P O L S K A   A K A D E M I A   N A U K
INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI

ARCHIWUM MECHANIKI STOSOWANEJ

ARCHIVES DE MECANIQUE APPLIQUEE

3
XI

VOL. XI   WARSZAWA 1959   No 3
PAŃSTWOWE WYDAWNICTWO NAUKOWE

TWO ONE-DIMENSIONAL PROBLEMS OF THERMOELASTICITY

WITOLD NOWACKI (WARSZAWA)

1. Introduction

The work on dynamic problems of thermoelasticity was started in 1950 with two papers by V. I. Danilovskaya, [1], [2], on the stress provoked by sudden heating of the plane bounding an elastic semi-space. The problem of stresses in a space or an infinite plate was treated by H. Parcus, [3], the author of the present paper, [4], and A. M. Katasonov, [5]. All these papers are based on the heat equation in which the terms taking account of the influence of the dilatation of the elastic body are rejected. Dynamic problems of thermoelasticity, where the influence of dilatation is taken into consideration in the heat equation, are dealt with in works by I. N. Sneddon, P. Chadwick, [6], and F. J. Lockett, [7].

In the present paper, we shall consider the action of a continuously distributed plane source of heat acting in an elastic semi-space in the $x=\xi$ plane. In the second part, stress propagation in a perfectly elastic semi-space will be considered, and in the third — such propagation in a viscoelastic semi-space. The location of the heat source in the $x=\xi$ plane considered here is characterized by the fact that it involves the necessity of taking into consideration the longitudinal wave reflected from the plane bounding the elastic semi-space. The considerations of the present paper will be based on the heat equation, the influence of the dilatation on the temperature field not being taken into consideration. This influence will be discussed in the next paper.

The heat equation has the form

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{\varkappa}\frac{\partial T}{\partial t} = -\frac{Q}{\varkappa}, \tag{1.1}$$

where T denotes the temperature, t — time, and $\varkappa = \lambda_0/\varrho c$, where λ_0 is the coefficient of heat conduction, ϱ — density and c — specific heat, and $Q = W/\varrho c$, where W is the heat quantity transmitted by the heat source per unit time and volume.

The displacement equation of the theory of elasticity has the form

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-\varrho\frac{\partial^2 u}{\partial t^2}=(2\mu+3\lambda)\,a_t\frac{\partial T}{\partial x}, \tag{1.2}$$

where λ, μ are L a m é' s constants, u — displacement, a_t — coefficient of linear thermal expansion.

Let us introduce the potential of thermoelastic strain Φ, where

$$u=\frac{\partial \Phi}{\partial x}. \tag{1.3}$$

The Eq. (1.2) takes, after integration with respect to x, the following form, [4]:

$$\frac{\partial^2\Phi}{\partial x^2}-\sigma^2\frac{\partial^2\Phi}{\partial t^2}=\vartheta_0 T, \tag{1.4}$$

where

$$\sigma^2=\frac{\varrho}{2\mu+\lambda}=\frac{1}{c_1^2},\qquad \vartheta_0=\frac{3\lambda+2\mu}{\lambda+2\mu}\,a_t=\frac{1+\nu}{1-\nu}\,a_t,$$

and c_1 is the velocity of propagation of an elastic longitudinal wave and ν — P o i s s o n' s ratio. In the one-dimensional case under consideration, the stresses $\bar{\sigma}_{ii}$ $(i=1,2,3)$ will be the only stresses appearing. They are expressed by the equations

$$\bar{\sigma}_{11}=\varrho\frac{\partial^2\Phi}{\partial t^2},\qquad \bar{\sigma}_{22}=\bar{\sigma}_{33}=-2G\frac{\partial^2\Phi}{\partial x^2}+\varrho\frac{\partial^2\Phi}{\partial t^2}, \tag{1.5}$$

or

$$\bar{\sigma}_{11}=\varrho\frac{\partial^2\Phi}{\partial t^2},\qquad \bar{\sigma}_{22}=\bar{\sigma}_{33}=-2G\vartheta_0 T+\lambda\sigma^2\frac{\partial^2\Phi}{\partial t^2}. \tag{1.6}$$

