POLSKA AKADEMIA NAUK
INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI

ARCHIWUM MECHANIKI STOSOWANEJ

ARCHIVES DE MECANIQUE APPLIQUEE

4
XIII

VOL. XIII WARSZAWA 1961 No 4
PAŃSTWOWE WYDAWNICTWO NAUKOWE

ARCHIWUM
MECHANIKI
STOSOWANEJ
4, 13 (1961)

APPLICATION OF DIFFERENCE EQUATIONS IN THE THEORY OF PLATES (I)

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1. Introduction

The calculus of finite differences found a broad application in the theory of plates owing to the works by N.J. NIELSEN, [1], H. MARCUS, [2], P.M. WARWAK, [3]. By replacing the derivatives in the differential equation of the theory of plates by difference quotients, a differential equation is replaced by a partial difference equation, the continuous deflection surface thus being represented — in an approximate manner by means of a polyhedron. Treating the partial difference equation as a system of linear algebraic equations, and solving these equations by known methods (the Gaussian elimination method or various iteration methods) approximate values of plate deflection are obtained. The partial difference equation may also be treated as a matrix equation [4], [5] and solved by means of matrix methods in a manner shown by E. EGERVÁRY, [6].

Another way, which will be followed in the present paper, is to solve the partial difference equation of plate deflection by means of methods of finite differences successfully applied to plane gridworks by H. BLEICH and E. MELAN, [7]. In this way full analogy between the solution of the differential equation and that of a difference equation of plate deflection is obtained. By letting the net become more and more dense we can always pass to the results obtained in the domain of differential equations.

In the second and the third section of the present paper we shall be concerned with the solution of the differential equation of plate deflection by means of double finite series for both forces and free vibration, simultaneous bending and compression, and buckling. The results obtained are fully analogous to those of the generalized Navier method in the differential theory of plates. In Sec. 4 an orthogonalization method will be given, analogous to B. G. Galerkin's method for differential equations, [8].

Sec. 5 is concerned with the application of simple finite series to the determination of the deflection of a plate, principally for a plate strip, making use of a Fourier integral transformation devised by I. BABUŠKA for difference equations.

Finally, the last section is concerned with difference-differential equations of a plate the application of which is convenient in many cases.

The second part of the paper will be devoted to a number of plate problems concerning mixed boundary conditions, and the application of double Fourier transformation to the difference equation of plate deflection.

The solution methods described in the present paper may be transferred to a considerable degree to problems of plates loaded in their planes and a number of static and quasi-static space problems.

2. Free and Forced Vibration of a Rectangular Plate

Let us consider the equation of forced vibration of the plate

$$N\nabla^4 \overline{w}(x, y, t) + \varrho h \frac{\partial^2 \overline{w}(x, y, t)}{\partial t^2} = \overline{q}(x, y, t) \tag{2.1}$$

assuming homogeneous boundary conditions. In this equation $\overline{w}$ denotes the deflection, $\overline{q}$ — the excitation load, N — the bending rigidity of the plate, ϱ — the density per unit area of the middle surface and h — the plate thickness. In the case of a periodic load $\overline{q}(x, y, t) = q(x, y)e^{i\omega t}$ we have also $\overline{w}(x, y, t) = w(x, y)e^{i\omega t}$, where ω is the excitation frequency.

Introducing these relations in (2.1), we obtain the amplitude equation for plate deflection

$$N\nabla^4 w(x, y) - \varrho h\omega^2 w(x, y) = q(x, y). \tag{2.2}$$

Let us replace the derivatives in (2.2) by difference quotients. Dividing the edge a of the rectangle into n equal segments Δx, the edge b into m equal segments Δy, we reduce the Eq. (2.2) to the form

$$(L_{xy} - \tau^2) w_{xy} = \varkappa q_{xy} \quad (x = 0, 1, 2, \ldots, n;\ y = 0, 1, 2, \ldots, m), \tag{2.3}$$

where

$$L_{xy}(w_{xy}) = (\Delta_x^4 + 2\varepsilon^2 \Delta_x^2 \Delta_y^2 + \varepsilon^4 \Delta_y^4) w_{xy}, \qquad \varepsilon = \frac{\Delta x}{\Delta y} = \frac{a}{b}\frac{m}{n}, \tag{2.4}$$

and Δ_x^2, Δ_x^4 denote the second and the fourth difference in the x-direction and Δ_y^2, Δ_y^4 the second and the fourth difference in the y-direction, respetively, where

$$\begin{cases} \Delta_x^2(w_{xy}) = w_{x-1,y} - 2w_{xy} + w_{x+1,y}, \\ \Delta_x^4(w_{xy}) = w_{x-2,y} - 4w_{x-1,y} + 6w_{xy} - 4w_{x+1,y} + w_{x+2,y}. \end{cases} \tag{2.5}$$

The symbols in the Eq. (2.3) are

$$\tau^2 = \frac{\varrho h\omega^2 \Delta x^4}{N}, \qquad \varkappa = \frac{\Delta x^4}{N}.$$

The solution of the Eq. (2.3) will be sought in the form of a double finite serie;

$$w_{xy} = \sum_{\nu,\mu}^{n,m} A_{\nu\mu} \varphi_{xy}^{\nu\mu}, \tag{2.6}$$

where the quantities $A_{\nu\mu}$ are the unknown coefficients and $\varphi_{xy}^{\nu\mu}$ are the eigenfunctions of the difference equations

$$L_{xy}(\varphi_{xy}^{\nu\mu}) = \sigma_{\nu\mu}\varphi_{xy}^{\nu\mu} \tag{2.7}$$

assuming that the functions $\varphi_{xy}^{\nu\mu}$ satisfy the same boundary conditions as the function w_{xy}.

The quantities $\sigma_{\nu\mu}$ are the eigenvalues ($\nu = 0, 1, 2, \ldots, n$; $\mu = 0, 1, 2, \ldots, m$) corresponding to the eigenfunctions $\varphi_{xy}^{\nu\mu}$. The latter constitute a complete set of orthonormal functions, therefore they satisfy the conditions

$$\sum_{x,y}^{n,m} \varphi_{xy}^{\nu\mu}\varphi_{xy}^{ik} = \delta_{\nu i}\delta_{\mu k}, \tag{2.8}$$

where $\delta_{i\nu}$, $\delta_{\mu k}$ are Kronecker's deltas, or

$$\delta_{i\nu} = \begin{cases} 1 & \text{if} \quad i = \nu, \\ 0 & \text{if} \quad i \neq \nu; \end{cases} \qquad \delta_{\mu k} = \begin{cases} 1 & \text{if} \quad \mu = k, \\ 0 & \text{if} \quad \mu \neq k. \end{cases} \tag{2.9}$$

If the series (2.6) is to constitute an accurate solution of the differential equation (2.3), the functions $(L_{xy} - \tau^2)w_{xy} - q_{xy}\varkappa$ should be orthogonal to every function $\varphi_{xy}^{\nu\mu}$. Therefore

$$\sum_{x,y}^{n,m}\left[(L_{xy} - \tau^2)\sum_{\nu,\mu}^{n,m} A_{\nu\mu}\varphi_{xy}^{\nu\mu} - \varkappa q_{xy}\right]\varphi_{xy}^{ik} = 0. \tag{2.10}$$

Changing the summation order and bearing in mind (2.7) we obtain

$$\sum_{\nu,\mu}^{n,m} A_{\nu\mu}(\sigma_{\nu\mu} - \tau^2)\sum_{x,y}^{n,m}\varphi_{xy}^{\nu\mu}\varphi_{xy}^{ik} = \varkappa q_{ik}, \qquad q_{ik} = \sum_{x,y}^{n,m} q_{xy}\varphi_{xy}^{ik}. \tag{2.11}$$

Making use of the orthogonality condition (2.7), we obtain finally

$$A_{ik}(\sigma_{ik} - \tau^2) = \varkappa q_{ik} \quad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m). \tag{2.12}$$

Introducing A_{ik} from the last equation in the Eq. (2.6) we find

$$w_{xy} = \varkappa\sum_{\nu,\mu}^{n,m}\frac{q_{\nu\mu}}{\sigma_{\nu\mu} - \tau^2}\varphi_{xy}^{\nu\mu}. \tag{2.13}$$

Observe that for $\tau^2 \to \sigma_{\nu\mu}$ the amplitudes increase indefinitely. Thus we are concerned with the phenomenon of resonance. If $\tau \to 0$ ($\omega \to 0$), (2.13) represents the deflection produced by the static load. If the plate performs free vibration (that is if $q_{xy} = 0$, $q_{\nu\mu} = 0$), the values of the natural frequencies are obtained from (2.12) with $A_{ik} \neq 0$

$$\sigma_{ik} = \tau_{ik}^2 \qquad (i = 0, 1, \ldots, n;\ k = 0, 1, \ldots, m). \tag{2.14}$$

The solutions represented here for forced vibration (2.13) and free vibration (2.14) are valid assuming that the functions $\varphi_{xy}^{\nu\mu}$ can be expressed in the form of a product $X_x^\nu Y_y^\mu$ or $X_x^\nu Y_y^{\nu\mu}$ or $X_x^{\nu\mu} Y_y^\mu$. It will be found (see Appendix) that functions of the type $\varphi_{xy}^{\nu\mu} = X_x^\nu Y_y^\mu$ appear in the case of a rectangular plate simply supported on the entire contour, and functions of the type $\varphi_{xy}^{\nu\mu} = X_x^\nu Y_y^\mu$—in the case of a plate simply supported on the edges $x = 0$, $x = n$, and supported in an arbitrary manner or free along the remaining edges.

Consider a number of particular cases of application of the double series method.

Let the plate be acted on by a load q_{xy}, and let it have an additional immovable support at the point (ξ, η). The deflection of the plate will be composed of a deflection due to the load q_{xy} and R at the point (ξ, η) the value of R being selected in such a manner that $w_{\xi\eta} = 0$. Therefore

$$w_{xy} = \varkappa \sum_{\nu,\mu}^{n,m} \frac{\varphi_{xy}^{\nu\mu}}{\sigma_{\nu\mu} - \tau^2} [q_{\nu\mu} + q_{\nu\mu}^*], \tag{2.15}$$

where

$$q_{\nu\mu}^* = R \sum_{x,y}^{n,m} \delta_{x\xi} \delta_{y\eta} \varphi_{xy}^{\nu\mu} = R\varphi_{\xi\eta}^{\nu\mu}.$$

Inserting $q_{\nu\mu}^*$ in (2.15) and requiring that $w_{\xi\eta} = 0$, we obtain the equation

$$R \sum_{\nu,\mu}^{n,m} \frac{(\varphi_{\xi\eta}^{\nu\mu})^2}{\sigma_{\nu\mu} - \tau^2} + \sum_{\nu,\mu}^{n,m} \frac{q_{\nu\mu}}{\sigma_{\nu\mu} - \tau^2} = 0, \tag{2.16}$$

from which the amplitude of the support reaction R can be found.

If in the Eq. (2.16) it is assumed that $q_{\nu\mu} = 0$, the equation

$$\sum_{\nu,\mu}^{n,m} \frac{(\varphi_{\xi\eta}^{\nu\mu})^2}{\sigma_{\nu\mu} - \tau^2} = 0 \tag{2.17}$$

constitutes the condition of free vibration of a rectangular plate supported on the contour and at the additional point ξ, η.

