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# PROPAGATION OF THERMOELASTIC WAVES IN PLATES

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Notations

| | |
|---|---|
| $\bar{u}(u, v, w)$ | displacement vector, |
| $\mu, \lambda$ | Lamé's constants, |
| $\nu$ | Poisson's ratio, |
| $\varrho$ | density, |
| $a_t$ | coefficient of thermal expansion, |
| $k$ | coefficient of thermal conduction, |
| $\varkappa$ | coefficient of temperature conduction: $\varkappa = k/\varrho c_v$, |
| $c_v$ | specific heat at constant volume, |
| $\eta =$ | $(3\lambda + 2\mu) a_t T/k$, |
| $\vartheta_0 =$ | $(3\lambda + 2\mu) a_t/(\lambda + 2\mu)$, |
| $T$ | absolute temperature of the initial state, |
| $\Theta$ | increment of the absolute temperature, |
| $t$ | time, |
| $2a$ | thickness of the plate, |
| $c_1, c_2$ | velocity of the longitudinal and transverse waves, |
| $c_R$ | velocity of Rayleigh's surface waves, |
| $\omega$ | frequency of forced vibrations, |
| $\varepsilon$ | non-dimensional parameter, |
| $\varepsilon = \eta\varkappa\vartheta_0 =$ | $\left(\frac{1+\nu}{1-\nu}\right)^2 c_1^2 \frac{a_t^2 T}{c_v}$, |
| c | phase velocity, |
| $\zeta =$ | $(c/c_2)^2$. |

Let us consider the problem of propagation of plane waves in an infinite elastic plate bounded by the surfaces $z = \pm a$. If the coupling of the equations of motion with the heat equation be taken into account, [1], the problem is described by the system of equations:

$$
(1) \qquad \begin{cases} \mu \nabla^2 \bar{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \bar{u} - \varrho \dfrac{\partial^2 \bar{u}}{\partial t^2} = \gamma \operatorname{grad} \Theta, \\[2ex] \nabla^2 \Theta - \dfrac{1}{\varkappa} \dfrac{\partial \Theta}{\partial t} - \eta \dfrac{\partial}{\partial t} \operatorname{div} \bar{u} = 0, \end{cases}
$$

where $\bar{u}$ is the displacement vector and $\Theta$ the absolute temperature of the points of the plate measured from the initial state $T$ which is free of stresses.

Let us assume that the direction of propagation of the waves is parallel to the $x$-axis; then the system (1) can be reduced to three scalar equations. To this end let us introduce the scalar potential $\Phi$ and the vector potential $\bar{\psi}$

$$\bar{u} = \operatorname{grad} \Phi + \operatorname{rot} \bar{\Psi}. \tag{2}$$

In the simplified form we have:

$$\Phi = \Phi(x, z, t), \qquad \bar{\Psi} = \Psi[0, \Psi_2(x, z, t), 0].$$

Thus, we arrive at the equations:

$$\begin{cases} \nabla^2 \Psi - \dfrac{1}{c_2^2} \dfrac{\partial^2 \Psi}{\partial t^2} = 0, \\ \nabla^2 \Phi - \dfrac{1}{c_1^2} \dfrac{\partial^2 \Phi}{\partial t^2} = \vartheta_0 \Theta, \quad \nabla^2 \Theta - \dfrac{1}{\varkappa} \dfrac{\partial \Theta}{\partial t} = \eta \dfrac{\partial \nabla^2 \Phi}{\partial t}, \end{cases} \tag{3}$$

where $\Psi_2(x, z, t) = \Psi$ and $c_1, c_2$ denote the velocities of propagation of longitudinal and transverse waves in an elastic body:

$$c_1 = \sqrt{(\lambda + 2\mu)/\varrho}, \qquad c_2 = \sqrt{\mu/\varrho}.$$

Assuming that the vibrations are harmonic in time, i.e. that the displacement functions $\Phi$ and $\Psi$ and the temperature $\Theta$ can be represented in the form

$$\Phi(x, z, t) = \Phi^*(x, z)\, e^{i\omega t}, \quad \Psi(x, z, t) = \Psi^*(x, z)\, e^{i\omega t}, \quad \Theta(x, z, t) = \Theta^*(x, z)\, e^{i\omega t}, \tag{4}$$

the system (3) can now be replaced by the following:

$$\begin{cases} \nabla^2 \Psi^* + \dfrac{\omega^2}{c_2^2} \Psi^* = 0, \\ \nabla^2 \Phi^* + \dfrac{\omega^2}{c_1^2} \Phi^* = \vartheta_0 \Theta^*, \\ \nabla^2 \Theta^* - \dfrac{i\omega}{\varkappa} \Theta^* = \eta i \omega \nabla^2 \Phi^*. \end{cases} \tag{5}$$