In general, the function Φ will not satisfy all the boundary conditions. To the stresses $\bar{\sigma}_{ii}$ the stresses $\bar{\bar{\sigma}}_{ii}$ should be added, expressed in terms of the function φ satisfying the homogeneous equation

$$\frac{\partial^2\varphi}{\partial x^2}-\sigma^2\frac{\partial^2\varphi}{\partial t^2}=0, \tag{1.7}$$

together with the corresponding boundary conditions. The stresses $\bar{\bar{\sigma}}_{ii}$ $(i=1,2,3)$ are expressed by the equations:

$$\begin{cases}\bar{\bar{\sigma}}_{11}=\lambda\bar{\bar{\Theta}}+2\mu\dfrac{\partial u}{\partial x}=(\lambda+2\mu)\dfrac{\partial^2\varphi}{\partial x^2}=\varrho\dfrac{\partial^2\varphi}{\partial t^2},\\[2ex] \bar{\bar{\sigma}}_{22}=\bar{\bar{\sigma}}_{33}=\lambda\bar{\bar{\Theta}}=\lambda\dfrac{\partial^2\varphi}{\partial x^2}=\lambda\sigma^2\dfrac{\partial^2\varphi}{\partial t^2}.\end{cases} \tag{1.8}$$

The resulting stresses will be obtained by adding $\bar{\sigma}_{ii}$ to σ_{ii}

$$(1.9)\quad \begin{cases} \sigma_{11} = \varrho\left(\dfrac{\partial^2 \Phi}{\partial t^2} + \dfrac{\partial^2 \varphi}{\partial t^2}\right), \\ \sigma_{22} = \sigma_{33} = -\,2GT\vartheta_0 + \lambda\sigma^2\left(\dfrac{\partial^2 \Phi}{\partial t^2} + \dfrac{\partial^2 \varphi}{\partial t^2}\right) = -\,2\,GT\vartheta_0 + \dfrac{\lambda\sigma^2}{\varrho}\,\sigma_{11}. \end{cases}$$

2. Plane Source of Heat in a Perfectly Elastic Semi-Space

Let a plane, continuous source of heat act at the distance $x = \xi$ from the $x = 0$ plane bounding the semi-space. Let us assume that $\partial T/\partial x = 0$ in the $x = 0$ plane.

Using the reflection method, we can represent the heat equation in the form

$$(2.1)\qquad \frac{\partial^2 T}{\partial x^2} - \frac{1}{\varkappa}\frac{\partial T}{\partial t} = -\frac{Q}{\varkappa}\left[\delta(x-\xi) + \delta(x+\xi)\right]\eta(t),$$

with the boundary conditions

$$(2.2)\qquad \frac{\partial T}{\partial x} = 0 \quad \text{for} \quad x = 0,$$

and $T = 0$ at infinity and the initial condition

$$(2.3)\qquad T(x, 0) = 0.$$

$\eta(t)$ in the Eq. (2.1) is Heaviside's function, δ-Dirac's function. Performing in the Eq. (2.1) Laplace transformation, and solving the transformed equation, (2.2) and (2.3) being taken into consideration, we obtain

$$(2.4)\qquad \begin{cases} \tilde{T}(x, p) = \dfrac{Q}{2\varkappa p\sqrt{q}}\left[e^{\mp(x-\xi)\sqrt{q}} + e^{-(x+\xi)\sqrt{q}}\right], \\ x \gtrless \xi, \qquad \xi > 0, \qquad q = p/\varkappa, \end{cases}$$

where

$$\tilde{T}(x, p) = \int_0^\infty e^{-pt}\,T(x, t)\,dt.$$

The upper (minus) sign in the Eq. (2.4) concerns the interval $\xi \leqslant x < \infty$, the lower (plus) sign — the interval $0 \leqslant x \leqslant \xi$.

Performing the inverse Laplace transformation, we obtain

$$(2.5)\quad T = \frac{Q}{2\sqrt{\varkappa}}\left\{f_1\left[\pm(x-\xi), t\right] + f_1\left[x+\xi, t\right]\right\}, \qquad x \gtrless \xi, \qquad \xi > 0,$$

where

$$f_1(z,t) = 2\sqrt{\frac{t}{\pi}}\, e^{-\frac{z^2}{4\varkappa t}} - \frac{z}{\sqrt{\varkappa}} \operatorname{erfc} \frac{z}{\sqrt{4\varkappa t}}.$$

The upper (plus) sign concerns the interval $\xi < x < \infty$, the lower (minus) sign the interval $0 < x < \xi$.

Next, we solve the equation (1.4). Performing the Laplace transformation we have:

$$\frac{d^2\tilde{\Phi}}{dx^2} - \sigma^2 p^2 \tilde{\Phi} = \vartheta_0 \tilde{T}. \tag{2.6}$$

This equation will be solved assuming that $\partial\tilde{\Phi}/\partial x = 0$ for $x = 0$ and $\tilde{\Phi} = 0$ at infinity.