Let now the plate be acted on by, in addition to the load q_{xy}, a load $R_y \delta_{x\xi}$ along the line $x = \xi$. The deflection amplitude of the plate is given by (2.15) where

$$q_{\nu\mu}^* = \sum_{x,y}^{n,m} R_y \delta_{x\xi} \varphi_{xy}^{\nu\mu}.$$

For a plate simply supported on the entire contour we have

$$\varphi_{xy}^{\nu\mu} = X_x^\nu Y_y^\mu.$$

Therefore, in this case, we have

$$q_{\nu\mu}^* = X_\xi^\nu \sum_{y}^{m} R_y Y_y^\mu = X_\xi^\nu b_\mu, \qquad b_\mu = \sum_{y}^{m} Y_y^\mu R_y. \tag{2.18}$$

Substituting (2.18) in (2.15) and requiring that the deflection of the plate along the line $x = \xi$ be zero, we obtain the following equation for the coefficients b_μ:

$$b_\mu \sum_\nu^n \frac{(X_\xi^\nu)^2}{\sigma_{\nu\mu} - \tau^2} + \sum_\nu^n \frac{q_{\nu\mu} X_\xi^\nu}{\sigma_{\nu\mu} - \tau^2} = 0 . \tag{2.19}$$

Knowing b_μ, the support reaction R_y can easily be found

$$R_y = \sum_\mu^m b_\mu Y_y^\mu .$$

Substituting $q_{\nu\mu} = 0$ in (2.19), we obtain the equation

$$\sum_\nu^n \frac{(X_\xi^\nu)^2}{\sigma_{\nu\mu} - \tau^2} = 0, \tag{2.20}$$

from which successive natural frequencies $\omega_{\nu\mu}$ of the plate can be found for a two-span plate.

Analogous solution will be obtained, in this case also, if

$$\varphi_{xy}^{\nu\mu} = Y_y^\mu X_x^{\nu\mu} .$$

If the plate is acted on by, in addition to q_{xy}, a load $R_y \delta_{x\xi}$ along the line $x = \xi$, and a load $Q_x \delta_{y\eta}$ along the line $y = \eta$, the deflection amplitude, assuming the plate to be simply supported on the entire contour, takes the form

$$\left\{ \begin{aligned} & w_{xy} = \varkappa \sum_{\nu,\mu}^{n,m} \frac{X_x^\nu Y_y^\mu}{\sigma_{\nu\mu} - \tau^2} (q_{\nu\mu} + b_\mu X_\xi^\nu + c_\nu Y_\eta^\mu) , \\ & b_\mu = \sum_y R_y Y_y^\mu , \qquad c_\nu = \sum_x Q_x X_x^\nu . \end{aligned} \right. \tag{2.21}$$

Requiring that the deflection along the lines $x = \xi$ and $y = \eta$ be zero we obtain the following system of two equations:

$$b_\mu \sum_\nu \frac{(X_\xi^\nu)^2}{D_{\nu\mu}} + Y_\eta^\mu \sum_\nu \frac{c_\nu X_\xi^\nu}{D_{\nu\mu}} + \sum_\nu \frac{q_{\nu\mu} X_\xi^\nu}{D_{\nu\mu}} = 0, \tag{2.22}$$

$$X_\xi^\nu \sum_\mu \frac{b_\mu Y_\eta^\mu}{D_{\nu\mu}} + c_\nu \sum_\mu \frac{(Y_\eta^\mu)^2}{D_{\nu\mu}} + \sum_\mu \frac{q_{\nu\mu} Y_\eta^\mu}{D_{\nu\mu}} = 0, \tag{2.23}$$

$$D_{\nu\mu} = \sigma_{\nu\mu} - \tau^2,$$

from which the coefficients c_ν and b_μ can be found. Knowing these, we can find the functions R_y and Q_x, because

$$R_y = \sum_\mu b_\mu Y_y^\mu , \quad Q_x = \sum c_\nu X_x^\nu . \tag{2.24}$$

The procedure just described may be generalized to the case of more linear supports in the region of the plate and to the case of a plate with ribs.

Let us consider a plate simply supported on the contour and stiffened by means of a rib of rigidity B along the line $x = \xi$. The differential equation of the rib deflection is

$$B\frac{d^4W}{dy^4}-\varrho_0 A\omega^2 W = -R_y(y)\,, \tag{2.25}$$

where A is the cross-section of the bar, ϱ_0 — the density per unit area and R_y — the interaction between the plate and the rib. Replacing the differential equation (2.25) with a difference equation we obtain

$$\Delta_y^4 W_y - \tau_0 W_y = -\beta R_y, \qquad \tau_0^2 = \frac{\varrho A\omega^2}{B}\Delta y^4, \qquad \beta = \frac{\Delta y^4}{N}. \tag{2.26}$$

The solution of the Eq. (2.26) has, assuming simple end supports, the form

$$W_y = -\beta\sum_{\mu}\frac{b_\mu Y_y^\mu}{D_\mu}, \quad b_\mu = \sum_{y} R_y Y_y^\mu\,. \tag{2.27}$$

From the compatibility condition of the plate and the rib in the section $x = \xi$, we obtain the following equation for b_μ

$$b_\mu\left[\varkappa\sum_{\nu}\frac{(X_\xi^\nu)^2}{D_{\nu\mu}}+\frac{\beta}{D_\mu}\right]+\varkappa\sum_{\nu}\frac{q_{\nu\mu}X_\xi^\nu}{D_{\nu\mu}} = 0. \tag{2.28}$$

The value of b_μ being known, the deflection of the plate is found from the Eq. (2.15), where $q^*_{\nu\mu}$ is given by the Eq. (2.18). The deflection of the rib will be determined from the Eq. (2.27).

With increasing rigidity of the rib ($b \to \infty$), the Eq. (2.28) becomes (2.19). In the case of natural vibration of a plate stiffened with a rib, $q_{\nu\mu} = 0$ should be assumed in (2.28).

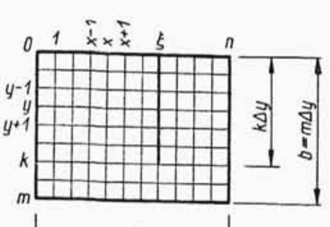

Fig. 1

Let us consider the case of free vibration of a plate simply supported on the entire contour and having an additional support along the segment $c_1 = k\Delta y$, $k < m$, of the line $x = \xi$ (Fig. 1).

If at the point (ξ, η) there acts a concentrated force $Pe^{i\omega t}$ the deflection amplitude is

$$w_{xy} = \varkappa P\sum_{\nu,\mu}\frac{X_\xi^\nu Y_\eta^\mu}{D_{\nu\mu}}X_x^\nu Y_y^\mu\,. \tag{2.29}$$

Requiring that the deflection amplitude at the points $y = 1, 2, \ldots, k$ along the line $x = \xi$ be zero, we obtain, making use of (2.29), the equation

(2.30) $$\sum_{\eta=1}^{k} P_\eta \sum_{\nu,\mu} \frac{(X_\xi^\nu)^2}{D_{\nu\mu}} Y_\xi^\mu Y_y^\mu = 0 \qquad (y = 1, 2, \ldots, k).$$

Thus, k equations, homogeneous in P_η, have been obtained. Setting the determinant of (2.30) equal to zero, we obtain the vibration condition. In the particular case of $k = m$, the condition of free vibration assumes the form of the Eq. (2.20). Indeed, on expanding P_η in a series of eigenfunctions

(2.31) $$P_\eta = \sum_j^m C_j Y_\eta^j$$

and substituting in (2.30) (where the summation is done from 1 to m) and making use of the relations

(2.32) $$\sum_{\eta=1}^{n} Y_\eta^j Y_\eta^\mu = \delta_{j\mu}, \qquad \sum_j C_j \delta_{j\mu} = C_\mu,$$

the Eq. (2.20) is obtained from (2.30).

Consider now the case of a plate clamped along the edge $x = 0$ and simply supported on the remaining edges. This problem will be solved thus. Let a linear load $P_y \delta_{x\xi} e^{i\omega t}$ act along the line $x = \xi$ and a linear load $-P_y \delta_{x,\xi-1} e^{i\omega t}$ along the line $x = \xi - 1$. Let the plate be simply supported on the entire contour. The deflection amplitude of the plate is expressed thus

(2.33) $$w_{xy} = \varkappa \sum_{\nu,\mu} \frac{b_\mu (X_\xi^\nu - X_{\xi-1}^\nu)}{D_{\nu\mu}} X_x^\nu Y_y^\mu .$$

Let us assume now that $\xi = 0$. Thus the couple $P_y \Delta x$ has been shifted to the edge of the plate. Since $X_0^\nu = 0$, therefore

(2.34) $$w_{xy}^P = -\varkappa \sum_{\nu,\mu} \frac{b_\mu X_{-1}^\nu}{D_{\nu\mu}} X_x^\nu Y_y^\mu .$$

The clamping condition will be realized by requiring that the deflection amplitude due to the load $q_{xy} e^{i\omega t}$ and the couple $P_y \Delta x e^{i\omega t}$ be zero on the line $x = -1$. This leads to the equation

(2.35.1) $$-b_\mu \sum_\nu \frac{(X_{-1}^\nu)^2}{D_{\nu\mu}} + \sum_\nu \frac{q_{\nu\mu} X_{-1}^\nu}{D_{\nu\mu}} = 0 .$$

Since $X_{-1}^\nu = -X_1^\nu$, therefore the Eq. (2.35) may be given the form

(2.35.2) $$b_\mu \sum_\nu \frac{(X_1^\nu)^2}{D_{\nu\mu}} + \sum_\nu \frac{q_{\nu\mu} X_1^\nu}{D_{\nu\mu}} = 0 .$$

From this equation b_μ can be found.

The knowledge of the coefficients b_μ enables the determination of the function P_y from the equation

$$P_y = \sum_{\mu} b_\mu Y_y^\mu .$$

If there is no excitation, then, for $q_{\nu\mu} = 0$, we obtain from (2.35.2)

$$\sum_{\nu} \frac{(X_1^\nu)^2}{D_{\nu\mu}} = 0 \tag{2.36}$$

as a condition of free vibration of the plate clamped along the edge $x = 0$, the remaining edges being simply supported. Let us consider, finally, the case of mixed boundary conditions. Let the plate be clamped at the points $y = 0, 1, 2, \ldots, k$ of the edge $x = 0$, the remaining points being simply supported. The deflection of the plate loaded by the load $q_{xy}e^{i\omega t}$ and the moments $P_y \Delta y e^{i\omega t}$ along the segment $k\Delta_y$ of edge $x = 0$ will be expressed by the equation

$$w_{xy} = \varkappa \sum_{\nu,\mu} \frac{q_{\nu\mu} X_x^\nu Y_y^\mu}{D_{\nu\mu}} - \varkappa \sum_{\eta=1}^{k} P_\eta \sum_{\nu,\mu} \frac{X_{-1}^\nu Y_\eta^\mu}{D_{\nu\mu}} X_x^\nu Y_y^\mu . \tag{2.37}$$

From the clamping condition $w_{-1,y} = 0$, we obtain the system of equations

$$\sum_{\nu,\mu}^{n,m} \frac{q_{\nu\mu} X_1^\nu Y_y^\mu}{D_{\nu\mu}} + \sum_{\eta=1}^{k} P_\eta \sum_{\nu,\mu}^{n,m} \frac{(X_1^\nu)^2}{D_{\nu\mu}} Y_\eta^\nu Y_y^\mu = 0 \qquad (y = 1, 2, \ldots, k), \tag{2.38}$$

from which the quantities P_η $(\eta = 1, 2, \ldots, k)$ can be found. Knowing these we obtain from (2.37) the sought-for deflection amplitude of the plate.

