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MIXED BOUNDARY-VALUE PROBLEMS IN HEAT CONDUCTION

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1. Introduction

Ref. [1], by the same author, is devoted to the problem of mixed boundary conditions for a stationary heat flow in a solid. The method presented there will now be generalized to non-homogeneous mixed boundary conditions and to problems of non-stationary heat flow. The method will be described in a manner somewhat different from [1], different fundamental systems being used. A new solution will also be given for a body with slits.

Let us consider a simply connected body B, bounded by the surface S. Let this surface be composed of three regular surfaces S_1, S_2, S_3 with common edges α and β (Fig. 1). Let time-variable heat sources $W(P, t)$ $P \in B$ be located in the body, which

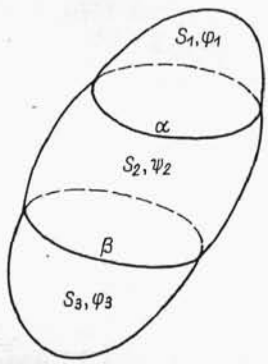

FIG. 1.

is heated on the surfaces S_i $(i = 1, 2, 3)$. The temperature field thus generated $T(P, t)$ is described by the heat equation

$$(1.1) \qquad \varkappa \nabla^2 T(P, t) - \dot{T}(P, t) = -M(P, t), \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In this equation, $\varkappa = \lambda/\varrho c$ is a material constant, where λ is the coefficient of heat conduction ϱ—the density and c — the specific heat. The function $M(P, t)$ expresses the intensity of the heat sources. We have $M(P, t) = \dfrac{1}{\varrho c} W(P, t)$, where $W(P, t)$ is the quantity of heat produced per unit volume and time and $\dot{T}$ is the time derivative of the temperature $\dot{T} = \partial T/\partial t$. It is assumed that $T(P, t)$ satisfies the initial condition

$$T(P, 0) = f(P), \quad P \in B, \tag{1.2}$$

and the boundary conditions

$$\begin{aligned} T(R_1, t) &= \varphi_1(R_1, t) \quad \text{on the surface} \quad S_1, \ R_1 \in S_1, \\ \frac{\partial T(R_2, t)}{\partial n} &= \psi_2(R_2, t) \quad \text{on the surface} \quad S_2, \ R_2 \in S_2, \\ T(R_3, t) &= \varphi_3(R_3, t) \quad \text{on the surface} \quad S_3, \ R_3 \in S_3. \end{aligned} \tag{1.3}$$

Let us denote, in general,

$$T(R_i, t) = \varphi_i(R_i, t), \qquad \frac{\partial T(R_i, t)}{\partial n} = \psi_i(R_i, t), \quad i = 1, 2, 3.$$

Let us observe that the functions φ_1, ψ_2, φ_3 prescribed on the surfaces S_1, S_2, S_3 are known, while ψ_1, φ_2 and ψ_3 are unknown functions on the same surfaces.

2. First Solution Method

Let us consider a "fundamental system" in which Green's function $G(P, Q, t)$ of the problem stated in the first section will be determined. Let us assume that the surfaces S_1 and S_2 are thermally insulated, the surface S_3 being kept at zero temperature. Let us observe also that it is impossible to have an insulation on the surface S_3, because then, heat exchange across the surface S being impossible, Green's functions would have no sense. Let us determine Green's function $G(P, Q, t)$ satisfying the heat equation

$$\varkappa \nabla^2 G(P, Q, t) - \dot{G}(P, Q, t) = -\delta(P-Q)\delta(t), \quad P, Q \in B, \tag{2.1}$$

with the homogeneous initial condition

$$G(P, Q, 0) = 0, \tag{2.2}$$

and the homogeneous boundary conditions

$$\begin{aligned} \frac{\partial G(R_1, Q, t)}{\partial n} &= 0 \quad \text{on the surface} \quad S_1, \ R_1, \in S_1, \\ \frac{\partial G(R_2, Q, t)}{\partial n} &= 0 \quad \text{on the surface} \quad S_2, \ R_2 \in S_2, \\ G(R_3, Q, t) &= 0 \quad \text{on the surface} \quad S_3, \ R_3 \in S_3. \end{aligned} \tag{2.3}$$

On the right-hand side of (2.1), Dirac function is involved to express the action of an instantaneous point heat at Q, where

$$\int_0^t d\tau \iiint_B \delta(P-Q)\delta(\tau) dB_P = 1.$$

Let us perform on Eqs. (1.1) and the boundary conditions (1.3), one-sided Laplace transformation as defined by equation

$$\bar{F}(P,p)=\int\limits_0^\infty F(P,t)e^{-pt}\,dt,\qquad p>0. \tag{2.4}$$

It is assumed that the action of the heat sources and the surface heating on S starts at the moment $t=0^+$. As a result, we obtain Eq. (1) in the transformed form

$$\varkappa\nabla^2\bar{T}(P,p)-[p\bar{T}(P,p)-T(P,0)]=-\bar{M}(P,p),\qquad T(P,0)=f(P). \tag{2.5}$$

The Laplace transformation performed on the boundary conditions (1.3) yields

$$\begin{aligned}\bar{T}(R_1,p)&=\bar{\varphi}_1(R_1,p),\qquad R_1\in S_1,\\ \frac{\partial\bar{T}(R_2,p)}{\partial n}&=\bar{\psi}_1(R_2,p),\qquad R_2\in S_2,\\ \bar{T}(R_3,p)&=\bar{\varphi}_3(R_3,p),\qquad R_3\in S_3.\end{aligned} \tag{2.6}$$

For Eq. (2.1) and the boundary conditions (2.3) we proceed in a similar way. We obtain

$$\varkappa\nabla^2\bar{G}(P,Q,p)-p\bar{G}(P,Q,p)=-\delta(P-Q), \tag{2.7}$$

$$\begin{aligned}\frac{\partial\bar{G}(R_1,Q,p)}{\partial n}&=0,\qquad R_1\in S_1,\\ \frac{\partial\bar{G}(R_2,Q,p)}{\partial n}&=0,\qquad R_2\in S_2,\\ \bar{G}(R_3,Q,p)&=0,\qquad R_3\in S_3.\end{aligned} \tag{2.8}$$

Let us make use of Green's formula

$$\iiint\limits_B(\bar{G}\nabla^2\bar{T}-\bar{T}\nabla^2\bar{G})\,dB=\iint\limits_S\left(\bar{G}\frac{\partial\bar{T}}{\partial n}-\bar{T}\frac{\partial\bar{G}}{\partial n}\right)dS, \tag{2.9}$$

by substituting appropriate values from Eqs. (2.5)–(2.8). Bearing in mind that

$$\iiint\limits_B\bar{T}(P,p)\delta(P-Q)\,dB_P=\bar{T}(Q,p),$$

we obtain from (2.9)

$$\begin{aligned}\bar{T}(Q,p)=&\iiint\limits_B f(P)\bar{G}(P,Q,p)\,dB_P+\iiint\limits_B\bar{G}(P,Q,p)\bar{M}(P,p)\,dB_P\\ &+\varkappa\iint\limits_{S_1}\bar{\psi}_1(R_1,p)\bar{G}(R_1,Q,p)\,dS_{R_1}+\varkappa\iint\limits_{S_2}\bar{\psi}_2(R_2,p)\bar{G}(R_2,Q,p)\,dS_{R_2}\\ &-\varkappa\iint\limits_{S_3}\bar{\varphi}_3(R_3,p)\frac{\partial\bar{G}(R_3,Q,p)}{\partial n}\,dS.\end{aligned} \tag{2.10}$$

Let us observe that the integrals on the right-hand side of this equation can be determined, except for $\iint\limits_{S_1} \psi_1 \bar{G} dS_{R_1}$.