It is assumed that:

$$\Phi(x, 0) = 0, \quad \left[\frac{\partial \Phi}{\partial t}\right]_{t=0} = 0.$$

The solution of the Eq. (2.6) is the function:

$$\tilde{\Phi}(x, p) = -\frac{\vartheta_0 Q}{2\varkappa p(\sigma^2 p^2 - q)} \left\{ \frac{1}{\sqrt{q}} \left[e^{\mp(x-\xi)\sqrt{q}} + e^{-(x+\xi)\sqrt{q}}\right] - \right.$$
$$\left. - \frac{1}{\sigma p}\left[e^{\mp(x-\xi)\sigma p} + e^{-(x+\xi)\sigma p}\right] \right\} \qquad x \gtrless \xi, \qquad \xi > 0. \tag{2.7}$$

The above result was obtained on the basis of the following solutions. Let us consider the two systems of equations

$$\frac{d^2\tilde{T}}{dx^2} - q\tilde{T} = -\frac{Q}{\varkappa p}\left[\delta(x-\xi) + \delta(x+\xi)\right], \qquad \frac{d^2\tilde{\Phi}}{dx^2} - p^2\sigma^2\tilde{\Phi} = \vartheta_0 \tilde{T}, \tag{2.8}$$

and two other systems:

$$\frac{d^2\tilde{\Psi}}{dx^2} - p^2\sigma^2\tilde{\Psi} = -\frac{Q}{\varkappa p}\left[\delta(x-\xi) + \delta(x+\xi)\right], \qquad \frac{d^2\tilde{F}}{dx^2} - q\tilde{F} = \vartheta_0 \tilde{\Psi}. \tag{2.9}$$

We assume that the boundary and initial conditions for the function $\tilde{T}$ and $\tilde{\Psi}$ are the same. This applies also to the function $\tilde{\Phi}$ and $\tilde{F}$.

Let us observe that eliminating from the system of equations (2.8) the function $\tilde{T}$, and from the equations (2.9) the function $\tilde{\Psi}$, we obtain equations of the same type and the same right-hand number. In view of the identity of the boundary conditions for the functions $\tilde{F}$ and $\tilde{\Phi}$, we have $\tilde{\Phi} = \tilde{F}$. The second equation of the system (2.8) can be represented in the form:

$$\frac{d^2\tilde{\Phi}}{dx^2} - q\tilde{\Phi} = \vartheta_0 \tilde{T} + (\sigma^2 p^2 - q)\tilde{\Phi}. \tag{2.10}$$

The left-hand member of this equation is, in view of the second of the Eqs. (2.9) and the relation $\widetilde{F}=\widetilde{\Phi}$, equal to $\vartheta_0\widetilde{\Psi}$. Therefore:

$$\widetilde{\Phi}=-\frac{\vartheta_0}{\sigma^2 p^2-q}(\widetilde{T}-\widetilde{\Psi}). \tag{2.11}$$

The function $\widetilde{\Psi}$ will be obtained directly from the Eq. (2.9) if q is replaced by $\sigma^2 p^2$,

$$\widetilde{\Psi}=\frac{Q}{2\varkappa\sigma^2 p^2}[e^{\mp(x-\xi)\sigma p}+e^{-(x+\xi)\sigma p}] \qquad x\gtrless\xi, \qquad \xi>0. \tag{2.12}$$

Knowing the function $\widetilde{\Phi}$, the stress $\widetilde{\widetilde{\sigma}}_{11}$ will be found from the Eq. (1.6):

$$\widetilde{\widetilde{\sigma}}_{11}=\varrho p^2\widetilde{\Phi}. \tag{2.13}$$

In the cross-section $x=0$ we obtain:

$$[\widetilde{\widetilde{\sigma}}_{11}]_{x=0}=-\frac{\vartheta_0 Q\varrho p}{\varkappa(\sigma^2 p^2-q)}\left(\frac{1}{\sqrt{q}}e^{-\xi\sqrt{q}}-\frac{1}{\sigma p}e^{-\xi\sigma p}\right). \tag{2.14}$$

This stress is different from zero. Let us determine $\widetilde{\varphi}$ from the Eq. (1.7):

$$\frac{d^2\widetilde{\varphi}}{dx^2}-\sigma^2 p^2\widetilde{\varphi}=0. \tag{2.15}$$

Taking the condition at infinity into consideration, the solution of the Eq. (2.15) has the form:

$$\widetilde{\varphi}(x,p)=A(p)e^{-p\sigma x}, \qquad x>0. \tag{2.16}$$

From the boundary condition

$$\widetilde{\widetilde{\sigma}}_{11}+\widetilde{\sigma}_{11}=0 \quad \text{for} \quad x=0,$$

we obtain:

$$A(p)=\frac{\vartheta_0 Q}{\varkappa(\sigma^2 p^2-q)p}\left(\frac{e^{-\xi\sqrt{q}}}{\sqrt{q}}-\frac{e^{-\xi\sigma p}}{\sigma p}\right). \tag{2.17}$$

