above formulae are the sought relations between potentials Φ, Σ, $\boldsymbol{\Psi}$, $\mathbf{H}$ and stress functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$.

It is necessary to check whether the relations (4.9)-(4.10) satisfy the wave Eqs. (2.3). One can be easily convinced that it really occurs.

5. Stress Functions for Nonsymmetric Thermoelasticity

The basic differential equations of coupled thermoelasticity have a form [9]

$$\Box_2\mathbf{u}+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u}+2\alpha\operatorname{rot}\boldsymbol{\omega}+\mathbf{X} = \nu\operatorname{grad}\theta, \tag{5.1}$$

$$\Box_4\boldsymbol{\omega}+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\omega}+2\alpha\operatorname{rot}\mathbf{u}+\mathbf{Y} = 0, \tag{5.2}$$

$$D\theta-\eta\operatorname{div}\dot{\mathbf{u}} = -\frac{Q}{\varkappa}, \quad D = \nabla^2-\frac{1}{\varkappa}\partial_t. \tag{5.3}$$

Here $\theta = T-T_0$ is an increase of temperature, T denotes an absolute temperature and T_0 — a temperature of natural state in which the stresses and deformations are equal to zero. Q is connected with an intensity of heat source, and quantities $\varkappa$, ν, η are material constants. Equation (5.3) constitute an generalised equation of a heat conduction.

Here also, two main ways of separation of the system of Eqs. (5.1)-(5.3) may be distinguished. First way consist of the reduction of the Stokes-Helmholtz representation (1.3), (1.4); a second manner is to introduce two vector functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and scalar functions ϑ.

First method, with a decomposition of the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ according to formulae (1.3)-(1.4) and body forces and body couples according to formulae

$$\mathbf{X} = \varrho(\operatorname{grad}\Pi+\operatorname{rot}\boldsymbol{\Lambda}), \tag{5.4}$$

$$\mathbf{Y} = J(\operatorname{grad}\Xi+\operatorname{rot}\mathbf{P}), \tag{5.5}$$

leads to the system of the wave equations [9]:

$$\Box_1 \Phi + \varrho \Pi = \nu \theta, \tag{5.6}$$

$$\Box_3 \Sigma + J \Xi = 0, \tag{5.7}$$

$$\Box_2 \mathbf{\Psi} + 2\alpha \operatorname{rot} \mathbf{H} = 2\alpha J \operatorname{rot} \mathbf{P} - \varrho \Box_4 \mathbf{\Lambda}, \tag{5.8}$$

$$\Box_4 \mathbf{H} + 2\alpha \operatorname{rot} \mathbf{\Psi} = 2\alpha \varrho \operatorname{rot} \mathbf{\Lambda} - J \Box_2 \mathbf{P}. \tag{5.9}$$

To these equations it is necessary to add a heat conduction equation, which, in virtue of Eq. (1.3), will take a form:

$$D\theta - \eta \partial_t \nabla^2 \Phi = -\frac{Q}{\varkappa}. \tag{5.10}$$

Let us notice, that Eqs. (5.6), (5.10) and (5.8)–(5.9) are mutually coupled.

Eliminating from Eqs. (5.6) and (5.10), first a temperature, and second time the function Φ, we have

$$M\Phi = -\frac{\nu}{\varkappa} Q - \varrho D \Pi, \tag{5.11}$$

$$M\theta = -\eta \varrho \partial_t \nabla^2 \Pi - \frac{1}{\varkappa} \Box_1 Q, \tag{5.12}$$

where

$$M = \Box_1 D - \nu \eta \partial_t \nabla^2.$$

The second way of procedure in a separation of the system of Eqs. (1.1)–(1.2) was presented in a Sec. 1. Here, in a theory of thermoelasticity, this way will be generalized. Let us eliminate from Eqs. (5.1)–(5.2), first—the displacements $\mathbf{u}$, the second time—rotations $\boldsymbol{\omega}$. As a result we shall obtain

$$\Omega \mathbf{u} + \operatorname{grad} \operatorname{div} \Gamma \mathbf{u} + \Box_4 (\mathbf{X} - \nu \operatorname{grad} \theta) - 2\alpha \operatorname{rot} \mathbf{Y} = 0, \tag{5.13}$$

$$\Omega \boldsymbol{\omega} + \operatorname{grad} \operatorname{div} \Theta \boldsymbol{\omega} + \Box_4 \mathbf{Y} - 2\alpha \operatorname{rot} \mathbf{X} = 0, \tag{5.14}$$

$$D\theta - \eta \partial_t \operatorname{div} \mathbf{u} = -\frac{Q}{\varkappa}. \tag{5.15}$$

All notations here are the same as in Sec. 1.