For, the functions ψ_2, φ_3, f and M are prescribed and G has already been found from (2.1). Equation (2.13) may be represented in the form

$$\overline{T}(Q,p) = \overline{T}_0(Q,p) + \varkappa \iint\limits_{S_1} \overline{\psi}_1(R_1,p)\overline{G}(R_1,Q,p)dS_{R_1} \tag{2.11}$$

because in the fundamental system described the temperature $T_0(Q,t)$ can be treated as a solution of the equation

$$\varkappa \nabla^2 T_0(P,t) - \dot{T}(P,t) = -M(P,t), \quad P \in B, \tag{2.12}$$

with the initial condition $T(P,0) = f(P)$ and the boundary conditions

$$\begin{gathered} \frac{\partial T_0(R_1,t)}{\partial n} = 0,\ R_1 \in S_1, \qquad \frac{\partial T_0(R_2,t)}{\partial n} = \psi_2(R_2,t), \qquad R_2 \in S_2, \\ T_0(R_3,t) = \varphi_3(R_3,t), \quad R_3 \in S_3, \end{gathered} \tag{2.13}$$

If one-sided Laplace transformation is performed on Eq. (2.12) and the boundary conditions (2.13) and Green's formula

$$\iiint (\overline{G}\nabla^2\overline{T}_0 - \overline{T}_0\nabla^2\overline{G})dB = \iint\limits_S \left(\overline{G}\frac{\partial \overline{T}_0}{\partial n} - \overline{T}_0\frac{\partial \overline{G}}{\partial n}\right)dS, \tag{2.14}$$

then, after some simple transformations, we obtain

$$\begin{aligned} \overline{T}_0(Q,p) = & \iiint\limits_B f(P)\overline{G}(P,Q,p)dB_P + \iiint\limits_B \overline{M}(P,p)\overline{G}(P,Q,p)dB_P \\ & + \varkappa \iint\limits_S \overline{\psi}_2(R_2,p)\overline{G}(R_2,Q,p)dS_{R_2} - \varkappa \iint\limits_{S_3} \overline{\varphi}_3(R_3,p)\frac{\partial \overline{G}(R_3,Q,p)}{\partial n}dS_{R_3}. \end{aligned} \tag{2.15}$$

Performing on the expression (2.15) the inverse Laplace transformation, we have

$$\begin{aligned} T_0(Q,t) = & \iiint\limits_B f(P)G(P,Q,t)dB_P + \int\limits_0^t d\tau \iiint\limits_B M(P,\tau)G(P,Q,t-\tau)dB_P \\ & + \varkappa \int\limits_0^t d\tau \iint\limits_{S_2} \psi_2(R_2,\tau)G(R_2,Q,t-\tau)dS_{R_2} \\ & - \varkappa \int\limits_0^t d\tau \iint\limits_{S_3} \varphi_3(R_3,\tau)\frac{\partial G(R_3,Q,t-\tau)}{\partial n}dS_{R_3}. \end{aligned} \tag{2.16}$$

Let us consider the expression (2.11) on which the inverse Laplace transformation will be performed

$$T(Q,t) = T_0(Q,t) + \varkappa \int\limits_0^t d\tau \iint\limits_{S_1} \psi_1(R_1,\tau)G(R_1,Q,t-\tau)dS_{R_1}, \tag{2.17}$$

ψ_1, (R_1, τ) is an unknown function on the surface S_1. Let us make the point $Q, \in B$ tend to the point $R_1', \in S_1$, on the surface S_1. Remembering that $T(R_1', t) = \varphi_1(R_1', t)$ is the boundary condition on S_1, we obtain from (2.17)

$$(2.18) \qquad \varphi_1(R_1', t) = T_0(R_1', t) + \varkappa \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau) G(R_1, R_1', t-\tau) dS_{R_1}.$$

In this integral equation, the only unknown function is ψ_1 (R_1, t). On determining it, we obtain the temperature $T(Q, t)$ from (2.17). Let us observe that the Green's function $G(P, Q, t)$ may be expressed by another Green's function defined in the following way:

Let us consider the function $K(P, R_1', t)$ which satisfies the heat equation

$$(2.19) \qquad \varkappa \nabla^2 K(P, R_1', t) - \dot{K}(P, R_1', t) = 0,$$

with the initial condition $K(P, R_1', t) = 0$ and the boundary conditions

$$(2.20) \qquad \begin{aligned} &\frac{\partial K(R_1, R_1', t)}{\partial n} = \delta(R_1 - R_1')\delta(t), && R_1, R_1' \in S_1, \\ &\frac{\partial K(R_2, R_1', t)}{\partial n} = 0, && R_2 \in S_2, \\ &K(R_3, R_1', t) = 0, && R_3 \in S_3. \end{aligned}$$

The first of the conditions (2.20) tells us that a concentrated and instantaneous heat flow takes place at the point $R_1', \in S_1$ of thermal insulation, and

$$\int_0^t d\tau \iint_{S_1} \delta(R_1 - R_1')\delta(\tau) dS_{R_1} = 1.$$

If now Green's formula (2.9) is applied to the functions $\overline{K}$ and $\overline{G}$, we obtain the relation

$$(2.21) \qquad \iiint_B \overline{K}(P, R_1', p)\delta(P-Q) dB_P = \varkappa \iint_{S_1} \overline{G}(R_1, Q, p)\delta(R_1 - R_1') dS_{R_1}.$$

Hence

$$(2.22) \qquad K(Q, R_1', t) = \varkappa G(R_1', Q, t).$$

Making use of this relation in (2.18), we obtain

$$(2.23) \qquad \varphi_1(R_1', t) = T_0(R_1', t) + \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau) K(R_1, R_1', t-\tau) dS_{R_1}.$$

Let us consider the particular case in which the temperature field varies harmonically with time. Then, with

$$(2.24) \qquad M(P, t) = L(P)e^{i\omega t}, \quad T(P, t) = \theta(P, \omega)e^{i\omega t}$$

Eq. (1.1) takes the form

$$(2.25) \qquad \varkappa \nabla^2 \theta(P, \omega) - i\omega\theta(P, \omega) = -L(P).$$