If the plate is clamped along the entire edge $x = 0$ of the plate, and therefore if $k = n$, the Eq. (2.38) will reduce to Eq. (2.35.2). This can easily be shown, by assuming that $P_\eta = \sum_j b_j Y_\eta^j$ and making use of the relations

$$\sum_{\eta=1}^{n} Y_\eta^j Y_\eta^\mu = \delta_{j\mu}, \qquad \sum_j b_j \delta_{j\mu} = b_\mu .$$

Consider a rectangular plate with the edges $x = 0$, $y = 0$ clamped, the remainig edges being simply supported. Generalizing the method described to the case of a plate clamped along the edge $x = 0$ the deflection amplitude of the plate may be represented in the form

$$w_{xy} = \varkappa \sum_{\nu,\mu} \frac{X_x^\nu Y_y^\mu}{D_{\nu\mu}} (q_{\nu\mu} - b_\mu X_{-1}^\nu - c_\nu Y_{-1}^\mu) . \tag{2.39}$$

From the clamping conditions of the edge $w_{-1,y} = 0$, $w_{x,-1} = 0$ we obtain the system of equations

(2.40) $$\sum_{\nu} \frac{X^{\nu}_{-1}}{D_{\nu\mu}} (q_{\nu\mu} - b_{\mu} X^{\nu}_{-1} - c_{\nu} Y^{\mu}_{-1}) = 0,$$

(2.41) $$\sum_{\mu} \frac{Y^{\mu}_{-1}}{D_{\nu\mu}} (q_{\nu\mu} - b_{\mu} X^{\nu}_{-1} - c_{\nu} Y^{\mu}_{-1}) = 0,$$

from which the quantities b_μ and c_ν can be found. Knowing these, we shall determine the forces R_y and Q_x constituting the support reactions along the lines $x = -1$, $y = -1$.

For,

(2.42) $$P_y = \sum_{\mu} b_{\mu} Y^{\mu}_{y}, \qquad Q_x = \sum_{\nu} c_{\nu} X^{\nu}_{x}.$$

This solution method may be generalized to the case of a plate with three or four edges clamped. Another solution method may be devised in the case of a plate with two adjacent edges clamped. This method will be discussed by means of the example of a plate clamped along the entire contour.

Let us take a complete set of orthonormal functions $[\eta^i_x]$ $(i = 0, 1, \ldots, m)$ and a set of functions $[\xi^k_y]$ $(k = 0, 1, 2, \ldots, m)$ satisfying the difference equation

(2.43) $$\Delta^4_x(\eta^i_x) = \gamma_i \eta^i_x, \qquad \Delta^4_y(\xi^k_y) = \vartheta_k \xi^k_y,$$

and the clamping conditions. The functions η^i_x may be treated as the natural vibration models of a bar with its ends clamped, the derivatives in the deflection equation being replaced with difference quotients.

Let us expand the deflection w_{xy} and the load q_{xy} in series of functions η^i_x, ξ^k_y

(2.44) $$w_{xy} = \sum_{i,k} A_{ik} \eta^i_x \xi^k_y, \qquad q_{xy} = \sum_{i,k} q_{ik} \eta^i_x \xi^k_y, \qquad q_{ik} = \sum_{x,y} q_{xy} \eta^i_x \xi^k_y.$$

Let us insert the deflection and the load thus expressed in the difference equations of the plate (2.3).

We find

(2.45) $$\sum_{i,k} A_{ik}[(\gamma_i + \vartheta_k \varepsilon^4 - \tau^2)\eta^i_x \xi^k_y + 2\varepsilon^2 \Delta^2_x(\eta^i_x)\Delta^2_y(\xi^k_y)] = \sum_{i,k} q_{ik} \eta^i_x \xi^k_y.$$

Let us expand the expression $\Delta^2_x(\eta^i_x)\Delta^2_y(\xi^k_y)$ in a series of eigenfunctions

$$\Delta^2_x(\eta^i_x)\Delta^2_y(\xi^k_y) = \sum_{\nu,\mu} c_{ik\nu\mu} \eta^i_x \xi^k_y,$$

$$c_{ik\nu\mu} = \sum_{x,y} \Delta^2_x(\eta^i_x)\Delta^2_y(\xi^k_y) \eta^i_x \xi^k_y,$$

and substitute it in (2.45). As a result, the following system of equations is obtained

$$A_{ik}(\gamma_i+\vartheta_k\varepsilon^4-\tau^2)+2\varepsilon^2\sum_{\nu,\mu}c_{\nu\mu ik}A_{\nu\mu}^{\square}=q_{ik}, \tag{2.46}$$

from which the quantities A_{ik}, can be found to be used later for the determination of the function w_{xy} in Eq. (2.46).

In the case of free vibration, $q_{ik}=0$ should be assumed in equation (2.46). Thus, a system of equations homogeneous in A_{ik} is obtained. Setting its determinant equal to zero, we obtain the free vibration condition from which the successive frequencies can be found.

3. Combined Bending and Compression of a Plate Buckling of a Rectangular Plate

Similar to, but more involved than the problem of forced vibration of a plate is that of combined bending and compression. This problem is governed by the differential equation of plate deflection

$$N\nabla^4 w+\bar{p}\frac{\partial^2 w}{\partial x^2}+\bar{t}\frac{\partial^2 w}{\partial y^2}+2\bar{s}\frac{\partial^2 w}{\partial x\partial y}=\bar{q}, \tag{3.1}$$

where $w, \bar{p}, \bar{t}, \bar{q}, \bar{s}$ are, in general, functions of both x and y.

Replacing the derivatives by difference quotients, the Eq. (3.1) is reduced to the following difference equation

$$L_{xy}(w_{xy})+D_{xy}(w_{xy})=q_{xy}, \tag{3.2}$$

where

$$L_{xy}=\Delta_x^4+2\varepsilon^2\Delta_x^2\Delta_y^2+\varepsilon^4\Delta_y^4, \qquad \varepsilon=\Delta_x/\Delta_y,$$
$$D_{xy}=p_{xy}\Delta_x^2+t_{xy}\Delta_y^2+2s_{xy}\Delta_x\Delta_y,$$

and

$$p_{xy}=\bar{p}_{xy}\frac{\Delta x^2}{N}, \quad t_{xy}=\bar{t}_{xy}\frac{\Delta x^2\varepsilon^2}{N}, \quad s_{xy}=\bar{s}_{xy}\frac{\varepsilon\Delta x^2}{N}, \quad q_{xy}=\bar{q}_{xy}\frac{\Delta x^4}{N}.$$

The solution of Eq. (3.2) will be sought for in the form of a double finite sum

$$w_{xy}=\sum_{\nu,\mu}A_{\nu\mu}\varphi_{xy}^{\nu\mu}, \tag{3.3}$$

where the eigenfunctions $\varphi_{xy}^{\nu\mu}$ satisfy the equation

$$L_{xy}(\varphi_{xy}^{\nu\mu})=\sigma_{\nu\mu}\varphi_{xy}^{\nu\mu} \tag{3.4}$$

with the same boundary conditions as the function w_{xy}.

From the requirement that the function $(L_x+D_x)w_{xy}-q_{xy}$ be orthogonal to every function $\varphi_{xy}^{\nu\mu}$, we obtain $n+m$ conditions

$$\sum_{x,y}[(L_{xy}+D_{xy})\sum_{\nu,\mu}A_{\nu\mu}\varphi_{xy}^{\nu\mu}-q_{xy}]\varphi_{xy}^{ik}=0. \tag{3.5}$$

Bearing in mind (3.4), and the fact that the functions $\varphi_{xy}^{\nu\mu}$ are orthonormal, let us transform the Eq. (3.5) to obtain

$$(3.6) \qquad A_{ik}\sigma_{ik} + \sum_{\nu,\mu} A_{\nu\mu} b_{ik\nu\mu} = q_{ik} \qquad (i = 0, 1, \dots, n;\ k = 0, 1, \dots, m),$$

where

$$(3.7) \qquad b_{ik\nu\mu} = \sum_{x,y} \varphi_{xy}^{ik} D_{xy}(\varphi_{xy}^{\nu\mu}), \qquad q_{ik} = \sum_{x,y} q_{xy}\varphi_{xy}^{ik}.$$

Thus, a system of non-homogeneous equations is obtained, from which A_{ik} will be determined. The knowledge of these quantities enables the determination of the deflection of the plate according to the Eq. (3.3).

If $q_{xy} = 0$, we are concerned with a buckling problem. The critical values will be obtained by setting the determinant of the non-homogeneous equations

$$(3.8) \qquad A_{ik}\sigma_{ik} + \sum_{\nu,\mu} A_{\nu\mu} b_{ik\nu\mu} = 0$$

equal to zero.

The solution of the system of equations (3.6) is simplified considerably in a number of particular cases. We proceed now to discuss two of them.

Let us assume that $q_{xy} = q = \text{const}$, $t_{xy} = t = \text{const}$, $s_{xy} = 0$ and that the plate is simply supported on the entire contour. In this particular case we have

$$\varphi_{xy}^{\nu\mu} = X_x^{\nu} Y_y^{\mu}, \quad \Delta_x^2(X_x^{\nu}) = a_\nu X_x^{\nu}, \quad \Delta_y^2(Y_y^{\mu}) = b_\mu Y_y^{\mu},$$

where

$$a_\nu = 2(\cos\alpha_\nu - 1), \quad b_\mu = 2(\cos\beta_\mu - 1), \quad \alpha_\nu = \frac{\nu\pi}{n}, \quad \beta_\mu = \frac{\mu\pi}{m}.$$

Therefore

$$D_{xy}(\varphi_{xy}^{\nu\mu}) = (pa_\nu + tb_\mu)\varphi_{xy}^{\nu\mu},$$

$$\sum_{x,y}^{n,m} \varphi_{xy}^{ik} D_{xy}(\varphi_{xy}^{\nu\mu}) = (pa_\nu + tb_\mu)\delta_{i\nu}\delta_{k\mu} = b_{ik\nu\mu}.$$

Substituting the final expression in (3.6) we obtain the simplified system of equations

$$(3.9) \qquad A_{ik}\varkappa_{ik} = q_{ik},$$

where

$$\varkappa_{ik} = \sigma_{ik} + pa_\nu + tb_\mu.$$

In the buckling problem the equation

$$(3.10) \qquad \varkappa_{ik} = 0$$

is the buckling condition.

In the case of simultaneous bending and compression, the deflection is expressed by the series

$$w_{xy} = \sum_{\nu, \mu} \frac{q_{\nu\mu}}{\varkappa_{\nu\mu}} \varphi_{xy}^{\nu\mu} . \tag{3.11}$$

This equation has a form analogous to the Eq. (2.13) relating to forced vibration of the plate. By means of considerations analogous to those of the former section we can, taking as the point of departure Eq. (3.11), analyse the cases of additional point and linear supports within the plate region, cases of clamping of one and more edges, and those of mixed boundary conditions.

Consider now the case of a plate compressed by a concentrated force P along the line $y = \eta$. It is assumed, here also, that the plate is simply supported on the contour. Then

$$p_{xy} = p\delta_{y\eta} , \quad t_{xy} = 0, \quad s_{xy} = 0 .$$

Next,

$$D_{xy}(\varphi_{xy}^{\nu\mu}) = p\delta_{y\eta} Y_y^{\mu} \Delta_x^2 (X_x^{\nu}) = pa_{\nu}\delta_{y\eta} Y_y^{\mu} X_x^{\nu} .$$

Therefore

$$\sum_{x, y} \varphi_{xy}^{ik} D_{xy}(\varphi_{xy}^{\nu\mu}) = pa_{\nu}\delta_{i\nu} Y_{\eta}^{\mu} Y_{\eta}^{k} ,$$

because

$$\sum_{x} X_x^{i} X_x^{\nu} = \delta_{i\nu} , \qquad \sum_{y} Y_y^{k} Y_y^{\mu} \delta_{y\eta} = Y_{\eta}^{k} Y_{\eta}^{\mu} .$$

The Eq. (3.8) takes the form:

$$A_{ik} + \frac{pa_i}{\sigma_{ik}} Y_{\eta}^{k} \sum_{\mu} A_{i\mu} Y_y^{\mu} = 0 . \tag{3.12}$$

Let us multiply this by Y_{η}^{k}, and sum up with respect to k. Then

$$\sum_{k} A_{ik} Y_{\eta}^{k} + pa_i \sum_{k} \frac{(Y_{\eta}^{k})^2}{\sigma_{ik}} \sum_{\mu} A_{i\mu} Y_y^{\mu} = 0 . \tag{3.13}$$

Observe that the first and the last sum in this equation is the same. Rejecting these sums, we obtain the following expression for the critical force p

$$p + \left[a_i \sum_{k} \frac{(Y_{\eta}^{k})^2}{\sigma_{ik}} \right]^{-1} = 0 , \tag{3.14}$$

which is the buckling condition.