The resultant stresses will be obtained from the Eq. (1.9). After performing the Laplace transformation, we have:

$$\widetilde{\sigma}_{11}=\varrho p^2(\widetilde{\Phi}+\widetilde{\varphi}), \qquad \widetilde{\sigma}_{22}=\widetilde{\sigma}_{33}=-2G\vartheta_0 T+\frac{p^2\lambda\sigma^2}{\varrho}\widetilde{\sigma}_{11}, \tag{2.18}$$

where

$$\widetilde{\sigma}_{11}=-\frac{\vartheta_0 Q\varrho}{2\varkappa\sigma^3\left(p-\frac{1}{\varkappa\sigma^2}\right)}\left\{\frac{\sigma\sqrt{\varkappa}}{\sqrt{p}}[e^{\pm(\xi-x)\sqrt{p\varkappa}}+ e^{-(x+\xi)\sqrt{p/\varkappa}}-2e^{-\xi\sqrt{p\varkappa}-x\sigma p}]-\frac{1}{p}[e^{\pm(\xi-x)\sigma p}-e^{-(x+\xi)\sigma p}]\right\}, \tag{2.19}$$

$$x\gtrless\xi, \qquad \xi>0.$$

Performing the inverse transformation, we obtain

$$(2.20)\quad \sigma_{11}(x,t) = -\frac{\vartheta_0 Q \varrho}{2\varkappa\sigma^3}\{\sigma\sqrt{\varkappa}\,[F_1[\pm(x-\xi),t]+ F_1[x+\xi,t]-2F_2(\xi,x,t)]-[F_3[\pm(x-\xi),t]-F_3(x+\xi,t)]\},$$

with the following notations:

$$(2.21)\quad \begin{cases} F_1(z,t) = \frac{1}{2}\,\sigma\sqrt{\varkappa}\,e^{\frac{t}{\varkappa\sigma^2}}\left[e^{-\frac{z}{\varkappa\sigma}}\operatorname{erfc}\left(\frac{z}{\sqrt{4\varkappa t}}-\sqrt{\frac{t}{\varkappa\sigma^2}}\right)- e^{\frac{z}{\varkappa\sigma}}\operatorname{erfc}\left(\frac{z}{\sqrt{4\varkappa t}}+\sqrt{\frac{t}{\varkappa\sigma^2}}\right)\right], \\ F_2(\xi,x,t) = \begin{cases} 0 & \text{for } t<\sigma x, \\ F_1(\xi, t-\sigma x) & \text{for } t>\sigma x, \end{cases} \\ F_3(z,t) = \begin{cases} 0 & \text{for } t<z, \\ \varkappa\sigma^2\left(e^{-\frac{1}{\varkappa\sigma^2}(t-z)}-1\right) & \text{for } t>z. \end{cases} \end{cases}$$

Let us consider the action of a plane source of heat in an elastic semi--space in the cross-section $x=\xi$. Assume that the plane $x=0$ is free from stress and that in this plane we have $T=0$. Using the method of reflections, placing in the infinite plate a positive plane source of heat in the $x=\xi$ plane and a negative source in the $x=-\xi$ plane, we obtain the following functions:

$$(2.22)\quad \widetilde{T} = \frac{Q}{2\varkappa p\sqrt{q}}\left(e^{\mp(x-\xi)\sqrt{q}}-e^{-(x+\xi)\sqrt{q}}\right), \qquad x \gtrless \xi, \qquad \xi>0,$$

$$(2.23)\quad \widetilde{\Phi} = -\frac{\vartheta_0 Q}{2\varkappa(\sigma^2p^2-q)p}\left\{\frac{1}{\sqrt{q}}\left[e^{\mp(x-\xi)\sqrt{q}}-e^{-(x+\xi)\sqrt{q}}\right]- \frac{1}{\sigma p}\left[e^{\mp(x-\xi)\sigma p}-e^{-(x+\xi)\sigma p}\right]\right\}, \qquad x\gtrless\xi, \qquad \xi>0.$$

Let us observe that in the $x=0$ plane we have $\widetilde{T}=0$, $\widetilde{\Phi}=0$ and $\widetilde{\sigma}_{11}=0$. Performing the inverse Laplace transform we obtain:

$$(2.24)\quad T(x,t) = \frac{Q}{2\sqrt{\varkappa}}\{f_1[\pm(x-\xi),t]-f_1(x+\xi,t)\}, \qquad x\gtrless\xi, \qquad \xi>0;$$

and

$$(2.25)\quad \sigma_{11} = \varrho\frac{\partial^2\Phi}{\partial t^2} = -\frac{\vartheta_0 Q\varrho}{2\varkappa\sigma^3}\{\sigma\sqrt{\varkappa}\,[F_1[\pm(x-\xi),t]-F_1(x+\xi,t)]- [F_3[\pm(x-\xi),t]-F_3(x+\xi,t)]\}, \qquad x\gtrless\xi, \qquad \xi>0.$$

The stresses σ_{22}, σ_{33} will be obtained from the Eq. (1.9).