Finally, eliminating the temperature $\Theta^*$ from the second and third equation (5), we obtain a system of two equations of the fourth and second degree:

$$(\nabla^2 - q)(\nabla^2 + \sigma^2)\Phi^* - q\varepsilon \nabla^2 \Phi^* = 0, \qquad (\nabla^2 + \tau^2)\Psi^* = 0. \tag{6}$$

This system, combined with appropriate boundary conditions, describes the course of the phenomenon under consideration. It should be emphasized that the coefficient $\varepsilon = \eta \varkappa \vartheta_0$ appearing in the equation (6), and defining the coupling of the thermal and elastic problems, is in the case of many materials encountered in practice much smaller than unity; this fact was stressed by Chadwick and Sneddon, [2]. For materials considered by those authors (Al, Fe, Cu, Pb) the value of $\varepsilon$ ranges between $3.5 \cdot 10^{-2}$ and $3.0 \cdot 10^{-4}$. In view of this fact, the quantitative effect of the coupling of the equations of motion and the heat equation is small and hardly perceptible in experimental investigations. On the other hand, to take into account the inequality $\varepsilon \ll 1$ considerably simplifies further discussion of the problem and investigation of the qualitative effect of the coupling.

The solution of the equations (6) is assumed to be in the form of harmonic functions with respect to the variable $x$:

$$\Phi^*(x,z) = f_1(z)\, e^{-i\alpha x}, \qquad \Psi^*(x,z) = f_2(z)\, e^{-i\alpha x}.$$

Hence we obtain two ordinary differential equations for the functions $f_1(z)$ and $f_2(z)$

$$(7) \quad f_1^{IV} - [(\alpha^2+q) + (\alpha^2-\sigma^2) + q\varepsilon]\, f_1'' + [(\alpha^2+q)(\alpha^2-\sigma^2) + q\alpha^2\varepsilon]\, f_1 = 0,$$

$$(8) \quad f_2'' - (\alpha^2 - \tau^2)\, f_2 = 0.$$

The solution of these equations leads us to the following form of the displacement functions $\Phi$ and $\Psi$ and the temperature $\Theta$, calculated by means of the second equation (5):

$$(9) \quad \Phi(x,z,t) = [A_1 \operatorname{ch} \lambda_1 z + A_2 \operatorname{ch} \lambda_2 z + A_3 \operatorname{sh} \lambda_1 z + A_4 \operatorname{sh} \lambda_2 z]\, e^{i(\omega t - \alpha x)},$$

$$(10) \quad \Psi(x,z,t) = [A_5 \operatorname{ch} \gamma z + A_6 \operatorname{sh} \gamma z]\, e^{i(\omega t - \alpha x)},$$

$$(11) \quad \Theta(x,z,t) = \frac{1}{\vartheta_0}[A_1(\lambda_1^2 + \sigma^2 - \alpha^2) \operatorname{ch} \lambda_1 z + A_2(\lambda_2^2 + \sigma^2 - \alpha^2) \operatorname{ch} \lambda_2 z + A_3(\lambda_1^2 + \sigma^2 - \alpha^2) \operatorname{sh} \lambda_1 z + A_4(\lambda_2^2 + \sigma^2 - \alpha^2) \operatorname{sh} \lambda_2 z]\, e^{i(\omega t - \alpha x)}.$$

The constants $A_i$ will be determined from the boundary conditions of the problem. In the above formulae the following notation has been assumed:

$$\gamma = \sqrt{\alpha^2 - \tau^2}, \qquad \sigma^2 = \frac{\omega^2}{c_1^2}, \qquad \tau^2 = \frac{\omega^2}{c_2^2}, \qquad q = \frac{i\omega}{\varkappa},$$

$\lambda_1$ and $\lambda_2$ denote two complex roots of the biquadratic equation:

$$(12) \quad \lambda^4 - [(\alpha^2+q) + (\alpha^2-\sigma^2) + q\varepsilon]\, \lambda^2 + [(\alpha^2+q)(\alpha^2-\sigma^2) + q\alpha^2\varepsilon] = 0.$$

Making use of the inequality $\varepsilon \ll 1$, the above roots may be simply determined in an approximate way by assuming

$$\lambda_1^2 = \alpha^2 + q + \varepsilon\delta_1, \qquad \lambda_2^2 = \alpha^2 - \sigma^2 + \varepsilon\delta_2,$$

and then determining the coefficients $\delta_1$ and $\delta_2$ from the equation (12), the terms containing $\varepsilon^2$ being disregarded. Thus, we obtain the approximate relations

$$\lambda_1 \approx \sqrt{\alpha^2 + q\left(1 + \frac{q\varepsilon}{\sigma^2+q}\right)}, \tag{13}$$

$$\lambda_2 \approx \sqrt{\alpha^2 - \sigma^2\left(1 - \frac{q\varepsilon}{\sigma^2+q}\right)}. \tag{14}$$

Out of four existing roots we consider the two satisfying the condition $\operatorname{Re}(\lambda_{1,2}) > 0$.