On introducing the notations

$$\varphi_1(R_1, t) = \Phi_1(R_1)e^{i\omega t}, \quad \psi_2(R_2, t) = \Psi_2(R_2)e^{i\omega t}, \quad \varphi_3(R_3, t) = \Phi_3(R_3)e^{i\omega t},$$

the boundary conditions (1.3) can be written thus:

$$\begin{aligned} \theta(R_1, \omega) &= \Phi_1(R_1), \quad R_1 \in S_1, \\ \frac{\partial\theta(R_2, \omega)}{\partial n} &= \Psi_2(R_2), \quad R_2 \in S_2, \\ \theta(R_3, \omega) &= \Phi_3(R_3), \quad R_3 \in S_3. \end{aligned} \tag{2.26}$$

Let us denote Green's function by

$$G(P, Q, t) = \Gamma(P, Q, \omega)e^{i\omega t},$$

and solve (2.1) with the boundary conditions (2.3). Similarly, let us solve (2.12) with the boundary conditions (2.13). We introduce the notation $T_0(P, t) = \theta_0(P, \omega)e^{i\omega t}$. Making use now of the Green's formula (2.9) for the functions Γ, and θ, we find the following expression for the amplitude $\theta(Q, \omega)$.

$$\theta(Q, \omega) = \theta_0(Q, \omega) + \varkappa \iint_{S_1} \Psi_1(R_1)\Gamma(R_1, Q, \omega)\,dS_{R_1}, \tag{2.27}$$

where

$$\theta_0(Q, \omega) = \iiint_B L(P)\Gamma(P, Q, \omega)\,dB_P + \varkappa \iint_{S_2} \Psi_2(R_2)\Gamma(R_2, Q, \omega)\,dS_{R_2} - \iint_{S_3} \Phi_3(R_3)\frac{\partial\Gamma(R_3, Q, \omega)}{\partial n}\,dS_{R_3}. \tag{2.28}$$

The function $\Psi_1(R_1)$ will be determined from the boundary condition

$$\theta_0(R_1', \omega) + \varkappa \iint_{S_1} \Psi_1(R_1)\Gamma(R_1, R_1', \omega)\,dS_{R_1} = \Phi_1(R_1'), \quad R_1, R_1' \in S_1. \tag{2.29}$$

The function $\Psi_1(R_1)$ having now been determined, we can find $\theta_0(Q, \omega)$ from the integral expression (2.27). It should be observed that a solution analogous to (2.27)–(2.29) can be obtained for the Helmholtz equation

$$\nabla^2 F + \lambda^2 F = -L, \tag{2.30}$$

with the boundary conditions (2.26) for the function F. In the solution of Eq. (2.25), we should only replace θ with F and ω with $+i\lambda^2$.

Returning now to the problem of heat conduction, let us observe that for $\omega \to 0$ — that is, for a heat wave with infinite period—the problem tends to that of stationary heat flow. Equation (1.1) becomes Poisson's equation. Denoting the temperature in this state by $T(P)$, the intensity of heat sources by $M(P)$, Green's function by $G(P, Q)$ etc., we obtain for the temperature equation

$$T(Q) = T_0(Q) + \varkappa \iint_{S_1} \psi_1(R_1)G(R_1, Q)\,dS_{R_1}, \quad R_1 \in S_1, \tag{2.31}$$

where

$$(2.32)\qquad T_0(Q) = \iiint_B M(P)G(P,Q)\,dB_P + \iint_{S_2} \psi_2(R_2)G(R_2,Q)\,dS_{R_1}$$
$$- \varkappa \iint_{S_3} \varphi_3(R_3)\frac{\partial G(R_3,Q)}{\partial n}\,dS_{R_3}.$$

The unknown function $\psi_2(R_2)$ will be obtained from the boundary condition $\varphi_1(R_1') = T_0'(R_1')$, $R_1 \in S_1$

$$(2.33)\qquad T_0(R_1') + \varkappa \iint_{S_1} \psi_1(R_1)G(R_1\, R_1',)\,dS_{R_1} = \varphi_1\,(R_1').$$

The above method for determining the temperature distribution with mixed boundary conditions is useful if Green's function $G(P, Q, t)$ can be obtained in the fundamental system assumed. For simple bodies such as a semi-infinite body, a slab, a sphere, a finite or semi-infinite cylinder, the form of the function $G(P, Q, t)$ is known. Let us consider as an example the case of a finite cylinder, in which S_1 will denote the lower bottom, S_3 — the upper bottom and S_2 — the lateral surface.

The fundamental system is the same cylinder, thermally insulated on S_1 and S_2 and kept at zero temperature on S_3. In this fundamental system, Green's function $G(P, Q, t)$ can be obtained relatively easily, as also the temperature $T_0(Q, t)$.

The unknown function $\psi_1(R_1, t)$ on the surface S_1 will be determined from (2.33). In the particular case of mixed boundary conditions under consideration, the solution presents no major difficulties.

It should be observed in addition that the problem treated here can be solved in a simpler way by direct integration of Eq. (1.1), the integral Eq. (2.23) not being considered.

Much greater difficulties are encountered for solving the next problem, concerning the semi-infinite cylinder (Fig. 2) with mixed and discontinuous boundary conditions on the lateral surface.

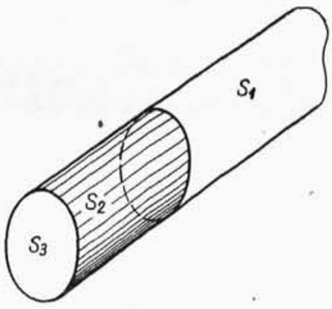

FIG. 2.

Let the temperature of the part S_1 of the lateral surface be zero, the part S_2 of that surface being insulated and let the temperature of S_3 be $\varphi_3\,(R_3, t)$, $R_3 \in S_3$. Discontinuity of boundary conditions occurs on the regular lateral surface $S_1 + S_2$. As a fundamental system, we assume a semi-infinite cylinder thermally insulated over the entire area S_1+S_2. In this fundamental system, we find easily the functions $G(P, Q, t)$ and $T_0(Q, t)$. The unknown function $\psi_1(R_1, t)$ can be determined from the integral Eq. (2.23) only. However, an accurate solution of this integral equation is connected with serious mathematical difficulties, which can be overcome only in a few simple cases — of stationary flow in particular. In more complex cases, we must have recourse to approximate solutions of Eq. (2.23).

A similar case of discontinuous boundary conditions is that of the semi-infinite cylinder of Fig. 3. The determination of Green's function $G(P, Q, t)$ and $T(Q, t)$

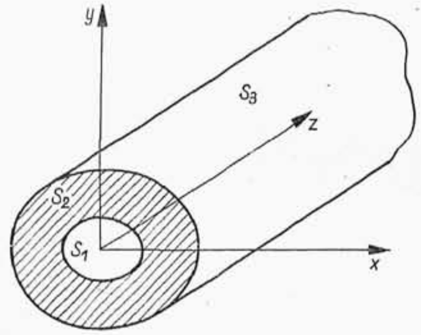

FIG. 3.

in the fundamental system (thermal insulation over S_1 and S_2 and zero temperature over S_3) is not difficult in this case either. Difficulties are first encountered in the solution of the integral Eq. (2.23).