Let us consider in addition the particular case of harmonic variation of the heat source in function of time $Q(t) = Q_0 e^{i\omega t}$. Since

$$T(x, t) = e^{i\omega t} H(x), \qquad \Phi(x, t) = e^{i\omega t}\Theta(x), \qquad \sigma_{ii}(x, t) = e^{i\omega t}\sigma_{ii}^{*}(x),$$

the Eqs. (2.1) and (1.4) will take the form:

$$\frac{d^2 H}{dx^2} - \frac{i\omega}{\varkappa} H = -\frac{Q_0}{\varkappa}[\delta(x-\xi) + \delta(x+\xi)], \tag{2.26}$$

$$\frac{d^2\Theta}{dx^2} - (\sigma i\omega)^2\Theta = \vartheta_0 H. \tag{2.27}$$

Solving the above equations under the assumption that $\partial T/\partial x = 0$, $\partial\Phi/\partial x = 0$ for $x = 0$, we obtain:

$$T(x,t) = \frac{Q_0 e^{i\omega t}}{2\varkappa\sqrt{i\eta}}[e^{\mp(x-\xi)\sqrt{i\eta}} + e^{-(x+\xi)\sqrt{i\eta}}], \quad \eta = \frac{\omega}{\varkappa}, \quad x \gtrless \xi, \quad \xi > 0, \tag{2.28}$$

$$\Phi(x,t) = \frac{Q_0 e^{i\omega t}\vartheta_0}{2\varkappa(\sigma^2\omega^2 + i\eta)}\left\{\frac{1}{\sqrt{i\eta}}[e^{\mp(x-\xi)\sqrt{i\eta}} + e^{-(x+\xi)\sqrt{i\eta}}] - \right.$$
$$\left. - \frac{1}{\sigma i\omega}[e^{\mp(x-\xi)i\sigma\omega} + e^{-(x+\xi)i\sigma\omega}]\right\}, \qquad x \gtrless \xi, \qquad \xi > 0. \tag{2.29}$$

Introducing, as before, in the case of a continuous source, a wave function $\varphi(x,t) = e^{i\omega t}\varphi^*(x)$, we obtain:

$$\varphi(x,t) = -e^{i\omega t}\frac{\vartheta_0 Q_0}{\varkappa(\sigma^2\omega^2 + i\eta)}\left(\frac{e^{-\xi\sqrt{i\eta}}}{\sqrt{i\eta}} - \frac{e^{-\xi\sigma i\omega}}{\sigma i\omega}\right)e^{-x\sigma i\omega}. \tag{2.30}$$

The stress σ_{11} will be found from the Eq. (1.9)

$$\sigma_{11} = -\frac{\vartheta_0 Q_0 \varrho\omega^2 e^{i\omega t}}{2\varkappa(\sigma^2\omega^2 + i\eta)}\left\{\frac{1}{\sqrt{i\eta}}[e^{\mp(x-\xi)\sqrt{i\eta}} + e^{-(x-\xi)\sqrt{i\eta}} - 2e^{-\xi\sqrt{i\eta} - \sigma i\omega x}] - \right.$$
$$\left. - \frac{1}{\sigma i\omega}[e^{\mp(\xi-x)\sigma i\omega} - e^{-(x+\xi)\sigma i\omega}]\right\}, \qquad x \gtrless \xi, \qquad \xi > 0. \tag{2.31}$$

If, in the plane $x = \xi$ a plane heat source acts, $Q(t) = Q_0 e^{i\omega t}$, and if we require that, $\sigma_{11} = 0$, $T = 0$ in the plane $x = 0$, then,

$$T(x,t) = \frac{Q_0 e^{i\omega t}}{2\varkappa\sqrt{i\eta}}[e^{\mp(x-\xi)\sqrt{i\eta}} - e^{-(x+\xi)\sqrt{i\eta}}], \qquad x \gtrless \xi, \qquad \xi > 0, \tag{2.32}$$

$$\Phi(x,t) = \frac{Q_0\vartheta_0 e^{i\omega t}}{2\varkappa(\sigma^2\omega^2 + i\eta)}\left\{\frac{1}{\sqrt{i\eta}}[e^{\mp(x-\xi)\sqrt{i\eta}} - e^{-(x+\xi)\sqrt{i\eta}}] - \right.$$
$$\left. - \frac{1}{\sigma i\omega}[e^{\mp(x-\xi)\sigma i\omega} - e^{-(x+\xi)\sigma i\omega}]\right\}, \qquad x \gtrless \xi, \qquad \xi > 0. \tag{2.33}$$

The stresses will be found from the equations:

$$\sigma_{11} = \varrho \frac{\partial^2 \Phi}{\partial t^2}, \qquad \sigma_{22} = \sigma_{33} = -2 G \vartheta_0 T + \lambda \sigma^2 \frac{\partial^2 \Phi}{\partial t^2}. \tag{2.34}$$

In the case of the source $Q(t) = Q_0 \cos \omega t$, we should take the real parts of the function $T, \varphi, \Phi, \sigma_{ii}$, and in the case of $Q(t) = Q_0 \sin \omega t$, the imaginary parts of these functions.