The above problem may be generalized to the case of more concentrated forces along the x-axis, therefore we can obtain an approximate solution in the case where p_{xy} depends on y only.

Another way of solving the difference equation (3.2) by means of double series consists in assuming a complete set of orthonormal functions η_x^i, ξ_y^k satisfying the equations

$$(3.15) \qquad \Delta_x^4(\eta_x^i) = \gamma_i \eta_x^i, \qquad \Delta_y^4(\xi_y^k) = \vartheta_k \xi_y^k,$$

and the same boundary conditions as the functions w_{xy}. Expanding the deflection and the load in series of functions η_x^i, ξ_y^k

$$(3.16) \qquad w_{xy} = \sum_{i,k} A_{ik} \eta_x^i \xi_y^k, \qquad q_{xy} = \sum_{i,k} q_{ik} \eta_x^i \xi_y^k, \qquad q_{ik} = \sum_{x,y} q_{xy} \eta_x^i \xi_y^k,$$

and substituting in the Eq. (3.2), we obtain the following system of equations

$$(3.17) \qquad A_{ik}(\gamma_i + \varepsilon^4 \vartheta_k) + \sum_{\nu,\mu} A_{\nu\mu} a_{\nu\mu ik} = q_{ik},$$

where

$$a_{\nu\mu ik} = \sum_{x,y} [2\varepsilon^2 \Delta_x^2(\eta_x^\nu) \Delta_y^2(\xi_y^\mu) + p_{xy} \Delta_x^2(\eta_x^\nu) \xi_y^\mu + t_{xy} \Delta_y^2(\xi_y^\mu) \eta_x^\nu + 2s_{xy} \Delta_x(\eta_x^i) \Delta_y(\xi_y^k)] \eta_x^i \xi_y^k .$$

In the particular case $p_{xy} = p = \text{const}$, $t_{xy} = t = \text{const}$, $s_{xy} = 0$ and assuming that the plate is simply supported on the contour, the system of equations (3.17) becomes (3.9).

Another solution method of Eq. (3.2) may be given in the case of one or more edges clamped. Let us consider first a plate with all the edges clamped, acted on by the forces p_{xy}, t_{xy}, s_{xy}.

Let us represent the solution of the homogeneous equation (3.2) (the case of buckling) in the form

$$(3.18) \qquad w_{xy} = -\sum_{\xi,\eta} w^*_{\xi\eta xy} D_{\xi\eta}(w_{\xi\eta}),$$

where $w^*_{xy\xi\eta}$ is the Green's function satisfying the equation

$$(3.19) \qquad L_{xy}(w^*_{xv\xi\eta}) = \delta_{x\xi} \delta_{y\eta}$$

with the same boundary conditions as the deflection w_{xy}.

A solution of the Eq. (3.19) is the function

$$w^*_{xy\xi\eta} = \sum_{\nu,\mu} \frac{\varphi_{\xi\eta}^{\nu\mu}}{\sigma_{\nu\mu}} \varphi_{xy}^{\nu\mu},$$

where the functions $\varphi_{xy}^{\nu\mu}$ satisfy the Eq. (3.4), and for simply supported edges take the form $\varphi_{xy}^{\nu\mu} = X_x^\nu Y_y^\mu$.

The solution of (3.18) will be sought for in the form

$$(3.20) \qquad w_{xy} = \sum_{i,k} A_{ik} \varphi_{xy}^{ik}.$$

Substituting (3.20) in both members of (3.18), and rearranging and changing the summation order, we obtain finally the system of equations

$$A_{ik}+\frac{1}{\sigma_{ik}}\sum_{\nu,\mu}A_{\nu\mu}\sum_{\xi,\eta}\varphi^{ik}_{\xi\eta}D_{\xi\eta}(\varphi^{\nu\mu}_{\xi\eta})=0\,, \tag{3.21.1}$$

or

$$A_{ik}+\frac{1}{\sigma_{ik}}\sum_{\nu,\mu}A_{\nu\mu}b_{ik\nu\mu}=0\,. \tag{3.21.2}$$

This is identical with the system of equations (3.8).

Consider now a plate with the edge $x=0$ clamped. The solution of the Eq. (3.2) will have the form

$$w_{xy}=-\sum_{\xi,\eta}w^{*}_{\xi\eta xy}D_{\xi\eta}(w_{\xi\eta})+w^{P}_{xy}\,, \tag{3.22}$$

where w^{P}_{xy} denotes the load of the plate along the edge $x=0$ by the couple P_y. Making use of the Eq. (2.34) where $\tau=0$ is substituted (the problem being static) we have

$$w^{P}_{xy}=-\varkappa\sum_{i,k}\frac{d_k X^{i}_{-1}}{\sigma_{ik}}\varphi^{ik}_{xy},\qquad d_k=\sum_{y}P_y Y^{k}_{y}\,. \tag{3.23}$$

The solution will be sought-for, as before, by means of the series (3.20). Requiring that the edge $x=0$ be clamped, we obtain the additional condition $w_{-1,\,y}=0$ or

$$\sum_{i,k}A_{ik}X^{i}_{-1}Y^{k}_{y}=0\,,\qquad\text{hence}\qquad\sum_{i}A_{ik}X^{i}_{-1}=0\,. \tag{3.24}$$

Substituting (3.20) in (3.22), we obtain, after some simple rearrangement the system of equations:

$$A_{ik}=-\frac{1}{\sigma_{ik}}\sum_{\nu,\mu}A_{\nu\mu}b_{ik\nu\mu}-\varkappa\frac{d_k X^{i}_{-1}}{\sigma_{ik}}\,. \tag{3.25}$$

Multiplying (3.25) by X^{i}_{-1}, and summing up with respect to i we obtain the equation

$$\sum_{i}A_{ik}X^{i}_{-1}=0=-\sum_{i}\frac{X^{i}_{-1}}{\sigma_{ik}}\sum_{\nu,\mu}A_{\nu\mu}b_{ik\nu\mu}-\varkappa d_k\sum_{i}\frac{(X^{i}_{-1})^2}{\sigma_{ik}}\,. \tag{3.26}$$

From the last equation we shall determine the quantity d_k. Replacing in (3.26) the summation with respect to i with the summation with respect to r and substituting d_k in the Eq. (3.25), we obtain the final system of equations

$$A_{ik}+\frac{1}{\sigma_{ik}}\sum_{\nu,\mu}A_{\nu\mu}\left[b_{ik\nu\mu}-X^{i}_{-1}\frac{\sum\limits_{r}\frac{b_{rk\nu\mu}X^{r}_{-1}}{\sigma_{rk}}}{\sum\limits_{r}\frac{(X^{r}_{-1})^2}{\sigma_{rk}}}\right]=0\,. \tag{3.27}$$

Setting the determinant of (3.27) equal to zero, we obtain the buckling condition of the plate.

A particularly simple solution of the system of equations (3.27) is obtained if $b_{ik\nu\mu} = \delta_{i\nu}\delta_{k\mu}b_{ik}$. Eliminating in this case the quantities A_{ik} from the Eqs. (3.2) and (3.26), we obtain the relation

$$(3.28) \qquad \sum_{i}^{n-1} \frac{(X_{-1}^{i})^2}{\sigma_{ik}+b_{ik}} = 0 \qquad (k = 1, 2, \dots, n-1) .$$

It can easily be shown that $b_{ik\nu\mu} = \delta_{i\nu}\delta_{k\mu}b_{ik}$ appears only in the case where $p_{xy}^{*} = p = \text{const}$, $t_{xy} = t = \text{const}$ and $s_{xy} = 0$ over the entire region of the plate.

The above procedure can easily be generalized to the case of two, three or four edges clamped. The above problems of plate with one edge $x = 0$ clamped, the remaining edges being simply supported, can be solved by assuming the solution of (3.2) in the form

$$w_{xy} = -\sum_{\xi,\eta} w_{\xi\eta xy}^{*} D_{\xi\eta}(w_{\xi\eta}) ,$$

where the Green's function $w_{\xi\eta xy}^{*}$ concerns the plate with the edge $x = 0$ clamped, the other being simply supported.

Let us consider finally the buckling case of a two-span beam simply supported on the contour. The solution of (3.2) will be sought-for in the form

$$(3.29) \qquad w_{xy} = -\sum_{\xi,\eta} w_{\xi\eta xy}^{*} D_{\xi\eta}(w_{\xi\eta}) + w_{\xi\eta}^{R} ,$$

where w_{xy}^{R} denotes the deflection of the plate on the line $x = \xi$, produced by the load $R_y^{*}\delta_{x\xi}$ not yet known. In agreement with the considerations of Sec. 2 we have

$$(3.30) \qquad w_{xy}^{R} = \varkappa \sum_{i,k} \frac{X_{\xi}^{i} d_k}{\sigma_{ik}} \varphi_{xy}^{ik} , \qquad d_k = \sum_{y} R_y Y_y^{k} .$$

Assuming the deflection surface of the plate in the form of the series (3.20) we obtain from (3.29) the following system of equations

$$(3.31) \qquad A_{ik} = -\frac{1}{\sigma_{ik}} \sum_{\nu,\mu} A_{\nu\mu} b_{ik\nu\mu} + \varkappa d_k \frac{X_{\xi}^{i}}{\sigma_{ik}} .$$

Requiring that the deflection be zero on the line $x = \xi$, we have the additional condition

$$(3.32) \qquad \sum_{i,k} A_{ik} X_{\xi}^{i} Y_y^{k} = 0 , \quad \text{hence} \quad \sum_{i} A_{ik} X_{\xi}^{i} = 0 .$$

From the latter equation we find the quantity d_k and insert it in the Eq. (3.31). As a result the following system of equations is obtained

$$A_{ik}+\frac{1}{\sigma_{ik}}\sum_{\nu,\mu}A_{\nu\mu}\left[b_{ik\nu\mu}-X_\xi^i\frac{\sum_r\frac{b_{rk\nu\mu}X_\xi^r}{\sigma_{rk}}}{\sum_r\frac{(X_\xi^r)^2}{\sigma_{rk}}}\right]=0\,. \tag{3.33}$$

Setting the determinant of these equations equal to zero we obtain the buckling condition for a two-span plate. In the particular case, where $q_{xy}=\text{const}$, $t_{xy}=\text{const}$, $s_{xy}=0$, we obtain $b_{ik\nu\mu}=\delta_{i\nu}\delta_{k\mu}b_{ik}$. Substituting this in the Eqs. (3.31) and (3.32) and eliminating A_{ik}, we obtain the buckling condition in the form

$$\sum_i^{n-1}\frac{(X_\xi^i)^2}{\sigma_{ik}+b_{ik}}=0 \quad (k=1,2,\ldots,m-1)\,. \tag{3.34}$$

If, in all the cases considered in Sec. 2 and 3, we pass to the limit for $m\to\infty$, $n\to\infty$ $\left(\lim\limits_{\substack{m\to\infty\\ \Delta y\to 0}} m\Delta y=b,\ \lim\limits_{\substack{n\to\infty\\ \Delta x\to\infty}} n\Delta x=a\right)$ we obtain the solution of the differential equation (2.1) or (3.1).

4. Approximate Solution of Problems of the Theory or Rectangular Plates

Let us consider the difference equation of plate deflection

$$H_{xy}(w_{xy})=\varkappa q_{xy}\,. \tag{4.1}$$

In this equation H_{xy} is a linear difference operator, which, in the case of forced vibration considered in Sec. 2, has the form $H_{xy}=L_{xy}-\tau^2$ and in the case of simultaneous bending and compression has the form $H_{xy}=L_{xy}+D_{xy}$.