If in the above considerations the effect of coupling be disregarded, i.e. if we set $\varepsilon = 0$, we should obtain:

$$\lambda_1 = \sqrt{\alpha^2 + \frac{i\omega}{\varkappa}}, \qquad \lambda_2 = \sqrt{\alpha^2 - \frac{\omega^2}{c_1^2}},$$

— the coefficients corresponding to the propagation of purely elastic or purely thermal disturbances. If, moreover, $\alpha$ were real, the wave motion would take place in the direction of the $x$-axis with a constant velocity and without dispersion.

In our case, however, $\varepsilon > 0$ and the coefficient $\alpha$ are to be determined from the boundary conditions of the problem. We shall consider here two kinds of conditions, namely:

(a) The surfaces of the plate free of tractions and maintained at the constant temperature $\Theta = 0$

$$\sigma_{zz} = \sigma_{xz} = \Theta = 0 \quad \text{for} \quad z = \pm a. \tag{15}$$

(b) The surfaces of the plate free of tractions and a perfect thermal isolation occurring:

$$\sigma_{zz} = \sigma_{xz} = \frac{\partial \Theta}{\partial z} = 0 \quad \text{for} \quad z = \pm a. \tag{16}$$

We shall discuss two types of vibrations — symmetric and antisymmetric (bending) — which corresponds to the assumption in the solutions (9), (10) and (11)

$$A_3 = A_4 = A_5 = 0 \quad \text{or} \quad A_1 = A_2 = A_6 = 0. \tag{17}$$

In order to compute the stresses in the plate, the displacements expressed by means of the functions $\Phi$ and $\Psi$

$$u = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial z}, \qquad w = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi}{\partial x}$$

should be inserted into H o o k e's equations

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\Theta\alpha_t\delta_{ij}, \qquad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

Since

$$\varepsilon_{kk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \nabla^2 \Phi,$$

we obtain, confining ourselves for the time being to the scalar potential $\Phi$:

$$\sigma'_{ij} = [\lambda \nabla^2 \Phi - (3\lambda + 2\mu) a_t \Theta]\, \delta_{ij} + 2\mu \frac{\partial^2 \Phi}{\partial x_i \partial x_j}.$$

Expressing in the above equation the temperature $\Theta$ by $\Phi$ according to the second equation (3), we have:

$$(3\lambda + 2\mu) a_t \Theta = (\lambda + 2\mu) \nabla^2 \Phi - \varrho \frac{\partial^2 \Phi}{\partial t^2};$$

and therefore:

$$\sigma'_{ij} = 2\mu \left( \frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij} \nabla^2 \right) \Phi + \varrho \frac{\partial^2 \Phi}{\partial t^2} \delta_{ij}.$$

Similarly, we can take into account the vector potential $\Psi$ leading to the stresses $\sigma''_{ij}$. Bearing in mind the representations (4), we obtain the following formulae for the stresses $\sigma_{zz}$ and $\sigma_{xz}$

$$\sigma_{zz} = -\mu \left( 2 \frac{\partial^2 \Phi^*}{\partial x^2} + \tau^2 \Phi^* - 2 \frac{\partial^2 \Psi^*}{\partial x \partial z} \right) e^{i\omega t},$$

$$\sigma_{xz} = \mu \left( 2 \frac{\partial^2 \Phi^*}{\partial x \partial z} + \frac{\partial^2 \Psi^*}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial z^2} \right) e^{i\omega t}.$$

Substituting these relations and the equation (11) into the conditions (15) or (16), and taking into account (9), (10) and (17), we arrive at a system of three homogeneous equations for the constants $A_i$. Equating to zero the determinant of this system, we obtain a transcendental equation enabling us to determine a sequence of eigenvalues of $\alpha$; then, we can determine the phase velocities of plane waves in the plate vibrating periodically with the frequency $\omega$.

In the four cases under consideration, the transcendental equations for $\alpha$ have the following form.