Let us consider two semi-infinite cylinders joined in the $z = 0$ plane. A sectional view of this system is represented by Fig. 4. The surfaces S_2^{I} and S_2^{II} are thermally

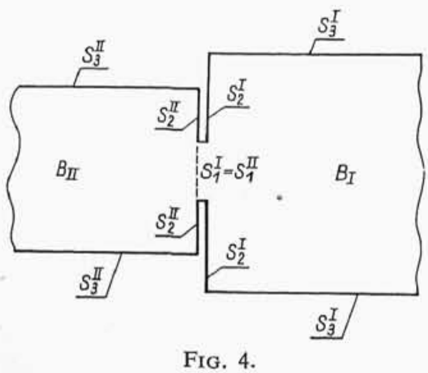

FIG. 4.

insulated; S_3^{I} and S_3^{II} are kept at zero temperature. The surface $S_1^{\mathrm{I}} = S_1^{\mathrm{II}}$ is that of joint between the regions B_{I} and B_{II}. The fundamental system is constituted by two semi-infinite cylinders: the cylinder B_{I}, thermally insulated on $S_1^{\mathrm{I}}+S_2^{\mathrm{I}}$ and kept at zero temperature on S_3^{I} and the cylinder B_{II} thermally insulated on $S_1^{\mathrm{II}}+S_2^{\mathrm{II}}$, and kept at zero temperature on S_3^{I}. Let us obtain the functions $G^{\mathrm{I}}(P, Q, t)$, $T_0^{\mathrm{I}}(Q, t)$, $P, Q \in B_{\mathrm{I}}$

and $G^{II}(P, Q, t)$, $T_0^{II}(Q, t)$ in the region B_{II}. The temperature gradient in the plane will be assumed to be unknown $\alpha - \alpha$.

$$\psi(R_1, t) = \frac{\partial T(R_1, t)}{\partial n}, \qquad R_1 \in S_1.$$

We write Eq. (2.17) first for the region B_I, and then for the region B_{II}. Letting the points $Q_I \in B_I$ and $Q_{II} \in B_{II}$ tend to the point R_1' on $S_1^I = S_2^{II}$ and making use of the condition of identical temperature on S_1^I, $[\varphi_1^I(R_1, t) = \varphi_1^{II}(R_1, t)]$, we obtain the required integral equation for the unknown function $\psi_1(R_1, t)$. The solution of (1.1) with the boundary conditions (1.3) leads to that of the integral Eq. (2.23). In more complex cases with mixed conditions on the surface $S(S = S_1 + S_2 + \ldots, S_k)$, a set of integral equations will be obtained.

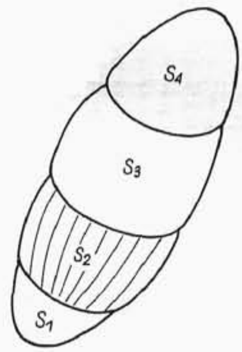

FIG. 5.

Let us consider the solid body represented in Fig. 5 in which thermal sources act, and mixed boundary conditions are prescribed on the surface $S = S_1 + S_2 + S_3 + S_4$

$$(2.34) \qquad \begin{aligned} &T(R_1, t) = \varphi_1(R_1, t) \;\; R_1 \in S_1; \quad && \frac{\partial T(R_2, t)}{\partial n} = \psi_2(R_2, t), \quad && R_2 \in S_2; \\ &T(R_3, t) = \varphi_3(R_3, t) \;\; R_3 \in S_3; \quad && T(R_4, t) = \varphi_4(R_4, t), \quad && R_4 \in S_4. \end{aligned}$$

Let us assume as a fundamental system the same body, thermally insulated on the surfaces S_1, S_2 and S_3, and kept at zero temperature on S_4. In this fundamental set, Green's function $G(P, Q, t)$ and temperature $T_0(Q, t)$ must be found. The latter will be obtained as a solution of (1.1) with the boundary conditions (2.34) from the equation

$$(2.35) \qquad T(Q, t) = T_0(Q, t) + \varkappa \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau) G(R_1, Q, t-\tau) dS_{R_1} + \varkappa \int_0^t d\tau \iint_{S_3} \psi_3(R_3, \tau) G(R_3, Q, t-\tau) dS_{R_3}.$$

Letting the point Q tend first to $R_1 \in S_1$, and then to Q, $R_3 \in S_3$, and taking the first and the third of the conditions (2.34), we obtain a set of two integral equations

$$
(2.36)\quad
\begin{aligned}
\varphi_1(R_1', t) &= T_0(R_1', t) + \varkappa \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau) G(R_1, R_1', t-\tau) dS_{R_1} \\
&\quad + \varkappa \int_0^t d\tau \iint_{S_3} \psi_3(R_3, \tau) G(R_3, R_1', t-\tau) dS_{R_3}, \quad R_1, R_1' \in S_1 \\
\varphi_3(R_3', t) &= T_0(R_3', t) + \varkappa \int_0^t d\tau \iint_{S_1} \psi_1(R_1, \tau) G(R_1, R_3', t-\tau) dS_{R_1} \\
&\quad + \varkappa \int_0^t d\tau \iint_{S_3} \psi_3(R_3, \tau) G(R_3, R_3', t-\tau) dS_{R_3}, \quad R_3, R_3' \in S_3 .
\end{aligned}
$$

On solving this for $\psi_1(R_1, t)$ and $\psi_3(R_3, t)$, we find the temperature $T(Q, t)$ from Eq. (2.35).

Let us return to the initial problem of determining the temperature in the body of Fig. 1. The boundary conditions will be somewhat modified, it being assumed that the heat exchange over the surface S_1 is free

$$
(2.37)\qquad \frac{\partial T(R_1, t)}{\partial n} + hT(R_1, t) = 0, \quad h = \text{const}, \quad R_1 \in S_1 .
$$

By determining in the fundamental system assumed the Green's function [Eqs. (2.1) and (2.2)], and the temperature T_0 [Eqs. (2.12) and (2.13)], we obtain Eqs. (2.16) and (2.17) with $\psi_1(R_1, t)$ replaced by $-hT(R_1, t)$. In this way, (2.17) takes the form

$$
(2.38)\qquad T(Q, t) = T_0(Q, t) - \varkappa h \int_0^t d\tau \iint_{S_1} T(R_1, \tau) G(R_1, Q, t-\tau) dS_{R_1} .
$$

Letting now the point Q tend to R_1' on S_1, we obtain an integral equation of the second kind

$$
(2.39)\qquad T(R_1' t) = T_0(R_1, t) - h\varkappa \int_0^t d\tau \int_{S_1} T(R_1, \tau) G(R_1, R_1', t-\tau) dS_{R_1} .
$$

Having determined $T(R_1, t)$ on S_1 from (2.39), we find temperature $T(Q, t)$ from (2.38).