Let us take a complete set of orthonormal functions $[\chi_{xy}^{\nu\mu}]$ $(\nu=0,1,2,\ldots,n;$ $\mu=0,1,2,\ldots,m)$ satisfying the boundary conditions of the Eq. (4.1), but not necessarily the deflection equation of the plate. Such an orthogonal set may for instance be constituted by the functions $\eta_x^\nu\xi_y^\mu=\chi_{xy}^{\nu\mu}$ of Sec. 2, satisfying the system of equations (2.43).

Let us assume the approximate solution of the Eq. (4.1) in the form of the finite series

$$\overline{w}_{\nu y}=\sum_{\nu,\mu}^{j,f}B_{x\mu}\chi_{xy}^{\nu\mu} \qquad (\nu=0,1,2,\ldots,j;\ \mu=0,1,2,\ldots,f;\ j<m;\ f<m)\,, \tag{4.2}$$

where $B_{\nu\mu}$ are unknown coefficients whose determination is our aim.

The requirement that $\overline{w}_{xy}$ should satisfy the Eq. (4.1) is identical with the orthogonality condition of the function $H_{xy}(\overline{w}_{xy})-q_{xy}$ with each of the functions $\chi_{xy}^{\nu\mu}$ $(\nu=0,1,2,\ldots,n;\ \mu=0,1,2,\ldots,m)$. Since $(j+f)$ functions $\chi_{xy}^{\nu\mu}$ are available, therefore the approximate solution of the Eq. (4.1) will consist in satisfying the $(j+f)$ orthogonality conditions

$$\sum_{x,y}^{n,m}\left[H_{xy}\left(\sum_{\nu,\mu}^{j,f}B_{\nu\mu}\chi_{xy}^{\nu\mu}\right)-\varkappa q_{xy}\right]\chi_{xy}^{ik}=0\,, \tag{4.3}$$

which leads to a system of $(j+f)$ non-homogeneous equations

$$\sum_{\nu,\mu}^{j,f}B_{\nu\mu}b_{ik\nu\mu}=\varkappa q_{ik} \quad (i=0,1,2,\ldots,j;\ k=0,1,2,\ldots,f)\,, \tag{4.4}$$

where

$$b_{ik\nu\mu}=\sum_{x,y}^{n,m}\chi_{xy}^{ik}H_{xy}(\chi_{xy}^{\nu\mu})\,,\qquad q_{ik}=\sum_{x,y}^{n,m}q_{xy}\chi_{xy}^{ik}\,.$$

From the system of equations (4.2) we find the quantities $B_{\nu\mu}$ and from the relation (4.2) the approximate solution of Eq. (4.1). In the particular case where $q_{xy}=0$ (the problem of free vibration or buckling) we obtain a homogeneous system of equations (4.4). Setting equal to zero the determinant of this system of equations we obtain the condition of free vibration or buckling.

The procedure just presented constitutes a generalization of the B. G. Galerkin's orthogonalization method to difference equations.

Let us observe that if the functions $\chi_{xy}^{\nu\mu}$ assumed satisfy the equation

$$H_{xy}(\chi_{xy}^{\nu\mu})=\tau_{\nu\mu}\chi_{xy}^{\nu\mu} \tag{4.5}$$

and the same boundary conditions as the functions w_{xy}, then

$$b_{ik\nu\mu}=\tau_{\nu\mu}\delta_{i\nu}\delta_{k\mu}$$

and the system of equations (4.4) will be simplified considerably. We obtain

$$B_{ik}\tau_{ik}=\varkappa q_{ik} \quad (i=0,1,2,\ldots,n;\quad k=0,1,2,\ldots,f)\,. \tag{4.6}$$

If a complete set of functions $\chi_{xy}^{\nu\mu}$ satisfying the Eq. (4.5) is assumed, the system of equations

$$B_{ik}\tau_{ik}=\varkappa q_{ik} \tag{4.7}$$

constitutes the accurate solution. In this case the deflection of the plate is expressed by the equation

$$w_{xy}=\sum_{\nu,\mu}^{n,m}B_{\nu\mu}\chi_{xy}^{\nu\mu}\,. \tag{4.8}$$

Let us consider two simple examples of application of the orthogonalization method just presented.

(a) Let the plate be acted on by an excitation load

$$q_{xy}e^{i\omega t}\,,\qquad \text{where}\qquad q_{xy}=q=\text{const.}$$

Let the plate be simply supported on the entire contour. In this case we have $H_{xy} = L_{xy} - \tau^2$, according to the Eq. (2.3). The functions $\chi_{xy}^{\nu\mu}$ will be assumed in the form of products $X_x^\nu \cdot Y_y^\mu$, where

(4.9) $$X_x^\nu = \sqrt{\frac{2}{n}} \sin \alpha_\nu x\,, \quad Y_y^\mu = \sqrt{\frac{2}{m}} \sin \beta_\mu y\,, \quad \alpha_\nu = \frac{\nu\pi}{n}, \quad \beta_\mu = \frac{\mu\pi}{m}.$$

These functions are orthonormal and satisfy the equation

(4.10) $$L_{xy}(\chi_{xy}^{\nu\mu}) = \sigma_{\nu\mu}\chi_{xy}^{\nu\mu}\,.$$

Substituting (4.9) in (4.10) we obtain

(4.11) $$\sigma_{\nu\mu} = (a_\nu + \varepsilon^2 b_\mu)^2, \quad a_\nu = 2(\cos\alpha_\nu - 1), \quad b_\mu = 2(\cos\beta_\mu - 1)\,.$$

Next, the quantities q_{ik} are found

(4.12) $$q_{ik} = q \sum_{x,\,y}^{n-1,\,m-1} X_x^i\, Y_y^k = \frac{2q}{\sqrt{nm}} \operatorname{ctg}\frac{i\pi}{2n} \operatorname{ctg}\frac{k\pi}{2m}\,.$$

Let us observe that from the equation

$$H_{xy}(\chi_{xy}^{\nu\mu}) = \tau_{\nu\mu}\chi_{xy}^{\nu\mu}$$

the following expression is obtained

(4.13) $$\tau_{\nu\mu} = \sigma_{\nu\mu} - \tau^2\,.$$

Therefore, the approximate equation of the problem is

(4.14) $$\overline{w}_{xy} = \frac{4\varkappa q}{nm} \sum_{\nu=1,\mu=1}^{j,\,f} \frac{\operatorname{ctg}\frac{\nu\pi}{2n} \operatorname{ctg}\frac{\mu\pi}{2m}}{(a_\nu + \varepsilon^2 b_\mu)^2} \sin\alpha_\nu x \sin\beta_\mu y\,.$$

It can easily be seen that the approximate solution just obtained gives the first $(j+f)$ terms of the accurate solution in which the summation is performed from 1 to $n-1$ for ν and from 1 to $m-1$ for μ.

(b) Let us consider the buckling problem of the rectangular plate simply supported on the contour, assuming that $p_{xy} = p$ =const, $t_{xy} = 0$, $s_{xy} = 0$. Assuming also that

(4.15) $$\overline{w}_{xy} = B_{11} X_x^1 Y_y^1\,,$$

where the functions X_x^1 and Y_y^1 are given by the Eqs. (4.9) we obtain from the Eq. (4.4) the buckling condition in the form

(4.16) $$B_{11} b_{1111} = 0\,,$$

or $b_{1111} = 0$, because $B_{11} \neq 0$. This condition may be written thus

$$\sum_{x,\,y}^{n-1,\,m-1} X_x^1 Y_y^1 (L_{xy} + D_{xy})(X_x^1 Y_y^1) = 0\,.$$

Bearing in mind that

$$L_{xy}(X_x^1 Y_y^1) = (a_1 + \varepsilon^2 b_1) X_x^1 Y_y^1 , \quad D_{xy}(X_x^1 Y_y^1) = a_1 q X_x^1 Y_y^1 ,$$

where a_1, b_1 are given by the Eqs. (4.11) the buckling condition is obtained in the form

$$[(a_1 + \varepsilon^2 b_1) + q a_1] \sum_{x, y}^{n-1, m-1} (X_x^1)^2 (Y_y^1)^2 = 0 ,$$

or

$$q = -\frac{a_1 + \varepsilon^2 b_1}{a_1} . \tag{4.17}$$

The approximate solution coincides with the accurate one for one-half-wave buckling in the x- and y-direction.

5. Application of Simple Finite Series to the Solution of Plate Problems

Let us consider a rectangular plate simply supported on two opposite edges, and supported in any way on the remaining edges. Our considerations will be confined to the static case, although there is no obstacle to generalize them to the problem of forced vibration and simultaneous bending and compression. The solution of the difference equation of the plate

$$L_{xy}(w_{xy}) = \varkappa q_{xy} \quad (x = 0, 1, \ldots, n; \quad y = 0, 1, 2, \ldots, m) \tag{5.1}$$

will be sought-for (assuming that the edges $y = 0$, $y = m$ are simply supported) in the form of the finite simple series

$$w_{xy} = \sum_{\mu=1}^{m-1} Y_y^\mu X_x^\mu , \tag{5.2}$$

where

$$Y_y^\mu = \sqrt{\frac{2}{m}} \sin \beta_\mu y , \quad \beta_\mu = \frac{\mu \pi}{m} \quad (\mu = 1, 2, \ldots, m-1). \tag{5.3}$$

The series (5.2) will constitute the accurate solution of the Eq. (5.1) if the functions $L_{xy}(w_{xy}) - q_{xy}$ are orthogonal to each of the functions Y_y^μ.

The following conditions must be satisfied

$$\sum_{y=0}^{m-1} \left[L_{xy} \left(\sum_{\mu=1}^{m-1} Y_y^k X_x^k \right) - q_{xy} \varkappa \right] Y_y^\mu = 0 \quad (\mu = 1, 2, \ldots, m-1) . \tag{5.4}$$

The Eq. (5.4) is reduced to the form

$$\sum_{\mu=1}^{m-1} \sum_{y=1}^{m-1} Y_y^\mu L_{xy}(Y_y^k X_x^k) = \varkappa q_x^\mu , \quad q_x^\mu = \sum_{y=1}^{m-1} q_{xy} Y_y^\mu . \tag{5.5}$$

Bearing in mind that

$$L_{xy}(Y_y^k X_x^k) = Y_y^k(\Delta_x^4 + 2\varepsilon^2 b_k \Delta_x^2 + \varepsilon^4 b_k^2), \quad b_k = 2(\cos\beta_k - 1) \tag{5.6}$$

we obtain from the Eq. (5.5) the following ordinary difference equation

$$(\Delta_x^4 + 2\varepsilon^2 b_\mu \Delta_x^2 + \varepsilon^4 b_\mu^2) X_x^\mu = \varkappa q_x^\mu \quad (x = 0, 1, 2, \ldots, n;\ y = 0, 1, 2, \ldots, m-1). \tag{5.7}$$

This equations may also be represented in the form

$$X_{x-2}^\mu - c_\mu X_{x-1}^\mu + d_\mu X_x^\mu - c_\mu X_{x+1}^\mu + X_{x+2}^\mu = \varkappa q_x^\mu, \tag{5.8}$$

where

$$c_\mu = 4 - 2\varepsilon^2 b_\mu, \quad d_\mu = 6 - 2\varepsilon^2 b_\mu + \varepsilon^4 b_\mu^2.$$

The solution of the Eq. (5.8) is composed of a particular solution and the general solution of the homogeneous equation

$$X_x^\mu = \overline{X}_x^\mu + (C_1^\mu + x C_2^\mu)\lambda_\mu^x + (C_3^\mu + C_4^\mu x)\lambda_\mu^{-x}, \tag{5.9}$$

where

$$\lambda_\mu = (1 + \varepsilon^2 \varrho_\mu^2) - \varepsilon \varrho_\mu \sqrt{2 + \varepsilon^2 \varrho_\mu^2}, \quad \varrho_\mu = \left|\sin\frac{\beta_\mu}{2}\right|,$$

or

$$X_x^\mu = \overline{X}_x^\mu + A_1^\mu \operatorname{ch}\vartheta_\mu x + A_2^\mu x \operatorname{sh}\vartheta_\mu x + A_3^\mu x \operatorname{ch}\vartheta_\mu x + A_4^\mu \operatorname{sh}\vartheta_\mu x, \tag{5.10}$$

where

$$\vartheta_\mu = \ln \varrho_\mu.$$

The function $\overline{X}_x^\mu$ is a particular solution of the Eq. (5.7). From the boundary conditions (two for each edge) we find the constants $A_1^\mu, \ldots, A_4^\mu$ and X_x^μ. The knowledge of this function enables the deflection w_{xy} to be obtained from the Eq. (5.2). This procedure is of importance for plates, of which one edge is very long or infinite (an infinite strip). In the latter case the Eq. (5.8) will be assumed in the form

$$X_x^\mu = \overline{X}_x^\mu + (D_1^\mu + x D_2^\mu) e^{-\vartheta_\mu x}. \tag{5.11}$$

If the number of segments is small ($m < 10$) it is more convenient to treat the Eq. (5.8) as a system of algebraic equations.