1. The boundary condition (a), symmetric form of vibrations:

$$\frac{(2\alpha^2 - \tau^2)^2 (\lambda_1^2 - \lambda_2^2)}{4\alpha^2 \gamma} = \frac{(\sigma^2 + \lambda_1^2 - \alpha^2)\lambda_2 \operatorname{th} \lambda_2 a - (\sigma^2 + \lambda_2^2 - \alpha^2)\lambda_1 \operatorname{th} \lambda_1 a}{\operatorname{th} \gamma a}. \tag{18}$$

2. The boundary condition (a), anti-symmetric form of vibrations:

$$\frac{(2\alpha^2 - \tau^2)^2 (\lambda_1^2 - \lambda_2^2)}{4\alpha^2 \gamma} = \frac{(\sigma^2 + \lambda_1^2 - \alpha^2)\lambda_2 \operatorname{cth} \lambda_2 a - (\sigma^2 + \lambda_2^2 - \alpha^2)\lambda_1 \operatorname{cth} \lambda_1 a}{\operatorname{cth} \gamma a}. \tag{19}$$

3. The boundary condition (b), symmetric form of vibrations:

$$\frac{(2\alpha^2 - \tau^2)^2}{4\alpha^2 \gamma} = \frac{\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2) \operatorname{cth} \gamma a}{(\sigma^2 + \lambda_1^2 - \alpha^2)\lambda_1 \operatorname{cth} \lambda_2 a - (\sigma^2 + \lambda_2^2 - \alpha^2)\lambda_2 \operatorname{cth} \lambda_1 a}. \tag{20}$$

4. The boundary condition (b), anti-symmetric form of vibrations:

$$\frac{(2\alpha^2-\tau^2)^2}{4\alpha^2\gamma}=\frac{\lambda_1\lambda_2(\lambda_1^2-\lambda_2^2)\ \mathrm{th}\ \gamma a}{(\sigma^2+\lambda_1^2-\alpha^2)\lambda_1\ \mathrm{th}\ \lambda_2 a-(\sigma^2+\lambda_2^2-\alpha^2)\lambda_2\ \mathrm{th}\ \lambda_1 a}. \tag{21}$$

If we assume that $\varepsilon=0$, the equations (18)-(21) reduce to well-known relations discussed at length by V. G. Gogoladze, [3], and quoted in the monograph [4]. In this case, obviously, $\Theta=0$ at all points of the plate.

Let us now consider in greater detail the case (a) of the plate the surfaces of which are held at the zero temperature. Introducing new simplifying notations we shall represent the equation (18), corresponding to the symmetric vibrations in the dimensionless form. We assume:

$$\left\{\begin{aligned}&\zeta=\left(\frac{c}{c_2}\right)^2,\qquad \beta_1=\left(\frac{c_2}{c_1}\right)^2=\frac{\mu}{\lambda+2\mu},\qquad c_0=\sqrt{\varkappa\omega},\\&\beta_0=\left(\frac{c_2}{c_0}\right)^2,\qquad h=\frac{q}{q+\sigma^2}=\frac{\beta_0}{\beta_0-i\beta_1},\end{aligned}\right. \tag{22}$$

where $c=\omega/\alpha$ is the phase velocity of the plane wave only in the case of real values of $\alpha$. For complex values of $\alpha$, $\alpha=\alpha_R-i\alpha_i$ the exponential term in the formulae (9) and (10) assumes the form:

$$e^{i(\omega t-\alpha x)}=e^{i(\omega t-\alpha_R x)-\alpha_i x}=e^{-i\alpha_R(\omega t/\alpha_R-x)}e^{-\alpha_i x}.$$

Thus, the phase velocity is here equal to $\omega/\mathrm{Re}\,\alpha$, $\mathrm{Im}\,\alpha$ constituting the dispersive term.

Making use of the notations (22), the equation (18) can be written in the form:

$$\begin{aligned}&\frac{(1-0.5\zeta)^2[\beta_1+i\beta_0-h\varepsilon(\beta_1-i\beta_0)]}{\sqrt{1-\zeta}}=\\&=\mathrm{cth}\sqrt{1-\zeta}\,\alpha a\left\{[\beta_1+i\beta_0+h\varepsilon i\beta_0]\sqrt{1-\beta_1\zeta+h\varepsilon\beta_1\zeta}\ \mathrm{th}\sqrt{1-\beta_1\zeta+h\varepsilon\beta_1\zeta}\,\alpha a-\right.\\&\left.\qquad-h\varepsilon\beta_1\sqrt{1+i\beta_0\zeta+h\varepsilon i\beta_0\zeta}\ \mathrm{th}\sqrt{1+i\beta_0\zeta+h\varepsilon i\beta_0\zeta}\,\alpha a\right\}.\end{aligned} \tag{23}$$

The investigation of the above equation in the general case is most cumbersome. We shall therefore confine ourselves to the consideration of the two extremal cases: $\alpha a\gg1$ and $\alpha a\ll1$. In the first case, the functions tgh appearing in the equation (23) may be replaced (if only $\zeta<1$) approximately by unity; in the second case by their arguments.