3. Plane Heat Source in a Viscoelastic Semi-Space

Let us consider a viscoelastic medium where the relations between the stresses and strains and the temperature are given by the equations, [8], [9]:

$$\sigma_{ij}^{(1)}(x_r, t) = 2 \int_0^t \mu(t-\tau) \frac{\partial}{\partial \tau} \varepsilon_{ij}^{(1)}(x_r, t)\, d\tau + \delta_{ij} \int_0^t \Big\{ \lambda(t-\tau) \frac{\partial \Theta^{(1)}(x_r, \tau)}{\partial \tau} - \\ - [3\lambda(t-\tau) + 2\mu(t-\tau)]\, a_t \frac{\partial T(x_r, \tau)}{\partial \tau} \Big\} d\tau, \tag{3.1}$$

$$P_1(D) P_3(D) \sigma_{ij}^{(2)}(x_r, t) = P_2(D) P_3(D) \varepsilon_{ij}^{(2)}(x_r, t) + \\ + \delta_{ij} \Big\{ \frac{1}{3} [P_1(D) P_4(D) - P_2(D) P_3(D)]\, \Theta^{(2)}(x_r, t) - P_1(D) P_4(D)\, a_t T \Big\}. \tag{3.2}$$

The relations (3.1) have been given by M. A. Biot, [9], and generalized by D. S. Berry, [10], to three-dimensional viscoelastic problems. To these relations are added terms taking the influence of the temperature into consideration. $\lambda(t)$, $\mu(t)$ are relaxation functions, which in the case of a perfectly elastic body reduce to Lamé's constants. In the relations (3.1), the operators $P_i(D)$ $(i = 1, 2, 3, 4)$ are represented by the equations, [9],

$$P_i(D) = \sum_{n=0}^{N_i} a_i^{(n)} D^n, \qquad a_i^{(N_i)} \neq 0, \tag{3.3}$$

where $D^n = \partial^n / \partial t^n$ denotes the n-th derivative with respect to the time. Substituting the Eqs. (3.1), (3.2), in the equations of equilibrium, expressing the stresses in terms of strains and these in terms of displacements and introducing the potential of thermoelastic strain Φ^*, we obtain:

$$\int_0^t [2\mu(t-\tau) + \lambda(t-\tau)] \frac{\partial}{\partial \tau} \nabla^2 \Phi^{*(1)}\, d\tau\, \varrho - \frac{\partial^2 \Phi^{(1)}}{\partial t^2} = \\ = \int_0^t [3\lambda(t-\tau) + 2\mu(t-\tau)] \frac{\partial T}{\partial \tau}\, d\tau, \tag{3.4}$$

$$(3.5)\quad \frac{1}{3}\left[2P_2(D)P_3(D)+P_4(D)P_1(D)\right]\nabla^2\Phi^{*(2)}-P_1(D)P_3(D)\varrho\frac{\partial^2\Phi^{*(2)}}{\partial t^2}=$$
$$=P_1(D)P_4(D)\alpha_t T.$$

Expressing also the Eqs. (3.1) and (3.2) in terms of the function Φ^*, and using the Eqs. (3.4) and (3.5), we obtain:

$$(3.6)\qquad \sigma_{ij}^{(1)}=\int_0^t 2\mu(t-\tau)\frac{\partial}{\partial\tau}\left(\frac{\partial^2}{\partial x_i\partial x_j}-\delta_{ij}\nabla^2\right)\Phi^{*(1)}d\tau+\varrho\delta_{ij}\frac{\partial^2\Phi^{*(1)}}{\partial t^2},$$

$$(3.7)\quad P_1(D)P_3(D)\sigma_{ij}^{(2)}=$$
$$=P_2(D)P_3(D)\left(\frac{\partial^2}{\partial x_i\partial x_j}-\delta_{ij}\nabla^2\right)\Phi^{*(2)}+P_1(D)P_3(D)\varrho\delta_{ij}\frac{\partial^2\Phi^{*(2)}}{\partial t^2}.$$