Let us notice the fact that in Eq. (5.14) a temperature does not appear. Let us eliminate from Eqs. (5.13), (5.15) first the temperature θ, second time the displacements $\mathbf{u}$. In this way we obtain a new system of equations

$$D\Omega \mathbf{u} + \operatorname{grad} \operatorname{div} N \mathbf{u} + D \Box_4 \mathbf{X} - 2\alpha \operatorname{rot} D \mathbf{Y} + \frac{\nu}{\varkappa} \operatorname{grad} (\Box_4 Q) = 0, \tag{5.16}$$

$$\Omega \boldsymbol{\omega} + \operatorname{grad} \operatorname{div} \Theta \boldsymbol{\omega} + \Box_2 \mathbf{Y} - 2\alpha \operatorname{rot} \mathbf{X} = 0, \tag{5.17}$$

$$\Box_4 M\theta + \eta \partial_t \operatorname{div} \Box_4 \mathbf{X} + \frac{1}{\varkappa} \Box_1 \Box_4 Q = 0, \quad N = \Gamma D - \nu \eta \partial_t \Box_4. \tag{5.18}$$

Applying the same procedure as in Sec. 1, we shall assume the following representation for displacements $\mathbf{u}$ and rotations $\boldsymbol{\omega}$

$$\mathbf{u} = \Box_4 M\boldsymbol{\xi} - \operatorname{grad}\operatorname{div} N\boldsymbol{\xi}, \tag{5.19}$$

$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\eta} - \operatorname{grad}\operatorname{div}\Theta\boldsymbol{\eta}. \tag{5.20}$$

Inserting the above formulae into Eqs. (5.16), (15.17), one can obtain the following differential equations:

$$D\Omega\Box_4 M\boldsymbol{\xi} + D\Box_4 X - 2\alpha \operatorname{rot} D\mathbf{Y} + \frac{\nu}{\varkappa}\operatorname{grad}\Box_4 Q = 0, \tag{5.21}$$

$$\Omega\Box_2\Box_3\boldsymbol{\eta} + \Box_2\mathbf{Y} - 2\alpha\operatorname{rot} X = 0. \tag{5.22}$$

Here the relations were used

$$\Omega = \Box_2\Box_4 + 4\alpha^2\nabla^2 = \Box_1\Box_4 - \Gamma\nabla^2 = \Box_2\Box_3 - \Theta\nabla^2, \quad D\Omega + N\nabla^2 = \Box_4 M.$$

In Eqs. (5.21), (5.23) the differential operations on the body forces and body moments figure what is an inconvenience when equations are solving.

Let us enrich the representation (5.19), (5.20) adding a rotational term and a gradient of a certain scalar function τ.

$$\mathbf{u} = \Box_4 M\boldsymbol{\varphi} - \operatorname{grad}\operatorname{div} N\boldsymbol{\varphi} - 2\alpha\operatorname{rot}\Box_3\boldsymbol{\psi} + \nu\operatorname{grad}\tau, \tag{5.23}$$

$$\boldsymbol{\omega} = \Box_2\Box_3\boldsymbol{\psi} - \operatorname{grad}\operatorname{div}\Theta\boldsymbol{\psi} - 2\alpha\operatorname{rot} M\boldsymbol{\varphi}, \tag{5.24}$$

$$\theta = \eta\partial_t \operatorname{div}\Omega\boldsymbol{\varphi} + \Box_1\tau. \tag{5.25}$$

Inserting Eqs. (5.23)–(5.25) into the system of equations (5.16)–(5.18), one can obtain the following wave equations:

$$\Omega M\boldsymbol{\varphi} + \mathbf{X} = 0, \tag{5.26}$$

$$\Omega\Box_3\boldsymbol{\psi} + \mathbf{Y} = 0, \tag{5.27}$$

$$M\tau + \frac{Q}{\varkappa} = 0. \tag{5.28}$$

We have obtained the system of equations in which the body forces, body couples and heat sources appear separately. Let us notice, that in an infinite, elastic space, the assumption $\mathbf{X} = 0$ with homogeneous initial conditions for $\boldsymbol{\varphi}$, leads to a corolary that $\boldsymbol{\varphi} = 0$ in a whole space. The same corolary concerns the function $\boldsymbol{\psi}$ with $\mathbf{Y} = 0$ and τ with $Q = 0$.