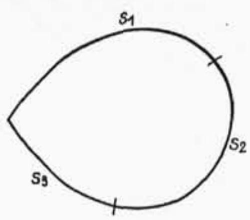

FIG. 6.

Let us proceed now to solve the two-dimensional problem. Let us consider the infinite cylinder with cross-section S. Let the contour of this cross-section be composed of sectionally regular arcs s_1, s_2, s_3 (Fig. 6). Let us assume that the temperature field is independent of the variables x_3 (the x_3 axis is parallel to the axis of the cylinder). Let the temperature field in the cylinder be produced by heat sources $W(P, t)$, and by surface heating. The temperature in the region S is determined by equation

$$\varkappa \nabla_1^2 T(P, t) - \dot{T}(P, t) = -M(P, t), \qquad \begin{aligned} &P = (x_1, x_2) \in S, \\ &\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \end{aligned} \tag{2.40}$$

with the boundary condition

$$T(P, 0) = 0, \tag{2.41}$$

and the boundary conditions

$$\begin{aligned} T(R_1, t) &= \varphi(R_1, t), \qquad R_1 \in s_1, \\ \frac{\partial T(R_2, t)}{\partial n} &= \psi_2(R_2, t), \qquad R_2 \in s_2, \\ T(R_3, t) &= \varphi_3(R_3, t), \qquad R_3 \in s_3. \end{aligned} \tag{2.42}$$

Let us determine Green's function $G(P, Q, t)$ from the heat equation

$$\varkappa \nabla_1^2 G(P, Q, t) - \dot{G}(P, Q, t) = -\delta(P-Q)\delta(t), \qquad P, Q \in S, \tag{2.43}$$

with the initial condition

$$G(P, Q, 0) = 0, \tag{2.44}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial G(R_1, t)}{\partial n} &= 0, \qquad R_1 \in s_1, \\ \frac{\partial G(R_2, t)}{\partial n} &= 0, \qquad R_2 \in s_2, \\ G(R_3, t) &= 0, \qquad R_3 \in s_3. \end{aligned} \tag{2.45}$$

Making use of Green's formula in the plane S, we obtain equations

$$T(Q, t) = T_0(Q, t) + \varkappa \int_0^t d\tau \int_{s_1} \psi_1(R_1, \tau) G(R_1, Q, t-\tau) ds_1, \qquad \begin{aligned} &Q \in S \\ &R_1 \in s_1, \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} T_0(Q, t) = &\iint_S f(P) G(P, Q, t) dS + \int_0^t d\tau \iint_S M(P, \tau) G(P, Q, t-\tau) dS \\ &+ \varkappa \int_0^t d\tau \int_{s_2} \psi_2(R_2, \tau) G(R_2, Q, t-\tau) ds_2 - \varkappa \int_0^t d\tau \int_{s_3} \varphi_3(R_3, \tau) \frac{\partial G(R_3, Q, t-\tau)}{\partial n} ds_3, \\ &P, Q \in S, \qquad R_1 \in s_1, \qquad R_3 \in s_3. \end{aligned} \tag{2.47}$$

The unknown function $\psi_1(R_1, t)$ will be found from equation

$$\varphi_1(R_1', t) = T_0(R_1', t) + \varkappa \int_0^t d\tau \int_{s_1} \psi_1(R_1, \tau) G(R_1, R_1', t-\tau) ds_1, \tag{2.48}$$

which will be obtained from (2.46) by letting the point Q tend to R_1' on the boundary s_1.

Similarly, the previous results, obtained for periodic and steady temperature, can be generalized to two-dimensional problems of heat conduction.

3. Second Solution Method

We shall give another variant of the solution method of (1.1) with mixed boundary conditions (1.3). The difference in the procedure will consist in a different choice of the fundamental system. While in the previous case thermal insulation extended over S_1 and S_2 for the function G, now, for the function G^*, we shall take the same body with the surfaces S_1 and S_2 kept at zero temperature. The function $G^*(P, Q, t)$ should therefore satisfy the heat equation

$$\varkappa \nabla^2 G^*(P, Q, t) - \dot{G}^*(P, Q, t) = -\delta(P-Q)\delta(t), \tag{3.1}$$

with the initial condition

$$G^*(P, Q, 0) = 0, \tag{3.2}$$

and the boundary conditions

$$\begin{aligned} G^*(R_1, Q, t) &= 0, \quad R_1 \in S_1; \\ G^*(R_2, Q, t) &= 0, \quad R_2 \in S_2; \\ G^*(R_3, Q, t) &= 0, \quad R \in S_3. \end{aligned} \tag{3.3}$$

If now Green's formula (2.9) is applied to the functions $\bar{T}$ and $\bar{G}^*$, and if the boundary conditions (1.3) and (3.3) are considered, we obtain

$$\begin{aligned} \bar{T}(Q, p) = \iiint_B f(p) \bar{G}^*(P, Q, p) dB_P + \iiint_B \bar{M}(P, p) G^*(P, Q, p) dB_P \\ - \varkappa \iint_{S_1} \bar{\varphi}_1(R_1, p) \frac{\partial \bar{G}^*(R_1, Q, p)}{\partial n} dS_{R_1} - \varkappa \iint_{S_2} \bar{\varphi}_2(R_2, p) \frac{\partial \bar{G}^*(R_2, Q, p)}{\partial n} dS_{R_2} \\ - \varkappa \iint_{S_3} \bar{\varphi}_3(R_3, p) \frac{\partial \bar{G}^*(R_3, Q, p)}{\partial n} dS_{R_3}, \end{aligned} \tag{3.4}$$

or

$$\bar{T}(Q, p) = \bar{T}_0(Q, p) - \varkappa \iint_{S_2} \bar{\varphi}_2(R_2, p) \frac{\partial \bar{G}^*(R_2, Q, p)}{\partial n} dS_{R_2}. \tag{3.5}$$

Let us observe that the quantities involved in $T_0^*(Q, p)$ are known. The function $T_0^*(Q, t)$ can be obtained by solving the heat equation

$$\varkappa \nabla^2 T_0^*(P, t) - \dot{T}_0^*(P, t) = -M(P, t), \quad P \in B, \tag{3.6}$$

with the initial condition $T_0^*(P, 0) = f(P)$ and the boundary condition

$$(3.7)\quad \begin{aligned} &T_0^*(R_1, t) = \varphi_1(R_1, t), \quad R_1 \in S_1, \\ &T_0^*(R_2, t) = 0, \qquad R_2 \in S_2; \quad T_0^*(R_3, t) = \varphi_3(R_3, t), \quad R_3 \in S_3. \end{aligned}$$