Performing on the Eqs. (3.4) and (3.5) the Laplace transformation we obtain, [11],

$$(3.8)\qquad \nabla^2\widetilde{\Phi}^*-p^2\sigma^2(p)\widetilde{\Phi}^*=\vartheta(p)\widetilde{T},$$

where the following notations have been introduced:

$$\sigma^2(p)=\frac{\varrho}{p\left[2\widetilde{\mu}(p)+\widetilde{\lambda}(p)\right]},\qquad \vartheta(p)=\frac{3\widetilde{\lambda}(p)+2\widetilde{\mu}(p)}{\widetilde{\lambda}(p)+2\widetilde{\mu}(p)}\alpha_t,\qquad G(p)=p\widetilde{\mu}(p),$$

for a viscoelastic body for which the relations between the state of stress and strain are given by the Eqs. (3.1);

$$\sigma^2(p)=\frac{3P_1(p)P_3(p)\varrho}{2P_2(p)P_3(p)+P_1(p)P_4(p)},\qquad \vartheta(p)=\frac{3P_1(p)P_4(p)\alpha_t}{2P_2(p)P_3(p)+P_1(p)P_4(p)}$$
$$G(p)=\frac{P_2(p)}{2P_1(p)},$$

for a viscoelastic body in which the Eqs. (3.2) are valid. In addition we assume for a viscoelastic body for which the Eqs. (3.2) are valid, that:

$$\Phi^{*(\beta-1)}(x_r,0)=0\quad \text{for}\quad \beta=1,2,\dots,\max\left[(N_1+N_3),(N_1+N_4),(N_1+N_3+2)\right],$$
$$T^{(\gamma-1)}(x_r,0)=0\quad \text{for}\quad \gamma=1,2,\dots,\max(N_1+N_4).$$

The initial conditions for Φ^* are at the same time the initial conditions for the displacements and stresses σ_{ij}.

After performing the Laplace transformation, the relations (3.6), (3.7) take the form

$$(3.9)\qquad \widetilde{\sigma}_{ij}(x_r,p)=2G(p)\left(\frac{\partial^2}{\partial x_i\partial x_j}-\delta_{ij}\nabla^2\right)\widetilde{\Phi}^*(x_r,p)+\delta_{ij}p^2\varrho\widetilde{\Phi}^*(x_r,p).$$

In our particular case of a one-dimensional problem, we have

$$\frac{d^2\tilde{\Phi}^*}{dx^2} - p^2\sigma^2(p)\tilde{\Phi}^* = \vartheta(p)\tilde{T}, \tag{3.10}$$

and

$$\tilde{\sigma}_{11} = p^2\varrho\Phi^*, \qquad \tilde{\sigma}_{22} = \tilde{\sigma}_{33} = -2G(p)\vartheta(p)\tilde{T} + [1-h(p)]\tilde{\sigma}_{11}, \tag{3.11}$$

where

$$h(p) = \frac{2G(p)\sigma^2(p)}{\varrho}.$$

The stresses $\tilde{\sigma}_{11}$, $\tilde{\sigma}_{22}$, $\tilde{\sigma}_{33}$ are expressed in terms of the function $\tilde{\varphi}$ by the equations

$$\tilde{\sigma}_{11} = p^2\varrho\tilde{\varphi}^*, \qquad \tilde{\sigma}_{22} = \tilde{\sigma}_{33} = [1-h(p)]\tilde{\sigma}_{11}, \tag{3.12}$$

and the function $\tilde{\varphi}^*$ satisfies the equation

$$\frac{d^2\tilde{\varphi}^*}{dx^2} - p^2\sigma^2(p)\tilde{\varphi}^* = 0. \tag{3.13}$$

Let a plane source of heat act in the $x=\xi$ plane of a viscoelastic semi-space. Let us assume that $\partial T/\partial x = 0$ in the plane $x=0$. The temperature field is given here also by the Eq. (2.5) and, after performing the Laplace transformation, by the Eq. (2.4). The function $\tilde{\Phi}^*$ will be found from the equation

$$\tilde{\Phi}^* = -\frac{\vartheta(p)}{p^2\sigma^2(p) - q}(\tilde{T} - \tilde{\Psi}^*), \tag{3.14}$$

where the function $\tilde{\Psi}^*$ satisfies the equation

$$\frac{d^2\tilde{\Psi}^*}{dx^2} - p^2\sigma^2(p)\tilde{\Psi}^* = -\frac{Q}{\varkappa p}[\delta(x-\xi) + \delta(x+\xi)], \tag{3.15}$$

with the boundary conditions $d\tilde{\Psi}^*/dx = 0$ for $x=0$ and $\tilde{\Psi}^* = 0$ at infinity.