Equations (5.26)–(5.28) are particularly usefull in a case of determining of the singular solutions in an infinite, micropolar space. Without any difficulty we shall prove the following theorem:

THEOREM. *Let the functions* $\mathbf{u}(\mathbf{x}, t)$, $\boldsymbol{\omega}(\mathbf{x}, t)$ *and temperature* $\theta(\mathbf{x}, t)$ *be particular solutions of Eqs.* (5.1)–(5.3) *in a region of space* V *for* $t_1 < t < t_2$. *Then, there exist such scalar functions* φ, Σ and vector functions $\boldsymbol{\Psi}$, $\mathbf{H}$, that displacements $\mathbf{u}(\mathbf{x}, t)$ and rotations $\boldsymbol{\omega}(\mathbf{x}, t)$ *may be expressed by relations*

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{rot}\boldsymbol{\Psi}, \quad \operatorname{div}\boldsymbol{\Psi} = 0, \tag{5.29}$$

$$\boldsymbol{\omega} = \operatorname{grad}\Sigma + \operatorname{rot}\mathbf{H}, \quad \operatorname{div}\mathbf{H} = 0. \tag{5.30}$$

Functions Φ, Σ, $\mathbf{\Psi}$, $\mathbf{H}$ *should satisfy the wave equations*

$$\Box_1\Phi = \nu\theta, \quad \Box_3\Sigma = 0, \quad \Omega\mathbf{\Psi} = 0, \quad \Omega\mathbf{H} = 0, \tag{5.31}$$

The proof of this theorem is a small modification of the proof presented in a Sec. 2. Namely, the formula $(2.9)_1$ will take now a form

$$\nabla^2(\Box_1 U - \nu\theta) = 0, \quad \nabla^2\Box_3 T = 0,$$

what leads to a modification of Eq. (2.16). Thus we have

$$\Box_1 U_1 = \nu\theta, \quad \Box_3 T_1 = 0.$$

In order to check whether the function Φ satisfies the wave Eq. $(5.31)_1$, we start with a formula

$$\Box_1 U_1 = \nu\theta, \quad \text{or} \quad \Box_1(\Phi - U_2 + \alpha(t)) = \nu\theta,$$

whence we obtain immediately the wave Eq. $(5.31)_1$.

Much more complicated is a proof of a theorem on the completeness of the stress functions $\boldsymbol{\xi}$, $\boldsymbol{\eta}$. This theorem has a following form.

THEOREM. *Let* $\mathbf{u}(\mathbf{x}, t)$ *and* $\boldsymbol{\omega}(\mathbf{x}, t)$ *be particular solutions of Eqs.* (5.1)–(5.3) *in a region of space* V *for* $t_1 < t < t_2$. *Then, there exist such vector functions* $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, *that displacements* $\mathbf{u}$ *and rotations* $\boldsymbol{\omega}$ *are determined by representations*

$$\begin{aligned} \mathbf{u} &= M\Box_4\boldsymbol{\xi} - \operatorname{grad}\operatorname{div} N\boldsymbol{\xi}, \\ \boldsymbol{\omega} &= \Box_2\Box_4\boldsymbol{\eta} - \operatorname{grad}\operatorname{div}\Theta\boldsymbol{\eta}, \end{aligned} \tag{5.32}$$

and functions $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ *satisfy the wave equation*

$$\begin{aligned} D\Omega\Box_4 M\boldsymbol{\xi} &= 2\alpha\operatorname{rot} D\mathbf{Y} - D\Box_4\mathbf{X} + \frac{\nu}{\varkappa}\operatorname{grad}\Box_4 Q, \\ \Omega\Box_2\Box_3\boldsymbol{\eta} &= 2\alpha\operatorname{rot}\mathbf{X} - \Box_2\mathbf{Y}. \end{aligned} \tag{5.33}$$