If $\alpha a\gg1$, then disregarding in the equation (23) the terms containing $\varepsilon^2$, we obtain a simplified equation:

$$\begin{aligned}&\frac{(1-0.5\zeta)^2[\beta_1+i\beta_0-h\varepsilon(\beta_1-i\beta_0)]}{\sqrt{1-\zeta}}=\\&\qquad=[(\beta_1+i\beta_0)+h\varepsilon i\beta_0]\sqrt{(1-\beta_1\zeta)+h\varepsilon\beta_1\zeta}-h\varepsilon\beta_1\sqrt{1+i\beta_0\zeta}.\end{aligned}$$

After simple transformations:

$$(24)\quad \frac{(1-0\,5\zeta)^2}{\sqrt{1-\zeta}\sqrt{1-\beta_1\zeta}}-1=h\varepsilon\left\{\frac{\beta_1-i\beta_0}{\beta_1+i\beta_0}\left[\frac{(1-0.5\zeta)^2}{\sqrt{1-\zeta}\sqrt{1-\beta_1\zeta}}-1\right]+\right.$$
$$\left.+\frac{\beta_1}{(\beta_1+i\beta_0)(1-\beta_1\zeta)}\left[1-0.5\,\beta_1\zeta+0.5\,i\beta_0\zeta-\sqrt{1-\beta_0\zeta}\sqrt{1+i\beta_0\zeta}\right]\right\}.$$

In the case $\varepsilon=0$ the-right hand side of (24) vanishes, and we obtain the well-known equation:

$$(25)\qquad F_1(\zeta)=\frac{(1-0.5\,\zeta)^2}{\sqrt{1-\zeta}\sqrt{1-\beta_1\,\zeta}}-1=0,$$

leading to the phase velocity $c=c_R$ corresponding to Rayleigh's surface waves, [4]. The corresponding value of $\zeta=c_R^2/c_2^2$ is denoted by $\zeta_R$.

In order to estimate the value of $\zeta$ corresponding to $\varepsilon>0$, we proceed as follows. Denoting by $h\varepsilon F_2(\zeta)$ the right hand side of the equation (24), and by $\delta$ the increment of $\zeta$ due to the coupling of the equations (1), we can write (24) in the form:

$$(26)\qquad F_1(\zeta_R+\delta)=h\varepsilon F_2(\zeta_R+\delta).$$

The postulate $\delta\ll\zeta_R$ enables us to expand both sides of the equation (26) into Taylor's series in the vicinity of the point $\zeta=\zeta_R$, preserving only two terms of the series:

$$(27)\qquad F_1(\zeta_R)+\delta\left[\frac{\partial F_1}{\partial\zeta}\right]_{\zeta=\zeta_R}\approx h\varepsilon F_2(\zeta_R).$$

Since $F_1(\zeta_R)=0$, we obtain from (27):

$$\delta\approx h\,\varepsilon\frac{F_2(\zeta_R)}{\left[\frac{\partial F_1}{\partial\zeta}\right]_{\zeta=\zeta_R}}.$$

Taking into account the real values of the functions $F_1$ and $F_2$ we have:

$$(28)\quad \delta=-\,\varepsilon\frac{i\beta_0\beta_1(\beta_0+i\beta_1)^2}{(\beta_0^2+\beta_1^2)^2(1-\beta_1\zeta_R)}\times$$
$$\times\frac{1-0.5\,\beta_1\zeta_R+0.5\,i\beta_0\zeta_R-\sqrt{1-\beta_1\zeta_R}\sqrt{1+i\beta_0\zeta_R}}{0.5\,\beta_1/(1-\beta_1\zeta_R)+0.5/(1-\zeta_R)-1/(1-0.5\,\zeta_R)}.$$

In the above equation the increment of the phase velocity c is represented in terms of dimensionless variables $\varepsilon$, $\beta_1$, $\zeta_R$ and $\beta_0$.

Let us bear in mind that the coefficient $\varepsilon=\eta\varkappa\vartheta_0$ depends on the thermal properties of the material, and satisfies the inequality $\varepsilon\ll1$. The values of $\beta_1$ and $\zeta_R$ depend exclusively on the elastic properties of the material, and they range from 0.5 to 0 or from 0.766 to 0.913 (cf. Table 1):

Table 1

| $\nu$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|---|
| $\beta_1$ | 0.500 | 0.444 | 0.375 | 0.286 | 0.166 | 0.000 |
| $\zeta_R$ | 0.766 | 0.797 | 0.830 | 0.860 | 0.888 | 0.913 |