Let us consider the case of a plate strip simply supported on the edges $y = 0$, $y = m$ and loaded by a concentrated force P at the point $(0, \eta)$. The solution of the Eq. (5.8) will be sought-for by making use of the Fourier integral transformation proposed by I. Babuška, [9], for difference equations.

The Fourier transformation in an infinite interval is defined by the infinite series

$$\mathcal{F}(X_x) = X^*(\alpha) = \sum_{x=-\infty}^{x=\infty} X_x e^{i\alpha x}, \tag{5.12}$$

where the sum $\sum_{x=-\infty}^{x=\infty} |X_x|$ should be bounded. The inverse Fourier transformation is defined by the equation

$$X_x = \mathcal{F}^{-1}(X_x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\alpha) e^{-i\alpha x} d\alpha . \tag{5.13}$$

From the definition (5.12) it follows, for instance, that the transformation of the Dirac function $\delta(x-\xi)$ is

$$\mathcal{F}[\delta(x-\xi)] = \sum_{x=-\infty}^{x=\infty} \delta(x-\xi) e^{i\alpha x} = e^{i\alpha\xi} . \tag{5.14}$$

In our consideration use will be made of the following rule:

$$\mathcal{F}(X_{x\pm p}) = \mathcal{F}(X_x) e^{\mp ip\alpha} . \tag{5.15}$$

Let us express the Eq. (5.8) in the form

$$X^\mu_{x-2} - c_\mu X^\mu_{x-1} + d_\mu X^\mu_x - c_\mu X^\mu_{x+1} + X^\mu_{x+2} = \varkappa Y^\mu_\eta P \delta(x) \tag{5.16}$$

in view of the relation

$$q^\mu_x = P \sum_{y=1}^{m-1} \delta(y-\eta) \delta(x) Y^\mu_y = P Y^\mu_\eta \delta(x) .$$

Performing on the Eq. (5.16) the Fourier transformation, we obtain, bearing in mind (5.12), (5.14) and (5.15)

$$[e^{2i\alpha} + e^{-2i\alpha} - c_\mu(e^{-i\alpha} + e^{i\alpha}) + d_\mu] X^{\mu *}(\alpha) = \varkappa Y^\mu_\eta P ,$$

or

$$[a(\alpha) + \varepsilon^2 b_\mu]^2 X^{\mu *}(\alpha) = \varkappa Y^\mu_\eta P, \quad a(\alpha) = 2(\cos\alpha - 1) .$$

Therefore

$$X^\mu_x = \frac{\varkappa P Y^\mu_\eta}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \alpha x d\alpha}{[a(\alpha) + \varepsilon^2 b_\mu]^2} . \tag{5.17}$$

Bearing in mind (5.2) the deflection of the plate can be represented thus

$$w_{xy} = \frac{\varkappa P}{2\pi} \sum_{\mu=1}^{m-1} Y^\mu_\eta Y^\mu_y \int_{-\pi}^{\pi} \frac{\cos \alpha x d\alpha}{[a(\alpha) + \varepsilon^2 b_\mu]^2} . \tag{5.18}$$

Making use of the result (5.18) a number of other problems may be solved. Thus, for a semi-strip acted on by a concentrated force at a point (ξ, η) we obtain, by superposition of two forces, a positive force at (ξ, η) and a negative force at $(-\xi, \eta)$, the expression

$$w_{xy} = \frac{\varkappa P}{\pi} \sum_{\mu=1}^{m-1} Y_y^\mu Y_\eta^\mu \int_{-\pi}^{\pi} \frac{\sin \alpha\xi \sin \alpha x d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2}. \tag{5.19}$$

Let an infinite strip simply supported on the edges $y = 0$ and $y = m$ be acted on by a force P at the points (ξ, η) and $(-\xi, \eta)$ and also by a distributed load $R_y \delta(x)$ along the line $x = 0$. The resultant deflection produced by these loads is

$$w_{xy} = \frac{\varkappa P}{\pi} \sum_{\mu=1}^{m-1} Y_\eta^\mu Y_y^\mu \int_{-\pi}^{\pi} \frac{\cos \alpha\xi \sin \alpha\xi d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2} + \frac{\varkappa}{2\pi} \sum_{\mu=1}^{m-1} f_\mu Y_y^\mu \int_{-\pi}^{\pi} \frac{\cos \alpha x d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2}, \quad f_\mu = \sum_{y=1}^{m-1} R_y Y_y^\mu. \tag{5.20}$$

From the condition of zero deflection in the cross-section $x = 0$ we find the coefficients f_μ:

$$f_\mu \int_{-\pi}^{\pi} \frac{d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2} + 2PY_\eta^\mu \int_{-\pi}^{\pi} \frac{\cos \alpha\xi d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2} = 0\,. \tag{5.21}$$

The quantities f_μ being known, we find the support reactions by the formula

$$R_y = \sum_{\mu=1}^{m-1} f_\mu Y_y^\mu. \tag{5.22}$$

Let now the plate strip be acted on by a load $q_{xy} = q_y$ (independent of y only) and by concentrated forces P_η along the segment $k\Delta y$, $(k < m)$ of the line $x = 0$. The deflection thus produced is

$$w_{xy} = \varkappa \sum_{\mu=1}^{m-1} \frac{q_\mu}{b_\mu^2} Y_y^\mu + \frac{\varkappa}{\pi} \sum_{\eta=1}^{k} P_\eta \sum_{\mu=1}^{m-1} Y_\eta^\mu Y_y^\mu \int_{-\pi}^{\pi} \frac{\cos \alpha x d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2}\,. \tag{5.23}$$

We require now that the deflection be equal to zero for $y = 1, 2, \ldots, \eta, \ldots, k$. From the condition $w_{0y} = 0$, we obtain

$$\begin{cases} \displaystyle\sum_{\mu=1}^{m-1} \frac{q_\mu}{b_\mu^2} Y_y^\mu + \frac{\varkappa}{\pi} \sum_{\eta=1}^{k} P_\eta \sum_{\mu=1}^{m-1} Y_\eta^\mu Y_y^\mu e_\mu = 0 \quad (y = 1, 2, \ldots, k), \\ \displaystyle e_\mu = \int_{-\pi}^{\pi} \frac{d\alpha}{[a(\alpha)+\varepsilon^2 b_\mu]^2}\,. \end{cases} \tag{5.24}$$

Thus, a system of k equations for the determination of the unknown quantities P_η has been obtained.

The Eq. (5.24) may be given still another form. Let us expand P_η in a series

$$P_\eta = \sum_{j=1}^{m-1} A_j Y_\eta^j \tag{5.25}$$

and insert in (5.24). We obtain

$$\sum_{j=1}^{m-1} A_j a_{j\mu} + \frac{q_\mu}{b_\mu^2 e_\mu} = 0 , \quad a_{j\mu} = \sum_{\eta=1}^{k} Y_\eta^j Y_\eta^k . \tag{5.26}$$

If the plate is supported over the entire width $b = m\Delta y$, we have $a_{j\mu} = \delta_{j\mu}$, and the Eq. (5.26) becomes

$$A_\mu = -\frac{q_\mu}{e_\mu b_\mu^2} \quad (\mu = 1, 2, \dots, m-1) . \tag{5.27}$$

The Eq. (5.26) constitutes the solution with mixed boundary conditions along the line $x = 0$. As the net becomes more dense along the line $x = 0$, the approximation to the reality becomes better and better. In the limit case for $m \to \infty$, the Eq. (5.24) becomes a Fredholm integral equation of the first kind, [10].

6. Application of Difference-Differential Equations to the Theory of Plates

In a number of plate problems, especially with mixed boundary conditions, it may be very useful to describe the deflection of the plate by means of a difference-differential equation. Let us divide the plate into m equal strips of width Δy in the direction of the y-axis. Let us denote the function, expressing the deflection along the lines $y = 0, 1, 2, \dots, m$ by $w_y(x)$. Then

$$\begin{cases} \dfrac{\partial^4 w_y(x)}{\partial x^4} + 2\varkappa^2 \dfrac{\partial^2}{\partial x^2}\left[\Delta_y^2\big(w_y(x)\big)\right] + \varkappa^4 \Delta_y^4\big(w_y(x)\big) = \dfrac{q_y(x)}{N}, \\ 0 \leqslant x \leqslant a , \quad y = 0, 1, 2, \dots, m , \quad \varkappa^2 = \dfrac{1}{\Delta y^2}, \quad \Delta y = \dfrac{b}{m} \end{cases} \tag{6.1}$$

is the difference-differential equation of deflection. Let us assume now that the edges $y = 0$, $y = m$ are simply supported.

Assuming the solution of (6.1) in the form

$$w_y(x) = \sum_{\mu=1}^{m-1} Y_y^\mu X_\mu(x), \quad Y_y^\mu = \sqrt{\frac{2}{m}} \sin \beta_\mu y , \quad \beta_\mu = \frac{\mu\pi}{m}, \tag{6.2}$$

expanding the load $q_y(x)$ in a series of functions Y_y^μ

$$q_y(x) = \sum_{\mu=1}^{m-1} q_\mu(x) Y_y^\mu , \quad q_\mu(x) = \sum_{y=1}^{m-1} q_y(x) Y_y^\mu , \tag{6.3}$$

and applying the orthogonalization method, we obtain the following ordinary linear differential equation

$$(6.4)\qquad \begin{cases} \left[\dfrac{d^4}{dx^4}-2\varkappa^2 c_\mu^2\dfrac{d^2}{dx^2}+\varkappa^4 c_\mu^4\right]X_\mu(x)=\dfrac{q_\mu(x)}{N}, \\ c_\mu^2=2(1-\cos\beta_\mu)=4\sin^2\dfrac{\beta_\mu}{2}. \end{cases}$$

The solution of this equation has the form

$$(6.5)\quad X_\mu(x)=\overline{X}_\mu(x)+C_{1,\mu}\operatorname{ch}\tau_\mu x+C_{2,\mu}x\operatorname{sh}\tau_\mu x+C_{3,\mu}\operatorname{sh}\tau_\mu x+C_{4,\mu}x\operatorname{ch}\tau_\mu x\,,$$

where $\overline{X}_\mu(x)$ is a particular integral of the Eq. (6.4) and

$$(6.6)\qquad \tau_\mu=\varkappa c_\mu=2\varkappa\left|\sin\frac{\beta_\mu}{2}\right|.$$

Let us consider an infinite plate strip acted on by a concentrated force P at the point $(0,\eta)$. Performing on the Eq. (6.4) the Fourier transformation, and bearing in mind the fact that $q_\mu(x)=PY_\eta^\mu\delta(x)$, we obtain

$$(6.7)\qquad X_\mu(x)=\frac{PY_\eta^\mu}{2\pi N}\int_0^\infty\frac{\cos\alpha x d\alpha}{(\alpha^2+\tau_\mu^2)^2},$$

or

$$(6.8)\qquad w_y(x)=\frac{P}{8N}\sum_{\mu=1}^{m-1}\frac{Y_\eta^\mu Y_y^\mu}{\tau_\mu^3}(1+x\tau_\mu)e^{-\tau_\mu x}\qquad x>0\,.$$