In a proof of this theorem we start with a representation of displacements $\mathbf{u}$ and rotations $\boldsymbol{\omega}$ by potentials Φ, Σ, $\mathbf{\Psi}$, $\mathbf{H}$ according to formulae (1.3)–(1.4). Putting Eqs. (1.3)–(1.4) into Eqs. (5.16)–(5.17) and taking into account Eq. (5.28), we have

$$\begin{aligned} \operatorname{grad}[M\Box_4(\Phi - \nu\tau)] + \operatorname{rot}(D\Omega\mathbf{\Psi}) + D\Box_4\mathbf{X} - 2\alpha\operatorname{rot} D\mathbf{Y} &= 0, \\ \operatorname{grad}(\Box_2\Box_3\Sigma) + \operatorname{rot}(\Omega\mathbf{H}) + \Box_2\mathbf{Y} - 2\alpha\operatorname{rot}\mathbf{X} &= 0. \end{aligned} \tag{5.34}$$

Let us express the vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ by their potential and solenoidal parts, respectively,

$$\begin{aligned} \boldsymbol{\xi} &= \operatorname{grad}\vartheta + \operatorname{rot}\boldsymbol{\lambda}, \quad \operatorname{div}\boldsymbol{\lambda} = 0, \\ \boldsymbol{\eta} &= \operatorname{grad}\sigma + \operatorname{rot}\boldsymbol{\chi}, \quad \operatorname{div}\boldsymbol{\chi} = 0. \end{aligned} \tag{5.35}$$

Inserting Eq. (5.35) into Eqs. (5.21)–(5.22), we obtain

$$\begin{aligned} \operatorname{grad}[M\Box_4(\Omega D\vartheta - \nu\tau)] + \operatorname{rot}(D\Omega\Box_4 M\boldsymbol{\lambda}) - 2\alpha\operatorname{rot} D\mathbf{Y} + \Box_4 D\mathbf{X} &= 0, \\ \operatorname{grad}(\Box_2\Box_3\Omega\sigma) + \operatorname{rot}(\Box_2\Box_3\Omega\boldsymbol{\chi}) - 2\alpha\operatorname{rot}\mathbf{X} + \Box_2\mathbf{Y} &= 0. \end{aligned} \tag{5.36}$$

Comparison of Eqs. (5.34) and (5.36) gives

$$D\Omega\vartheta = \Phi, \quad \Omega\sigma = \Sigma, \quad M\Box_4\boldsymbol{\lambda} = \mathbf{\Psi}, \quad \Box_2\Box_3\boldsymbol{\chi} = \mathbf{H}. \tag{5.37}$$

Let us perform the operation $M\square_4$ on the function $\boldsymbol{\xi}$ from the formula $(5.35)_1$ and apply the relation $(5.37)_3$. As a result, we shall obtain the relation

$$\square_4 M\boldsymbol{\xi} = \text{grad}(\square_4 M\vartheta)+\text{rot}\,\boldsymbol{\Psi}. \tag{5.38}$$

Eliminating the function $\boldsymbol{\Psi}$ from Eqs. (1.3) and (5.38), and taking into account relations

$$\Phi = D\Omega\vartheta, \quad \square_4 M = D\Omega+N\nabla^2, \quad \text{div}\boldsymbol{\xi} = \nabla^2\vartheta,$$

we obtain

$$\mathbf{u} = M\square_4\boldsymbol{\xi}-\text{grad}\,\text{div}\boldsymbol{\xi}. \tag{5.39}$$

Analogically, performing the operation $\square_2\square_3$ on the function $\boldsymbol{\eta}$, and using formulae $(5.37)_2$, $(5.37)_4$, (1.4) together with relation $\text{div}\boldsymbol{\eta} = \nabla^2\sigma$, one gets

$$\boldsymbol{\omega} = \square_2\square_4\boldsymbol{\eta}-\text{grad}\,\text{div}\,\Theta\boldsymbol{\eta}. \tag{5.40}$$

In this way we have obtained the representations (5.19)–(5.20); thus a proof of the completeness for the stress functions $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ is termined. In a completely analogous way one can demonstrate the completeness of the stress functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, τ on the basis of representations (5.23)–(5.25).