On performing on (3.1), (3.3), (3.6), (3.7) one-sided Laplace transformation, we obtain from Green's formula an equation, which, after inverse transformation, takes the form

$$(3.8)\quad \begin{aligned} T_0^*(Q, t) = \iiint\limits_B f(P) G^*(P, Q, t) dB_P + \int\limits_0^t d\tau \iiint\limits_B M(P, \tau) G^*(P, Q, t-\tau) dB_P \\ - \varkappa \int\limits_0^t d\tau \iint\limits_{S_1} \varphi_1(R_1, \tau) \frac{\partial G^*(R_1, Q, t-\tau)}{\partial n} dS_{R_1} \\ - \varkappa \int\limits_0^t d\tau \iint\limits_{S_3} \varphi_3(R_3, \tau) \frac{\partial G^*(R_3, Q, t-\tau)}{\partial n} dS_{R_3}. \end{aligned}$$

On performing now the inverse transformation on (3.5), we obtain

$$(3.9)\quad T(Q, t) = T_0^*(Q, t) - \varkappa \int\limits_0^t d\tau \iint\limits_{S_2} \varphi_2(R_2, \tau) \frac{\partial G^*(R_2, Q, t-\tau)}{\partial n} dS_{R_2}.$$

It is known, however, that the boundary condition assigned on S_2 is

$$(3.10)\quad \frac{\partial T(R_2, t)}{\partial n} = \psi_2(R_2, t), \quad R_2 \in S_2.$$

Let us make use of this condition by letting the point $Q \in B$ pass to the current point $R_2' \in S_2$, and by performing the equation $\partial/\partial n'$. Thus, from (3.9), we find

$$(3.11)\quad \psi_2(R_2', t) = \frac{\partial T_0^*(R_2', t)}{\partial n'} - \varkappa \int\limits_0^t d\tau \iint\limits_{S_2} \varphi_2(R_2, \tau) \frac{\partial^2 G^*(R_2, R_2', t-\tau)}{\partial n' \partial n} dS_{R_2},$$

where $\partial/\partial n'$ denotes the normal derivative at $R_2' \in S_2$. The integral Eq. (3.11) will be used to find the unknown function $\varphi_2(R_2, t)$. From (3.9) we can find the temperature $T(Q, t)$.

Let us consider also Green's function $K^*(P, R_2', t)$, satisfying, in our fundamental system, the heat equation

$$(3.12)\quad \varkappa \nabla^2 K^*(P, R_2', t) - \dot{K}^*(P, R_2', t) = 0, \quad P \in B,$$

with the initial condition $K^*(P, R_2', 0) = 0$, and the boundary conditions

$$(3.13)\quad \begin{aligned} K^*(R_1, R_2', t) = 0, \quad R_1 \in S_1, \quad K^*(R_2, R_2', t) = \delta(R_2 - R_2')\delta(t), \quad R_2, R_2' \in S_2, \\ K^*(R_3, R_3', t) = 0, \quad R_3 \in S_3, \end{aligned}$$

where

$$\int\limits_0^t d\tau \iint\limits_{S_2} \delta(R_2 - R_2')\delta(t) dS_{R_2} = 1.$$

By applying Green's formula to the functions G^* and K^* on which one-sided Laplace transformation has been performed, we find that

$$(3.14)\quad \iiint_B \bar{K}^*(P, R_2', p)\delta(P-Q)\,dB_P = -\varkappa \iint_{S_2} \delta(R_2 - R_2') \frac{\partial G^*(R_2, Q, p)}{\partial n}\, dS_{R_2};$$

hence, on performing the inverse Laplace transformation, we obtain

$$(3.15)\quad K^*(Q, R_2', t) = -\varkappa \frac{\partial \bar{G}^*(R_2', Q, t)}{\partial n}.$$

Bearing in mind (3.15), Eq. (3.6) can be represented in the form

$$(3.16)\quad \psi_2(R_2', t) = \frac{\partial T_0^*(R_2', t)}{\partial n'} + \int_0^t d\tau \iint_{S_2} \varphi_2(R_2, \tau) \frac{\partial K^*(R_2, R_2', t-\tau)}{\partial n'}\, dS_{R_2}.$$

Similarly to Sec. 2, we can easily pass from Eqs. (3.9)–(3.10) to those for harmonic temperature or steady-state flow.

Let us consider also the problem with the following boundary conditions on the surface S of the body

$$(3.17)\quad \begin{aligned} &T(R_1, t) = \varphi_1(R_1, t) && R_1 \in S_1,\\ &\frac{\partial T(R_2, t)}{\partial n} + hT(R_2, t) = 0, && R_2 \in S_2,\\ &T(R_3, t) = \varphi_3(R_3, t), && R_3 \in S_3. \end{aligned}$$

Making use of the integral Eq. (3.16) and bearing in mind that

$$\psi_2(R_2, t) = \frac{\partial T(R_2, t)}{\partial n}, \quad \varphi_2(R_2, t) = T(R_2, t),$$

and taking into consideration the second of the conditions (3.7), we find

$$(3.18)\quad -hT(R_2', t) = \frac{\partial T_0^*(R_2', t)}{\partial n'} + \int_0^t d\tau \iint_{S_2} T(R_2, \tau) \frac{\partial K^*(R_2, R_2', t-\tau)}{\partial n'}\, dS_{R_2}.$$

Thus in the case of the boundary conditions (3.17), an equation of the second kind is obtained.

4. Solution for a Body with Slits and Insulating Diaphragms

Let us consider a simly connected body bounded by the surface S and having a slit, of which the (upper and lower) surfaces are denoted by S_2' and S_2'', respectively the remaining part of the surface being denoted by S_1. Let the body be subject to internal heat sources and surface heating. The temperature $T(P, t)$ must satisfy the heat equation

$$(4.1)\quad \varkappa \nabla^2 T(P, t) - \dot{T}(P, t) = -M(P, t), \quad P \in B,$$

with the initial condition

$$(4.2)\quad T(P, 0) = f(P),$$

and the boundary conditions

$$(4.3)\qquad \begin{aligned} T(R_1, t) &= 0 \quad \text{on} \quad S_1, \quad R_1 \in S_1, \\ T(R_2, t) &= 0 \quad \text{on} \quad S_2' \text{ i } S_2''. \end{aligned}$$

Let us consider the function $G(P, Q, t)$ in the basic system constituted by the same body S_1 with no slit. The function $G(P, Q, t)$ must satisfy the heat equation

$$(4.4)\qquad \varkappa \nabla^2 G(P, Q, t) - \dot{G}(P, Q, t) = \delta(P-Q)\delta(t), \quad P \in B$$

with the initial condition

$$(4.5)\qquad G(P, Q, 0) = 0$$

and the boundary condition

$$(4.6)\qquad G(R_1, Q, t) = 0 \quad \text{on} \quad S_1.$$

Let us perform on (4.1)–(4.6) the one-sided Laplace transformation, and apply Green's formula

$$(4.7)\qquad \iiint\limits_B (\overline{G}\nabla^2\overline{T} - \overline{T}\nabla^2\overline{G})dB = \iint\limits_S \left(\overline{G}\frac{\partial \overline{T}}{\partial n} - \overline{T}\frac{\partial \overline{G}}{\partial n}\right)dS.$$