Introducing an auxiliary function

$$(6.9)\qquad \Phi_y(x)=-\frac{P}{4N}\sum_{\mu=1}^{m-1}\frac{Y_y^\mu Y_\eta^\mu}{\tau_\mu}e^{-\tau_\mu x}\qquad x>0\,,$$

we can express the quantities $\partial^2 w_y/\partial x^2$ and $\Delta_y^2(w_y)$ by the following simple equations

$$(6.10)\qquad \frac{\partial^2 w}{\partial x^2}=\frac{1}{2}\left(\Phi_y+x\frac{\partial\Phi_y}{\partial x}\right),\qquad \varkappa^2\Delta_y^2(w_y)=\frac{1}{2}\left(\Phi_y-x\frac{\partial\Phi_y}{\partial x}\right).$$

For an increasing number of strips $(m\to\infty)$, we obtain from (6.9)

$$(6.11)\qquad \begin{cases} \Phi_y(x)\to\Phi(x,\overline{y})=-\dfrac{P}{N\pi}\displaystyle\sum_{r=1}^{\infty}\frac{e^{-r\pi x/b}}{r}\sin\frac{r\pi\overline{\eta}}{b}\sin\frac{r\pi\overline{y}}{b}= \\ \qquad =\dfrac{P}{4\pi N}\ln\dfrac{\operatorname{ch}\dfrac{\pi x}{b}-\cos\dfrac{\pi}{b}(\overline{y}-\overline{\eta})}{\operatorname{ch}\dfrac{\pi x}{b}-\cos\dfrac{\pi}{b}(\overline{y}+\overline{\eta})},\qquad \lim\limits_{\substack{m\to\infty\\ \Delta y\to 0}}(y\Delta y)\to\overline{y} \end{cases}$$

being in agreement with the result obtained by NÁDAI, [11]. For $m \to \infty$ we have

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{2}\left(\Phi + x\frac{\partial \Phi}{\partial x}\right), \quad \frac{\partial^2 w}{\partial y^2} = \frac{1}{2}\left(\Phi - x\frac{\partial \Phi}{\partial x}\right). \tag{6.12}$$

Let the strip be acted on by a load $q_y(x) = q_y$ independent of x and forces $P_1, P_2, \ldots, P_\eta, \ldots, P_k$ at the nodes lying on the line $x = 0$. The resulting deflection is

$$w_y(x) = \frac{1}{N}\sum_{\mu=1}^{m-1} \frac{q_\mu}{\tau_\mu^4} Y_y^\mu + \frac{1}{8N}\sum_{\eta=1}^{k} P_\eta \sum_{\mu=1}^{m-1} \frac{Y_\eta^\mu Y_y^\mu}{\tau_\mu^3}(1 + x\tau_\mu)e^{-\tau_\mu x} \qquad x > 0. \tag{6.13}$$

From the deflection condition of the plate along the segment $k\Delta y$, we obtain

$$\sum_{\mu=1}^{m-1} \frac{q_\mu Y_y^\mu}{\tau_\mu^4} + \frac{1}{8}\sum_{\eta=1}^{k} \frac{Y_y^\mu Y_\eta^\mu}{\tau_\mu^3} = 0 \quad (y = 1, 2, \ldots, k). \tag{6.14}$$

If $k = m$, that is if the plate is supported along the line $x = 0$ over the entire width, we are concerned with the case of a semi-strip with the edge $x = 0$ clamped. In this case we assume

$$P_\eta = \sum_{j=1}^{m-1} A_j Y_\eta^j. \tag{6.15}$$

From the Eq. (6.14) we obtain $A_\mu = -8q_\mu/\tau_\mu$.

Introducing (6.15) in the Eq. (6.13), we find

$$w_y(x) = \frac{1}{N}\sum_{\mu=1}^{m-1} \frac{q_\mu Y_y^\mu}{\tau_\mu^4}[1 - (1 + x\tau_\mu)e^{-\tau_\mu x}] \qquad x > 0. \tag{6.16}$$

The clamping moment along the edge $x = 0$ takes the form

$$m_y^x(0) = -N\left[\frac{\partial^2 w_y}{\partial x^2}\right]_{x=0} = -\sum_{\mu}^{m-1} \frac{q_\mu}{\tau_\mu^2} Y_y^\mu.$$

If $q = \text{const}$, we have

$$m_y^x(0) = -\frac{q}{2m^2\varkappa^2}\sum_{\mu=1}^{m-1} \frac{\operatorname{ctg}\frac{\mu\pi}{2m}}{\sin^2\frac{\mu\pi}{m}} \sin\frac{\mu\pi y}{m}. \tag{6.17}$$

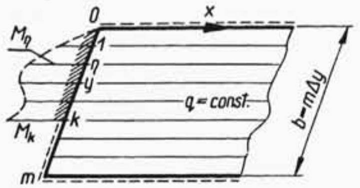

Fig. 2

Let us consider two auxiliary problems. Let a concentrated moment act at the point $(0, \eta)$ of the semi-strip simply supported on all edges (Fig. 2). Bearing in mind the boundary conditions

$$w_y(0) = 0, \quad m_y^x(0) = M\delta_{y\eta} \tag{6.18}$$

the deflection of the strip is obtained in the form

$$w_y^M(x) = \frac{xM}{2N}\sum_{\mu=1}^{m-1}\frac{Y_\eta^\mu Y_y^\mu}{\tau_\mu}e^{-\tau_\mu x} \qquad x>0\,. \tag{6.19}$$

If the semi-strip is acted on by the load $q_y(x) = q = \text{const}$, then

$$w_y^q(x) = \frac{1}{N}\sum_{\mu=1}^{m-1}\frac{q_\mu}{\tau_\mu^4}Y_y^\mu\left[1-\left(1+\frac{x\tau_\mu}{2}\right)e^{-\tau_\mu x}\right] \qquad x>0\,. \tag{6.20}$$

Let us consider a semi-strip simply supported along the egdes $y = 0$, $y = m$ and clamped along the segment $k\Delta y$ of the edge $x = 0$, the remaining part of that edge being also simply supported. Let the plate be acted on by a load $q_y(x) = q = \text{const}$ (Fig. 3). We are concerned with a problem of mixed boundary conditions. Denoting the unknown clamping moments by M_η, $1 \leqslant \eta \leqslant k$, the deflection of the plate is obtained in the form

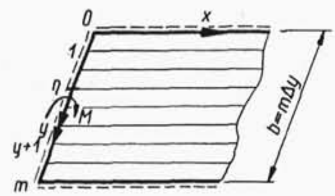

Fig. 3

$$w_y(x) = w_y^q(x) + \sum_{\mu=1}^{k} M_\eta w_y^M(x)\,. \tag{6.21}$$

From the condition $[\partial w/\partial x]_{x=0}$ on the segment $k\Delta y$ of the edge $x = 0$ the following system of equations is obtained

$$\sum_{\mu=1}^{m-1}\frac{q_\mu Y_y^\mu}{\tau_\mu^3} + \sum_{\eta=1}^{k} M_\mu \sum_{\mu=1}^{m-1}\frac{Y_y^\mu Y_\eta^\mu}{\tau_\mu} = 0 \qquad (y = 1, 2, \ldots, k)\,. \tag{6.22}$$

Solving this system of equations we find M_η, and from (6.21) the deflection $w_y(x)$.

Another solution procedure is such. Let us expand M_η in a series of functions Y_η^μ

$$M_\eta = \sum_j A_j Y_\eta^j\,, \qquad A_j = \sum_{\eta=1}^{m-1} M_\eta Y_\eta^j\,. \tag{6.23}$$

Inserting (6.23) in (6.22), we obtain the system of equations

$$\sum_j A_j a_{j\mu} + \frac{q_\mu}{\tau_\mu^2} = 0\,, \qquad a_{j\mu} = \sum_{\eta=1}^{k} Y_\eta^\mu Y_\eta^j\,. \tag{6.24}$$

If the plate is supported over the entire width, then, bearing in mind that $a_{j\mu} = \delta_{j\mu}$ for $k = m-1$, the Eq. (6.24) is reduced to the form

$$A_\mu + \frac{q_\mu}{\tau_\mu^2} = 0\,. \tag{6.25}$$

If (6.25) is substituted in (6.23) and (6.21) for $k = m-1$ then, as a result, the Eq. (6.16) is obtained.

The above problem of mixed boundary conditions can also be solved in another way. Let us assume, as a basic system the same plate simply supported on the edges $y = 0$, $y = m$, on the edge $x = 0$ being clamped. If in this system a load $q_y(x) = q = \text{const}$ acts, the deflection $w_y^q(x)$ is given by Eq. (6.16). Let now a concentrated angle

$$\left[\frac{\partial w}{\partial x}\right]_{x=0} = \varphi\delta_{y\eta}$$

act on the basic system. It can easily be shown that the deflection due to this state is given by the equation

$$w_y^\varphi(x) = \varphi x \sum_{\mu=1}^{m-1} Y_y^\mu Y_\eta^\mu e^{-\tau_\mu x} \qquad x > 0\,. \tag{6.26}$$

In the case of mixed boundary conditions the function expressing the angle φ is unknown in the interval $(m-k)\Delta y$ (see Fig. 4).

The deflection of the plate is expressed by the equation

$$w_y(x) = x \sum_{\mu=k+1}^{m-1} \varphi_\eta \sum_{\mu=1}^{m-1} Y_\eta^\mu Y_y^\mu e^{-\tau_\mu x} + \frac{1}{N} \sum_{\mu=1}^{m-1} \frac{q_\mu Y_y^\mu}{\tau_\mu^4} [1-(1+x\tau_\mu)e^{-\tau_\mu x}] \qquad x > 0\,. \tag{6.27}$$

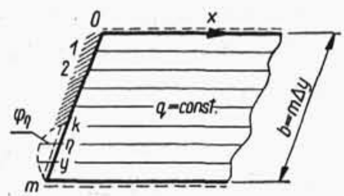

Fig. 4

From the conditions of zero bending moments $m_y^x(0)$ at the points $k+1$, $k+2, \ldots, m$, we obtain the following system of equations

$$\sum_{\eta=k+1}^{m-1} \varphi_\eta \sum_{\mu=1}^{m-1} \tau_\mu Y_y^\mu Y_\eta^\mu = \frac{1}{2N} \sum_{\mu=1}^{m=1} \frac{q_\mu}{\tau_\mu^2} Y_y^\mu \qquad (y = k+1, \ldots, m-1)\,, \tag{6.28}$$

from which the quantities φ_η can be found. Expanding φ_η in series

$$\varphi_\eta = \sum_{j=1}^{m-1} B_j Y_\eta^j\,, \qquad B_j = \sum_{y=1}^{m-1} \varphi_\eta Y_\eta^j \tag{6.29}$$

and substituting in (6.21) we obtain

$$\sum_{j=1}^{m-1} B_j b_{\mu j} = \frac{1}{2N} \frac{q_\mu}{\tau_\mu^3} \qquad (\mu = 1, 2, \ldots, m-1)\,. \tag{6.30}$$

If the plate is simply supported over the entire contour, then $b_{\mu j} = \delta_{\mu j}$ and the system of equations is simplified to the form

$$B_\mu = \frac{1}{2N} \frac{q_\mu}{\tau_\mu^3} \,. \tag{6.31}$$

If (6.31) is inserted in $w_y(x)$ (with $k = 0$) the Eq. (6.20) is obtained as a result.

Let us consider finally the plate of Fig. 5. Two semi-strips, of which one has the width $a = m\Delta y$ and a bending rigidity N_{I}, the other having the width $c = k\Delta y$

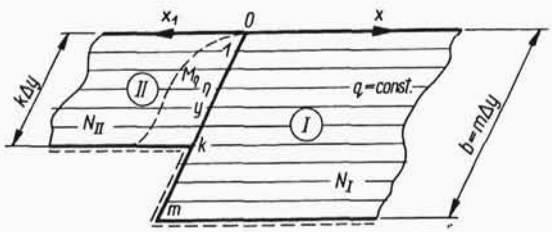

Fig. 5

and a bending rigidity N_{II}, are joined along the segment $k\Delta y$ of the line $x = 0$. The right-hand semi-strip is loaded by a uniform load q. In the coordinates assumed the following equations are obtained for deflection.