The dependence of $\delta$ on the parameter $\beta_0$, $\beta_0 = c_2^2/\varkappa\omega$, connected with the frequency of the forcing vibrations is analysed in greater detail; this dependence is given at Fig. 1, the real and imaginary parts of the function $\delta = \delta_r + i\delta_i$ being drawn separately. The graph has been constructed for a definite plate made of aluminium with the following properties:

$$E = 7 \cdot 10^5 \,\mathrm{kG\,cm^{-2}}, \quad \nu = 0.34, \quad \varrho = 2.75 \,\mathrm{g\,cm^{-3}},$$
$$\varkappa = 0.61 \,\mathrm{cm^2\,sek^{-1}}, \quad \varepsilon = 0.035, \quad c_1 = 6.17 \cdot 10^5 \,\mathrm{cm\,sek^{-1}},$$
$$c_2 = 3.04 \cdot 10^5 \,\mathrm{cm\,sek^{-1}}, \quad \beta_1 = 0.242, \quad \zeta_R = 0.870.$$

The initial temperature was assumed to be $T = 293°\mathrm{K} = 20°\mathrm{C}$.

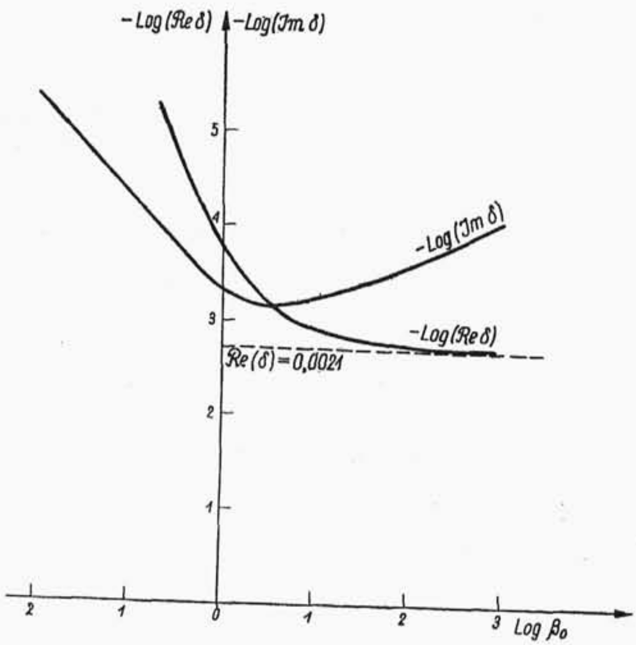


Fig. 1

On the basis of the formula (28) and Fig. 1, three ranges of $\beta_0$ may be distinguished, namely: for small values of $\beta_0 \to 0$ the absolute value $|\delta| \to 0$, and $\delta$ assumes purely imaginary values in accordance with the approximate relation

(29) $$\delta \approx 4.5 \cdot 10^{-4}\, i\beta_0.$$

The maximum influence of the coupling of the original equations takes place in the interval $1 < \beta_0 < 10$, where $\delta$ attains values of the order $6.4 \cdot 10^{-4}(1+i)$. For large values of $\beta_0$ the imaginary part $\delta_i$ tends to zero, and the real part $\delta_r$ tends asymptotically to the value:

$$\delta_r^\infty = \frac{0.5\,\varepsilon\beta_1\zeta_R}{(1-\beta_1\zeta_R)\,[0.5\,\beta_1/(1-\beta_1\zeta_R)+0.5/(1-\zeta_R)-1/(1-0.5\,\zeta_R)]} = 0.0021.$$

The validity of the graph in this part is restricted by the condition on which the formula (24) is based, namely by the inequality $\alpha a \gg 1$ equivalent approximately to the inequality

$$\alpha a = \frac{\alpha}{\omega}\,\omega a \approx \frac{\omega a}{c_R} \gg 1, \quad \text{i. e.} \quad \omega \gg \frac{c_R}{a}.$$

At the same time, however, the inequality $\beta_0 = c_2^2/\varkappa\omega \gg 1$ implies $\omega \ll c_2^2/\varkappa$. The right-hand side of the graph at Fig. 1 is valid only to $\beta_0 = ac_2^2/\varkappa c_R$ which in our case is approximately equal to $5.3 \cdot 10^5$.

The above result may be compared with that given by F. I. Lockett, [5], because in this particular case we deal, in fact, with a phenomenon similar to Rayleigh's surface waves. The equation (24) may be represented in the form:

$$\frac{(1-0.5\,\zeta)^2\left(1-h\varepsilon\dfrac{\beta_1-i\beta_0}{\beta_1+i\beta_0}\right)}{\sqrt{1-\zeta}\,\sqrt{(1-\beta_1\zeta)+h\varepsilon\beta_1\zeta}} = \left(1+h\varepsilon\frac{i\beta_0}{\beta_1+i\beta_0}\right) - h\varepsilon\beta_1\frac{\sqrt{1+i\beta_0\zeta}}{(\beta_1+i\beta_0)\sqrt{1-\beta_1\zeta+h\varepsilon\beta_1\zeta}}.$$