6. Relations Between Potentials and Stress Functions in Problems of Thermoelasticity

Let us consider the homogeneous equations of thermoelasticity (5.26)–(5.28).

$$M\Omega\boldsymbol{\varphi} = 0, \quad \square_3\Omega\boldsymbol{\psi} = 0, \quad M\tau = 0. \tag{6.1}$$

A solution of these equations may be presented on a basis of T. Boggio theorem [8] in a form

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}'+\boldsymbol{\varphi}'', \quad \boldsymbol{\psi} = \boldsymbol{\psi}'+\boldsymbol{\psi}'', \tag{6.2}$$

where "primed" functions satisfy the equation

$$\begin{aligned} M\boldsymbol{\varphi}' &= 0, \quad \Omega\boldsymbol{\varphi}'' = 0,\\ \square_3\boldsymbol{\psi}' &= 0, \quad \Omega\boldsymbol{\psi}'' = 0. \end{aligned} \tag{6.3}$$

Substituting Eq. (6.3) into Eqs. (5.23)–(5.25), we obtain

$$\begin{aligned} \mathbf{u} &= \square_4 M\boldsymbol{\varphi}''-\text{grad}\,\text{div}N(\boldsymbol{\varphi}'+\boldsymbol{\varphi}'')-2\alpha\,\text{rot}\,\square_3\boldsymbol{\psi}''+\nu\,\text{grad}\,\tau,\\ \boldsymbol{\omega} &= \square_2\square_3\boldsymbol{\psi}''-\text{grad}\,\text{div}\,\Theta(\boldsymbol{\psi}'+\boldsymbol{\psi}'')-2\alpha\,\text{rot}\,M\boldsymbol{\varphi}'',\\ \theta &= \eta\partial_t\,\text{div}\Omega\boldsymbol{\varphi}'+\square_1\tau. \end{aligned} \tag{6.4}$$

Using formula (3.7) and equations

$$(\square_4 M-N\nabla^2)\boldsymbol{\varphi}'' = D\Omega\boldsymbol{\varphi}'' = 0, \quad (\square_2\square_3-\nabla^2\Theta)\boldsymbol{\psi}'' = \Omega\boldsymbol{\psi}'' = 0,$$
$$\text{rot}\,\text{rot}\,\mathbf{U} = \text{grad}\,\text{div}\,\mathbf{U}-\nabla^2\mathbf{U},$$

we shall present the relations (6.4) in a form

$$\begin{aligned} \mathbf{u} &= -\text{grad}[\text{div}(N\boldsymbol{\varphi}')-\nu\tau]-\text{rot}[\text{rot}(N\boldsymbol{\varphi}'')+2\alpha\square_3\boldsymbol{\psi}''],\\ \boldsymbol{\omega} &= -\text{grad}\,\text{div}\,\Theta\boldsymbol{\psi}'-\text{rot}[\text{rot}(\Theta\boldsymbol{\psi}'')+2\alpha M\boldsymbol{\varphi}'']. \end{aligned} \tag{6.5}$$

By comparison of the above relations with representation (1.3)–(1.4), we obtain the following relations between potentials Φ, Σ, $\mathbf{\Psi}$, $\mathbf{H}$ and stress functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, τ:

$$\Phi = -\mathrm{div}(N\boldsymbol{\varphi}')-\nu\tau, \quad \mathbf{\Psi} = -\mathrm{rot}(N\boldsymbol{\varphi}'')-2\alpha\Box_3\boldsymbol{\psi}'',$$
$$\Sigma = -\mathrm{div}(\Theta\boldsymbol{\psi}'), \quad \mathbf{H} = \mathrm{rot}(\Theta\boldsymbol{\psi}'')-2\alpha M\boldsymbol{\varphi}''. \tag{6.6}$$

It remains to check whether the functions Φ, Σ, $\mathbf{\Psi}$, $\mathbf{H}$, expressed by the functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, τ, satisfy the homogeneous wave equations (5.6)–(5.7) and relation

$$\Omega\mathbf{\Psi} = 0, \quad \Omega\mathbf{H} = 0, \tag{6.7}$$

which is obtained from homogeneous Eqs. (5.8) and (5.9).