(a) Region I

$$w_y(x) = \frac{x}{2N_{\mathrm{I}}} \sum_{\eta=1}^{k-1} M_\eta \sum_{\eta=1}^{m-1} \frac{Y_y^\mu Y_\eta^\mu}{\tau_\mu} e^{-\tau_\mu x} + \frac{1}{N_{\mathrm{I}}} \sum_{\mu=1}^{m-1} \frac{q_\mu}{\tau_\mu^4} Y_y^\mu \left[1 - \left(1 + \frac{\tau_\mu x}{2}\right) e^{-\tau_\mu x}\right] \quad x > 0 \,; \tag{6.32}$$

(b) Region II

$$w_y(x_1) = \frac{x_1}{2N_{\mathrm{II}}} \sum_{\eta=1}^{k-1} M_\eta \sum_{\alpha=1}^{k-1} \frac{\varphi_y^\alpha \varphi_\eta^\alpha}{\sigma_\alpha} e^{-\sigma_\alpha x_1} \quad x_1 > 0 \tag{6.33}$$

where

$$\varphi_y^\alpha = \sqrt{\frac{2}{k}} \sin \frac{\alpha \pi y}{k} \,, \qquad \sigma_\alpha = \varkappa \bar{c}_\alpha = 2\varkappa \left| \sin \frac{\alpha \pi}{k} \right| \,.$$

It is assumed that the deflection along the line $x = 0$ is zero. From the continuity condition on the line $x = 0$ we obtain the following equation for M_η:

$$\sum_{\eta=1}^{k} M_\eta \left\{ \frac{1}{N_{\mathrm{II}}} \sum_\alpha \frac{\varphi_y^\alpha \varphi_\eta^\alpha}{\sigma_\alpha} + \frac{1}{N_{\mathrm{I}}} \sum_{\mu=1}^{m-1} \frac{Y_y^\mu Y_\eta^\mu}{\tau_\mu} \right\} + \frac{1}{N_{\mathrm{I}}} \sum_{\mu=1}^{m-1} \frac{q_\mu Y_y^\mu}{\tau_\mu^3} = 0 \tag{6.34}$$

$$(y = 1, 2, \ldots, k-1) \,.$$

The quantity M_η being found from (6.31), we obtain the deflection of the plate from the Eqs. (6.32) and (6.33).

However, the system of equations (6.34) may be replaced by another system. Let us expand M_η in a series of functions φ_η^β

$$M_\eta = \sum_{\beta=1}^{k-1} A_\beta \varphi_\eta^\beta . \tag{6.35}$$

Let us substitute (6.35) in the Eq. (6.34), multiply the Eq. (6.34) by φ_y^γ and sum up from $y=1$ to $y=k-1$. As a result, the following system of equations is obtained after some simple rearrangement

$$\frac{N_{\mathrm{I}}}{N_{\mathrm{II}}}\frac{A_\gamma}{\tau_\gamma} + \sum_{\beta=1}^{k-1} A_\beta \sum_{\mu=1}^{m-1} \frac{b_{\mu\gamma} c_{\beta\mu}}{\tau_\mu} + \sum_{\mu=1}^{m-1} \frac{q_\mu}{\tau_\mu^3} b_{\mu\gamma} = 0 \qquad (\gamma = 1, 2, \dots, k-1), \tag{6.36}$$

where

$$b_{\mu\gamma} = \sum_{y=1}^{m-1} Y_y^\mu \varphi_y^\gamma , \qquad c_{\beta\mu} = \sum_{\eta=1}^{m-1} Y_\eta^\mu \varphi_\eta^\beta . \tag{6.37}$$

If $a=c$ then $\varphi_y^\gamma = Y_y^\gamma$. Now, bearing in mind that $b_{\mu\gamma} = \delta_{\mu\gamma}$, $c_{\beta\mu} = \delta_{\mu\gamma}$ we obtain from the Eq. (6.36)

$$A_\gamma\left(1 + \frac{N_{\mathrm{I}}}{N_{\mathrm{II}}}\right) = -\frac{q_\gamma}{\tau_\gamma^2} \qquad (\gamma = 1, 2, \dots, m-1) . \tag{6.38}$$

If $N_{\mathrm{II}} \to \infty$ we are concerned with a plate clamped along the line $x=0$. The Eq. (6.38) becomes (6.25).

In the case of rectangular plates the Eq. (6.1) may also be solved by means of double series, one having a finite number of terms ($\mu = 0, 1, 2, \dots, m$) and the other an infinite number of terms. The solution methods do not differ from those discussed in Sec. 2.

7. Appendix. The Eigenfunctions of the Equation

$$L_{xy}(\varphi_{xy}^{\nu\mu}) = \sigma_{\nu\mu} \varphi_{xy}^{\nu\eta}$$

The most simple form of the function $\varphi_{xy}^{\nu\mu}$ is that in the case of a rectangular plate simply supported on the periphery. In this case the boundary conditions at the edges $x=0$, $x=n$ are

$$\varphi_{0,y}^{\nu\mu} = \varphi_{n,y}^{\nu\mu} = 0 , \quad \varphi_{-1,y}^{\nu\mu} + \varphi_{1,y}^{\nu\mu} = 0 , \quad \varphi_{n-1,y}^{\nu\mu} + \varphi_{n+1,y}^{\nu\mu} = 0 . \tag{7.1}$$

For the edges $y=0$, $y=m$ the boundary conditions are analogous to (7.1)

The difference equation

$$L_{xy}(\varphi_{xy}^{\nu\mu}) = \sigma_{\nu\mu} \varphi_{xy}^{\nu\mu} , \quad L_{xy} = \Delta_x^4 + 2\varepsilon^2 \Delta_x^2 \Delta_y^2 + \varepsilon^4 \Delta_y^4 \tag{7.2}$$

is satisfied by the functions

$$\varphi_{xy}^{\nu\mu} = X_x^\nu Y_y^\mu, \tag{7.3}$$

where

$$\begin{cases} X_\mu^\nu = \sqrt{\dfrac{2}{n}} \sin \alpha_\nu x, & \alpha_\nu = \dfrac{\nu\pi}{n} \quad (\nu = 1, 2, \dots, n-1; \quad x = 1, 2, \dots, n-1), \\ Y_y^\mu = \sqrt{\dfrac{2}{m}} \sin \beta_\mu y, & \beta_\mu = \dfrac{\mu\pi}{m} \quad (\mu = 1, 2, \dots, m-1; \quad y = 1, 2, \dots, m-1), \end{cases} \tag{7.4}$$

and the functions X_x^ν, Y_y^μ are orthonormal

$$\sum_{x=1}^{n-1} X_x^\nu X_x^{\nu'} = \delta_{\nu\nu'}, \qquad \sum_{y=1}^{m-1} Y_y^\mu Y_y^{\mu'} = \delta_{\mu\mu'}. \tag{7.5}$$

Inserting (7.3) in (7.2), we obtain

$$\sigma_{\nu\mu} = (a_\nu + b_\mu \varepsilon^2)^2, \quad a_\nu = 2(\cos \alpha_\nu - 1), \quad b_\mu = 2(\cos \beta_\mu - 1). \tag{7.6}$$

Let us assume that the edges $y = 0$, $y = m$ of the rectangular plate are the only simply supported. Substituting in (7.3) the expression

$$\varphi_{xy}^{\nu\mu} = Y_y^\mu X_x^{\nu\mu}, \tag{7.7}$$

where the function Y_y^μ is given by the Eq. (7.4) the following ordinary difference equations is obtained

$$(\Delta_x^4 + 2\varepsilon^2 b_\mu \Delta_x^2 + \varepsilon^4 b_\mu^2) X_x^{\nu\mu} = \sigma_{\nu\mu} X_x^{\nu\mu} \quad (x = 0, 1, 2, \dots, n). \tag{7.8}$$

This equation can also be represented in the form

$$X_{x-2}^{\nu\mu} - c_\mu X_{x-1}^{\nu\mu} + f_\mu X_x^{\nu\mu} - c_\mu X_{x+1}^{\nu\mu} + X_{x+2}^{\nu\mu} = \sigma_{\nu\mu} X_x^{\nu\mu} \quad (x = 0, 1, 2, \dots, n), \tag{7.9}$$

where

$$c_\mu = 4 - 2\varepsilon^2 b_\mu, \quad f_\mu = 6 - 2\varepsilon^2 b_\mu + \varepsilon^4 b_\mu^2 - \sigma_{\nu\mu}.$$

Setting the determinant of the system of equations (7.9) equal to zero and bearing in mind the boundary conditions for $x = 0$, $x = n$, we obtain for each value of μ a set of $n+1$ values of $\sigma_{\nu\mu}$. For each of these values the function $X_x^{\nu\mu}$ will be determined. It can easily be shown that these functions will be orthogonal. After normalization they will satisfy the condition

$$\sum_{x=0}^{n} X_x^{\nu\mu} X_x^{\nu'\mu} = \delta_{\nu\nu'}^\mu. \tag{7.10}$$

We can also consider the Eq. (7.9) to be a difference equation and solve the eigenvalue problem of the difference equation. This procedure leads to an involved transcendental equation, from which the values of the successive parameters $\sigma_{\nu\mu}$ are determined.

In the case of a rectangular plate clamped along two adjacent edges the functions $\varphi_{xy}^{\nu\mu}$ cannot be determined in the form (7.7) but only in the form of a finite series.

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Streszczenie

ZASTOSOWANIE RÓWNAŃ RÓŻNICOWYCH W TEORII PŁYT (I)

Przedmiotem pracy jest rozwiązanie cząstkowego równania różnicowego ugięcia płyty, które uzyskano przez zastąpienie pochodnych przez ilorazy różnicowe w równaniu różniczkowym ugięcia płyty.

W p. drugim i trzecim przedstawiono rozwiązanie równania różnicowego przy użyciu podwójnych skończonych szeregów i to w odniesieniu do drgań wymuszonych i własnych oraz do jednoczesnego zginania i ściskania oraz wyboczenia płyty prostokątnej. Uzyskane wyniki stanowią pełną analogię do rozszerzonej metody Naviera w terii różniczkowej płyty.

W p. piątym zajęto się stosowaniem pojedynczych szeregów do wyznaczania ugięcia płyty, głównie w odniesieniu do pasma płytowego, wykorzystując obmyśloną przez I. BABUŠKĘ transformację Fouriera dla równań różnicowych.

W ostatnim p. przedstawiono rozwiązanie kilku zagadnień statyki płyt przybliżając równanie różniczkowe ugięcia płyty równaniem różniczkowo-różnicowym.

Резюме

ПРИМЕНЕНИЕ РАЗНОСТНЫХ УРАВНЕНИЙ В ТЕОРИИ ПЛАСТИНОК (I)

Решается разностное уравнение в частных производных прогиба пластинки, полученное путем замены производных разностными частными в дифференциальном уравнении прогиба пластинки.

В п.п. 2 и 3 представлено решение разностного уравнения при использовании двойных конечных рядов и то в отнесении к вынуждающим и собственным колебаниям, а также к одновременному изгибу и сжатию и продольному изгибу прямоугольной пластинки. Полученные результаты являются полной аналогией расширенного метода Навие в дифференциальной теории пластинки.

В п. 5 рассматривается применение одинарных рядов к определению прогиба пластинки, главным образом в отнесении к пластинчатой полосе, используя предложенное И. Бабушкой преобразование Фурье для разностных уравнений.

В заключение представлено решение нескольких задач по статике пластинок, приближая дифференциальное уравнение прогиба пластинки дифференциально-разностным уравнением.

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Received March 6, 1961.