After some simple rearrangement, we obtain

$$(4.8)\qquad \overline{T}(Q, p) = \iiint\limits_B f(P)\overline{G}(P, Q, p)dB_p + \iiint\limits_B \overline{M}(P, p)\overline{G}(P, Q, p)dB_p + \iint\limits_{S_2'+S_2''} \overline{G}(R_2, Q, p)\frac{\partial \overline{T}(R_2, p)}{\partial n}dS_{R_2},$$

or

$$(4.9)\qquad \overline{T}(Q, p) = \overline{T}_0(Q, p) + \varkappa \iint\limits_{S_2} \overline{G}(R_2, Q, p)\left[\left(\frac{\partial \overline{T}}{\partial n}\right)_+ + \left(\frac{\partial \overline{T}}{\partial n}\right)_-\right]dS_{R_2},$$

where $+$ denotes the upper and $-$ the lower part of the surface S_2. The function G having been defined in the region B with no slit, this function is different from zero on S_2. The temperature gradients on the surfaces S_2' and S_2'' are also different from zero (the second boundary condition of the group (4.3) being that of zero emperature). The expression in brackets under the integration sign in (4.9) can be treated as an unknown function $\psi(R_2, t)$. Thus Eq. (4.9) takes, on performing the inverse Laplace transformation, the form

$$(4.10)\qquad T(Q, t) = T_0(Q, t) + \varkappa \int\limits_0^t d\tau \iint\limits_{S_2} \psi(R_2, \tau) G(R_2, Q, t-\tau) dS_{R_2},$$

where

$$(4.11)\quad T_0(Q, t) = \iiint\limits_B f(P)G(P, Q, t-\tau)dB + \int\limits_0^t d\tau \iiint\limits_B M(P, \tau)G(P, Q, t-\tau)dB.$$

Let us make the point Q tend to R_2' on S_2. Bearing in mind that (in agreement with the second boundary condition (4.3) we have $T(R_2, t) = 0$ on S_2, we obtain the following integral equation

$$(4.12)\quad T(R_2, t) = 0 = T_0(R_2, t) + \varkappa \int_0^t d\tau \iint_{S_2} \psi(R_2, \tau) G(R_2, R_2', t-\tau) dS_{R_2}.$$

By solving (4.12) we find the function $\psi(R_2, \tau)$ from which the temperature field according to (4.10) can be found[1].

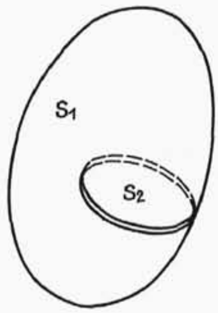

FIG. 7.

Let us consider in turn the solid of Fig. 7, subject also to a temperature field but with different conditions. Let the temperature $\vartheta(P, t)$ satisfy the heat equation

$$(4.13)\quad \varkappa \nabla^2 \vartheta(P, t) - \dot{\vartheta}(P, t) = -M(P, t), \qquad P \in B,$$

with the initial condition $\vartheta(P, 0) = 0$ and the boundary conditions

$$(4.14)\quad \begin{aligned} &\vartheta(R_1, t) = 0 \text{ on } S_1, \\ &\frac{\partial \vartheta(R_2, t)}{\partial n} = 0 \text{ on } S_2 = S_2' + S_2''. \end{aligned}$$

We have assumed here that the surface S_2 is thermally insulated. By applying Green's function $G(P, Q, t)$ as determined by Eq. (4.4) with the initial condition (4.5) and the boundary condition (4.3), we obtain, on applying Green's formula (4.7) for the following relation between the function ϑ and G

(4.15)

$$\vartheta(Q, t) = \vartheta_0(Q, t) + \varkappa \int_0^t d\tau \iint_{S_2} \left[\left(T(R_2, \tau)\right)_+ + \left(T(R_2, \tau)\right)_-\right] \frac{\partial G(R_2, Q, t-\tau)}{\partial n} dS_{R_2},$$

where

$$\vartheta_0(Q, t) = \iiint_B f(P) G(P, Q, t) dB + \int_0^t d\tau \iiint_B M(P, \tau) G(P, Q, t-\tau) dB.$$

[1] A similar solution method was applied in Ref. [2] to the problem of deflection of a membrane and torsion of a bar and in Ref. [3] to that of torsion of a bar—that is, to the solution of Poisson's equation in a plane.

Denoting by $\varphi(R_2, t)$ the unknown temperature function on S_2' and S_2'', let us rewrite Eq. (4.15) in the form

$$\vartheta(Q, t) = \vartheta_0(Q, t) + \int_0^t d\tau \iint_{S_2} \varphi(R_2, \tau) \frac{\partial G(R_2, Q, t-\tau)}{\partial n} dS_{R_2}. \tag{4.16}$$

Let us apply the condition $\dfrac{\partial \vartheta(R_2, t)}{\partial n} = 0$ on S_2. By letting Q pass to $R_2' \in S_2$, we obtain the following integral equation of the first kind

$$\frac{\partial \vartheta(R_2', t)}{\partial n'} = 0 = \frac{\partial \vartheta_0(R_2', t)}{\partial n'} + \varkappa \int_0^t d\tau \iint_{S_2} \varphi(R_2, \tau) \frac{\partial^2 G(R_2, R_2', t-\tau)}{\partial n \partial n'} dS_{R_2}. \tag{4.17}$$

The function $\varphi(R_2 \tau)$ being now determined, we can find the temperature $\vartheta(Q, t)$ from Eq. (4.16).

Let us consider the case in which the following conditions are required to be satisfied for the temperature $\theta(P, 0) = f(P)$, in addition to the initial condition

$$\begin{aligned} \theta(R_1, t) &= 0 \quad \text{on} \quad S_1, \\ \theta(R_2', t) &= 0 \quad \text{on} \quad S_2', \\ \frac{\partial \theta(R_2'', t)}{\partial n} &= 0 \quad \text{on} \quad S_2''. \end{aligned} \tag{4.18}$$

Making use of Green's function $G(P, Q, t)$ (4.4) and the conditions (4.5) and (4.6), we obtain for the temperature $\theta(Q, t)$ the formula

$$\begin{aligned} \theta(Q, t) = \theta_0(Q, t) &+ \varkappa \int_0^t d\tau \iint_{S_2'} \frac{\partial \theta(R_2', \tau)}{\partial n} G(R_2', Q, t-\tau) dS_{R_2}' \\ &- \varkappa \int_0^t d\tau \iint_{S_2''} \theta(R_2'', \tau) \frac{\partial G(R_2'', Q, t-\tau)}{\partial n} dS_{R_2}'', \end{aligned} \tag{4.19}$$

where

$$\theta_0(Q, t) = \iiint_B f(P) G(P, Q, t) dB + \varkappa \int_0^t d\tau \iiint_B M(P, \tau) G(P, Q, t-\tau) dB.$$