If we now assume that $\beta_0 \gg 1$, $h \approx 1$ and disregard the terms containing $\varepsilon$, except for the term $h\varepsilon\beta_1\zeta$ in the denominator in the left-hand side, we shall obtain the approximate relation:

$$\frac{1-0.5\,\zeta}{\sqrt{1-\zeta}\,\sqrt{1-(1-\varepsilon)\beta_1\zeta}} - 1 \approx 0.$$

Since we may take approximately

$$\beta_1(1-\varepsilon) \approx \frac{\beta_1}{1+\varepsilon},$$

our condition is identical to Lockett's condition. Lockett established that the velocity of the surface waves for small values of $\omega$ may be calculated from the well-known transcendental equation for Rayleigh's waves, in which the coefficient $\beta_1$ has been replaced by $(\beta_1/1+\varepsilon)$.

Passing now to the direct determination of the change of the phase velocity $c$, let us observe that for small values of $\delta$ we can set

$$\delta = \left(\frac{c_R + \Delta c}{c_2}\right)^2 - \left(\frac{c_R}{c_2}\right)^2 \approx \frac{2 c_R \Delta c}{c_2^2}$$

whence the relative increment of velocity follows:

$$\frac{\Delta c}{c_R} \approx \frac{\delta}{2\zeta_R} = \frac{\delta}{1.74}.$$

Let us now return to the formulae (9) and (10). Bearing in mind that

$$\frac{\omega}{a} = c_R\left(1 + \delta\frac{1}{2\zeta_R}\right) = c_R\left[1 + (\delta_r + i\delta_i)\frac{1}{2\zeta_R}\right],$$

the displacement function $\Phi(x, z, t)$ may be written in the form:

$$\Phi(x,z,t) = f_1(z)\exp\left[i(\omega t - a x)\right] =$$
$$= f_1(z)\exp\left\{i\frac{\omega}{c_R}\left(1 - \frac{\delta_r}{2\zeta_R}\right)\left[\left(1 + \frac{\delta_r}{2\zeta_R}\right)c_R t - x\right]\right\}\exp\left(-\frac{\omega\delta_i}{2\zeta_R c_R}x\right).$$

It follows clearly from the above results that $\operatorname{Re}\delta = \delta_r$ constitutes the factor increasing the phase velocity of the vibrations in the ratio $(1 + \delta_r/2\zeta_R)$; $\operatorname{Im}\delta = \delta_i$ on the other hand, being the dispersive factor. The coefficient of dispersion will be defined thus:

$$d = \frac{2\zeta_R c_R}{\omega\delta_i} = \frac{2\zeta_R^{3/2}\varkappa\beta_0}{\delta_i c_2} = 3.74\cdot 10^{-6}\frac{\beta_0}{\delta_i}.$$

Figure 2 represents in the logarithmic scale the relative increment of the velocity $\Delta c/c_R$ and the coefficient of dispersion $\alpha$ in terms of the frequency $\omega$ of the acting force in the aluminium plate under consideration.

Let us now consider the second extremum case $a\alpha \ll 1$. Making use of the first terms of the expansion of the function $\operatorname{tgh} x = x + \ldots$, the condition (23) may be written in the form:

$$(30)\quad (1 - 0.5\zeta)^2 - [\beta_1 + i\beta_0 - h\varepsilon(\beta_1 - i\beta_0)] =$$
$$= [(\beta_1 + i\beta_0) + h\varepsilon i\beta_0]\,[(1 - \beta_1\zeta) + h\varepsilon\beta_1\zeta] - h\varepsilon\beta_1(1 + i\beta_0\zeta),$$

if the terms containing $\varepsilon$ in the second and higher powers be disregarded. The equation (30) can be transformed as follows:

$$(31)\qquad \left[\frac{(1 - 0.5\zeta)^2}{1 - \beta_1\zeta} - 1\right]\left(1 - h\varepsilon\frac{\beta_1 - i\beta_0}{\beta_1 + i\beta_0}\right) = 0.$$

The last condition yields, irrespective of the value of $\varepsilon$, a value of $\zeta$ identical with that for the uncoupled problem (cf. e.g. [4]): $\zeta = 4(1 - \beta_1)$.