Let us perform an operation $\Box_1$ on Eq. $(6.6)_1$. Then

$$\Box_1\Phi = -\mathrm{div}\,\Box_1 N\boldsymbol{\varphi}'+\nu\Box_1\tau.$$

Taking into account Eq. (5.25), we obtain

$$\Box_1\Phi = -\mathrm{div}\,\Box_1 N\boldsymbol{\varphi}'+\nu(\theta-\eta\partial_t\mathrm{div}\,\Omega\boldsymbol{\varphi}').$$

Since

$$(\Box_1 N+\eta\partial_t\Omega)\boldsymbol{\varphi}' = M\boldsymbol{\varphi}' = 0,$$

then the homogeneous wave equation

$$\Box_1\Phi = \nu\theta \tag{6.8}$$

will be satisfied.

The remaining equations are also satisfied in virtue of Eq. (6.3). Hence, we have by turn

$$\Box_3\Sigma = -\mathrm{div}\,\Theta\Box_3\boldsymbol{\psi}' = 0,$$
$$\Omega\mathbf{\Psi} = -\mathrm{rot}(N\Omega\boldsymbol{\varphi}'')-2\alpha\Box_3\Omega\boldsymbol{\psi}'' = 0, \tag{6.9}$$
$$\Omega\mathbf{H} = -\mathrm{rot}(\Theta\Omega\boldsymbol{\psi}'')-2\alpha M\Omega\boldsymbol{\varphi}'' = 0.$$

We have considered here the coupled equations of thermoelasticity. A significant approximation we shall obtain within a frame of so named theory of thermal stresses, in which a term $\eta\partial_t\mathrm{div}\,\mathbf{u}$ — coupling a heat conduction equation with equations of motion. In this case a temperature is determined from parabolic equation and as a known function is introduced to Eq. (1.1).

In this way a thermoelastic problem is reduced to the elastic one. Instead of the body forces $\mathbf{X}$ it is necessary to apply the reduced body forces $\mathbf{X}^* = \mathbf{X}-\nu\,\mathrm{grad}\,\theta$.

References

1. B. A. Пальмов, *Основные уравнения теории несимметричной упругости*, Прикл. Мат. Мех. **28**, 1964, 401.

2. B. Galerkin, *Contribution à la solution générale du problème de la théorie de l'élasticité dans le cas de trois dimensions*, C. R. Acad. Sci., Paris, **190** (1930), 1047.

3. M. Iacovache, *O extindere a methodei lui Galerkin pentru sistemul ecuaţiilor elasticităţii*, Bul. stint. Acad. Rep. Pop. Române, Ser. A. **1** (1949), 593.

4. N. Sandru, *On some problems of the linear theory of the asymmetric elasticity*, Int. J. Engin. Sci., **4** (1966)

5. Gr. C. Moisil, *Aspura sistemelor de ecuaţii cu derivate parţiale lineare si cu coeficienţi constanti*, Bul., ştiint. Acad. Rep. Pop. Române, Ser. A, **1** (1949), 341.
6. P. Duhem, *Sur la généralisation d'un théorème de Clebsch*, J. Math. Pures et Appl., **6** (1900), 215.
7. E. Sternberg, *On the integration of the equations of motion in the classical theory of elasticity*, Arch. Rational. Mech. Anal., **6** (1959), 34.
8. T. Boggio, *Sull'integratione di alcune equationi lineari alle derivatti parziali*, An. Mat., Ser. III, **8** (1903), 181.
9. W. Nowacki, *Couple-stresses in the theory of thermoelasticity*, Proceed. of the IUTAM Symposia Vienna, June 22–28, 1966, Springer Verlag, Wien-New York 1968.

Streszczenie

O ZUPEŁNOŚCI POTENCJAŁÓW W MIKROPOLARNEJ SPRĘŻYSTOŚCI

W pracy wyprowadzono funkcje naprężeń (jako uogólnienie funkcji Galerkina) dla zagadnień elastokinetyki i termosprężystości w ośrodku sprężystym mikropolarnym. Podano dowody twierdzeń o zupełności dla potencjałów oraz dla funkcji naprężeń. Wreszcie podano zależności między potencjałami a funkcjami naprężeń.

Резюме

О ПОЛНОТЕ ПОТЕНЦИАЛОВ В МИКРОПОЛЯРНОЙ УПРУГОСТИ

В работе выводятся функции напряжений (в качестве функции Галеркина) для задач упругокинетики и термоупругости в упругомикрополярной среде. Даются доказательства теорем о полноте для потенциалов и для функций напряжений. Кроме того, даются зависимости между потенциалами и функциами напряжений.

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