Let us consider now the boundary conditions (4.18). Letting the point $Q \in B$ in (4.19) tend first to C_2' on S_2', and then to C_2'' on S_2'' we find the set of equations

$$\begin{aligned} \theta(C_2', t) = 0 = \theta_0(C_2', t) &+ \varkappa \int_0^t d\tau \iint_{S_2'} \frac{\partial \theta(R_2', \tau)}{\partial n} G(R_2', C_2', t-\tau) dS_{R_2}' \\ &- \varkappa \int_0^t d\tau \iint_{S_2''} \theta(R_2'', \tau) \frac{\partial G(R_2'', C_2', t-\tau)}{\partial n} dS_{R_2}'', \qquad C_2' \in S_2', \end{aligned} \tag{4.20}$$

$$(4.21)\qquad \frac{\partial\theta(C_2'',t)}{\partial n'}=0=\frac{\partial\theta_0(C_2'',t)}{\partial n'}+\varkappa\int_0^t d\tau\iint_{S_2'}\frac{\partial\theta(R_2',t)\partial G(R_2',C_2'',t-\tau)}{\partial n\,\partial n'}dS_{R_2}'$$
$$+\varkappa\int_0^t d\tau\iint_{S_2''}\theta(R_2'',\tau)\frac{\partial G(R_2'',C_2'',t-\tau)}{\partial n'\,\partial n}dS_{R_2}'',\qquad C_2''\in S_2''.$$

These are integral equations of the first kind with unknown functions $\frac{\partial\theta(R_2',t)}{\partial n}$ on S_2' and $\theta(R_2'',t)$ on S_2''. On solving them for these unknown functions we can determine the temperature from (4.19).

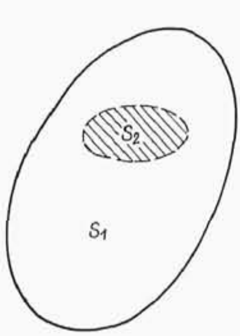

FIG. 8.

Let us consider now the simply connected body of Fig. 8 with an insulating diaphragm S_2 inside the body. Let the body be acted on by some heat sources inside the body which is also heated on the surface S_1. The diaphragm condition is $\partial T/\partial n$, there being no heat flow across S_2. The temperature should satisfy the heat equation

$$(4.22)\qquad \varkappa\nabla^2T(P,t)-T(P,t)=-M(P,t),\qquad P\in B,$$

with the initial condition

$$(4.23)\qquad T(P,0)=0$$

and the boundary conditions

$$(4.24)\qquad \begin{aligned}&T(R_1,t)=0\quad\text{on}\quad S_1,\quad R_1\in S_1,\\ &\frac{\partial T(R_2,t)}{\partial n}=0\quad\text{on}\quad S_2,\ S_2=S_2'+S_2'',\quad R_2\in S_2.\end{aligned}$$

Assuming in the fundamental set Green's function $G(P,Q,t)$, satisfying the equation (4.4) with the initial condition (4.5) and the boundary conditions (4.6), and, finally, applying Green's formula (4.7) to the functions G and T (obtained by solving (4.22), we obtain

$$(4.25)\qquad \overline{T}(Q,p)=\overline{T}_0(Q,p)-\varkappa\iint_{S_2'+S_2''}[\overline{T}_+(R_2,p)+\overline{T}_-(R_2,p)]\frac{\partial\overline{G}(R_2,Q,p)}{\partial n}\,dS_{R_2}$$

where $\overline{T}_0(Q, p)$ is given by (4.11) and T_+ T_- de ote the temperature on the upper and lower surface of the insulating diaphragm, respectively. These temperatures are unknown functions. Their sum is denoted by $\varphi_2(R_2, t)$. Thus, (4.25) takes the form

$$(4.26) \qquad \overline{T}(Q, p) = \overline{T}_0(Q, p) - \varkappa \iint\limits_{S_2} \overline{\varphi}_2(R_2, p) \frac{\partial \overline{G}(R_2, Q, p)}{\partial n} dS_{R_2}.$$

Letting the point $Q \in B$ tend to R_2' on S_2 and performing the operation $\partial/\partial n'$ on (4.26), we obtain

$$(4.27) \quad \frac{\partial \overline{T}(R_2', p)}{\partial n'} = 0 = \frac{\partial \overline{T}_0(R_2', p)}{\partial n'} - \varkappa \iint\limits_{S_2} \overline{\varphi}_2(R_2, p) \frac{\partial^2 \overline{G}(R_2, R_2', p)}{\partial n \partial n'} dS_{R_2}.$$

The left-hand side of (4.27) represents the normal heat flow at $R_2' \in S_2$. However, in view of the thermal insulation on S_2, this flow is zero. Eq. (4.27) is an integral equation of the first kind, from which the unknown function $\overline{\varphi}_2(R_2, p)$ can be determined, thus enabling us to find $\overline{T}(Q, p)$ from (4.26).

On performing the inverse Laplace transformation of (4.26), we obtain

$$(4.28) \quad T(Q, t) = T_0(Q, t) - \varkappa \int\limits_0^t d\tau \iint\limits_{S_2} \varphi(R_2, \tau) \frac{\partial G(R_2, Q, t-\tau)}{\partial n} dS_{R_2}.$$

The procedure just described can be generalized to the case of existence in the body of more than one insulating diaphragm. In the case of r diaphragms, we obtain a set of r integral equations.

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Streszczenie

MIESZANE WARUNKI BRZEGOWE W ZAGADNIENIACH PRZEWODNICTWA CIEPLNEGO

Nawiązując do swej dawniejszej pracy [1] autor rozszerza podaną tam metodę rozwiązywania zagadnień przewodnictwa cieplnego w ciele stałym z mieszanymi warunkami brzegowymi na nieustalone przepływy ciepła. Podano dwa warianty rozwiązania, przy użyciu dwu różnych tak zwanych układów podstawowych. W obu wariantach sprowadza się rozwiązanie zagadnienia do rozwiązania równania całkowego pierwszego rodzaju. W końcu omówiono tok postępowania dla przypadku istnienia w ciele stałym szczelin oraz przesłon izolacyjnych.

Резюме

СМЕШАННЫЕ КРАЕВЫЕ УСЛОВИЯ В ЗАДАЧАХ ТЕПЛОПРОВОДНОСТИ

В связи с одной из предыдущих работ [1] автор расширяет приведенный в ней метод решения вопросов, касающихся теплопроводности твердого тела с разрывными краевыми условиями для нестационарных потоков тепла. Даются два варианта решения при использовании двух различных, так называемых, основных систем. В обоих вариантах решение задачи сводится к решению интегрального уравнения первого рода. В заключение обсуждается процесс решения при наличии в твердом теле щелей и изоляционных экранов.

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