Depending on the Poisson ratio $\nu$, we obtain for the phase velocity $c$ values from $c_2$ to $2c_2$ (see Table 1), the coefficient $\varepsilon$ having no noticeable influence on this result. This can be explained if it is remembered that we are considering a plate the thickness of which is very small in comparison with the length of the wave; and the surfaces of the plate are held at zero temperature. Therefore the maximum value of the temperature $\Theta$ must be small, and hence its influence on the phenomenon under consideration is negligible. On the other hand, in the case

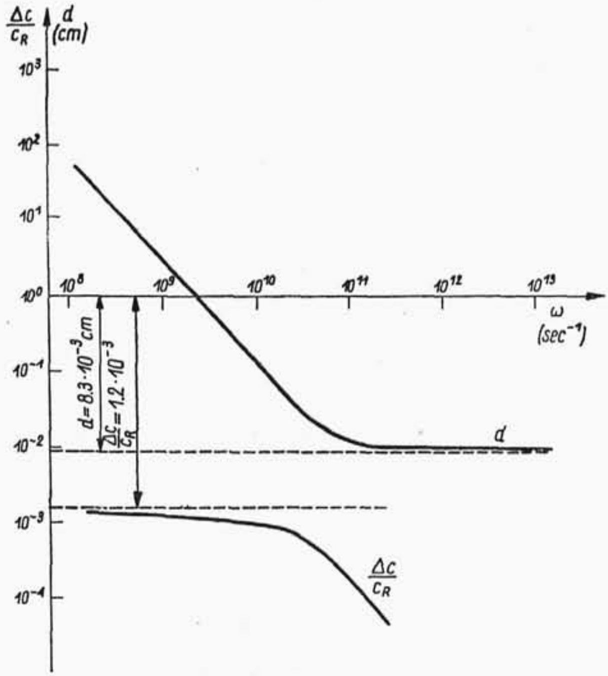


Fig. 2

of a thermal isolation of the surfaces of the plate, the representation of the appropriate transcendental equation in the form (31) is no longer possible and the temperature $\Theta$ may reach greater values.

Passing to the case of anti-symmetric form of the wave described by the equation (19), i.e. to the bending vibrations of the plate, let us observe that the first extremum case $\alpha a > 1$ leads to a relation identical with (24), since for the large arguments $\operatorname{tgh} x \approx \operatorname{ctgh} x \approx 1$. Accordingly, the discussion of the dependence of $\zeta$ on $\varepsilon$ carried out for the symmetric form of the vibrations remains valid, as well as the graphs at Fig. 1 and Fig. 2.

The discussion of the second extremum case $\alpha a \ll 1$ is also simple. In this case, in the expansion of the function ctgh $x$, two terms should be retained, according to the formula:

$$\operatorname{ctgh} x = \frac{1}{x} + \frac{1}{3} x \ldots .$$

Taking into account this expansion and introducing the notations (22), the equation (19) can be transformed to the following form:

$$\left[\frac{(1-0.5\zeta)^2}{1-\zeta}-\frac{1/(\alpha a)^2+(1-\beta_1\zeta)/3}{1/(\alpha a)^2+(1-\zeta)/3}\right]\left(1-h\varepsilon\frac{\beta_1-i\beta_0}{\beta_1+i\beta_0}\right)=0.$$

Comparing this result with that quoted in [4] on page 285, we observe that the phase velocity $c$ is unchanged in this case as well, and with the assumed degree of accuracy it is independent of $\varepsilon$. For we have approximately:

$$\zeta\approx\frac{4}{3}(\alpha a)^2(1-\beta_1).$$

Similarly a discussion can be conducted for the plate the surfaces of which are isolated. To this end an approximate analysis of the equations (20) and (21) should be made.

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## Streszczenie

### ROZCHODZENIE SIĘ FAL TERMOSPRĘŻYSTYCH W PŁYTACH

W pracy rozważono zagadnienie rozchodzenia się fal płaskich w płytach sprężystych przy uwzględnieniu sprzężenia równań ruchu i przewodnictwa cieplnego w postaci podanej przez Biota, [1]. Omówiono wpływ sprzężenia na charakter rozchodzenia się fal sprężystych w dwóch przypadkach skrajnych: płyty bardzo grubej albo bardzo cienkiej w stosunku do długości fali. Wpływ ten jest dwojaki: z jednej strony zwiększa się prędkość fazowa ruchu falowego, z drugiej zaś strony w rozwiązaniach dla przemieszczeń pojawiają się człony powodujące dyspersję. Przykład liczbowy dotyczący płyty aluminiowej pozwala na wyrobienie sobie pojęcia o ilościowym charakterze zjawiska.

Резюме

РАСПРОСТРАНЕНИЕ ТЕРМОУПРУГИХ ВОЛН В ПЛАСТИНКАХ

В работе рассматривается проблема распространения плоских волн в упругих пластинках с учетом сопряжения уравнений движения с уравнением теплопроводности в форме данной Биотом, [1]. Рассмотрено влияние этого сопряжения на характер распространения волн в двух крайних случаях: пластинки весьма толстой или весьма тонкой по сравнению с длиной волны. Это влияние проявляется двойным образом: увеличивается фазовая скорость волнового движения и в решениях для перемещений появляются члены, вызывающие дисперсию. Численный пример, касающийся алюминиевой пластинки, описывает численный характер явления.

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