Bearing in mind that

$$\tilde{\Psi}^* = \frac{Q}{2\varkappa p^2\sigma(p)}[e^{\mp(x-\xi)p\sigma(p)} + e^{-(x+\xi)p\sigma(p)}], \qquad x \gtrless \xi, \qquad \xi > 0, \tag{3.16}$$

we find:

$$\tilde{\Phi}^* = -\frac{\vartheta(p)Q}{2\varkappa p[p^2\sigma^2(p) - q]}\left\{\frac{1}{\sqrt{q}}[e^{\mp(x-\xi)\sqrt{q}} + e^{-(x+\xi)\sqrt{q}}] - \right.$$
$$\left. - \frac{1}{p\sigma(p)}[e^{\mp(x-\xi)p\sigma(p)} + e^{-(x+\xi)p\sigma(p)}]\right\}, \qquad x \gtrless \xi, \qquad \xi > 0. \tag{3.17}$$

The function $\widetilde{\varphi}^*$ will be obtained from the solution of the Eq. (3.13), assuming that $\widetilde{\varphi}^* = 0$ at infinity

$$\widetilde{\varphi}^* = A(p)\, e^{-p\sigma(p)x}. \tag{3.18}$$

The quantity $A(p)$ will be obtained from the boundary condition

$$\widetilde{\sigma}_{11} = \bar{\sigma}_{11} + \bar{\bar{\sigma}}_{11} = 0 \quad \text{for} \quad x = 0. \tag{3.19}$$

Hence:

$$A(p) = \frac{Q\,\vartheta(p)}{\varkappa p\,[p^2\sigma^2(p) - q]}\left[\frac{1}{\sqrt{q}}\, e^{-\xi\sqrt{q}} - \frac{1}{p\sigma(p)}\, e^{-\xi p\sigma(p)}\right].$$

The stress $\widetilde{\sigma}_{11}$ will be obtained from the equation

$$\widetilde{\sigma}_{11} = p^2 \varrho\,(\widetilde{\Phi}^* + \widetilde{\varphi}^*)$$

or

$$\bar{\sigma}_{11} = -\frac{\vartheta(p)\,Q p^2 \varrho}{2\varkappa p\,[p^2\sigma^2(p) - q]}\left\{\frac{1}{\sqrt{q}}\,[e^{\mp(x-\xi)\sqrt{q}} + e^{-(x+\xi)\sqrt{q}} - 2\,e^{-\xi\sqrt{q} - p\sigma(p)x}] - \frac{1}{p\sigma(p)}\,[e^{\mp(x-\xi)p\sigma(p)} - e^{-(x+\xi)p\sigma(p)}]\right\}, \quad x \gtrless \xi, \quad \xi > 0. \tag{3.20}$$

The stresses $\widetilde{\sigma}_{22}$, $\widetilde{\sigma}_{33}$ will be obtained from the equations

$$\widetilde{\sigma}_{22} = \widetilde{\sigma}_{32} = -2\,G(p)\,\vartheta(p)\,\widetilde{T} + \widetilde{\sigma}_{11}\,[1 - h(p)]. \tag{3.21}$$

Let us consider in a more detailed manner a viscoelastic body where the relations (3.1) are valid. Let us assume that the functions $\lambda(t)$, $\mu(t)$ are expressed by a simple exponential relation and have the same relaxation time ε^{-1}

$$\lambda(t) = \lambda_0\, e^{-\varepsilon t}, \qquad \mu(t) = \mu_0\, e^{-\varepsilon t}.$$

In the case considered we have:

$$\vartheta(p) = \frac{3\lambda_0 + 2\mu_0}{\lambda_0 + 2\mu_0}\,\alpha_t = \vartheta_* = \text{const}, \qquad \sigma^2(p) = \gamma\,\frac{p+\varepsilon}{p}, \qquad \gamma = \frac{\varrho}{\lambda_0 + 2\mu_0},$$

$$G(p) = \frac{\mu_0 p}{p+\varepsilon}, \qquad \beta = \frac{1}{\varkappa\gamma} - \varepsilon, \qquad h(p) = \frac{2\,G(p)\,\sigma^2(p)}{\varrho} = \frac{2\mu_0}{\lambda_0 + 2\mu_0} = \text{const}.$$

Therefore:

$$\widetilde{\sigma}_{11} = -\frac{Q\,\vartheta_*\,\varrho}{2\varkappa\gamma\sqrt{\gamma}\,(p-\beta)}\left\{\sqrt{\frac{\gamma\varkappa}{p}}\,[e^{\mp(x-\xi)\sqrt{q}} + e^{-(x+\xi)\sqrt{q}} - 2\,e^{-\xi\sqrt{q} - x\sqrt{\gamma p(p+\varepsilon)}}] - \frac{1}{\sqrt{p(p+\varepsilon)}}\,[e^{\mp(x-\xi)\sqrt{\gamma p(p+\varepsilon)}} - e^{-(x-\xi)\sqrt{p\gamma(p+\varepsilon)}}]\right\},$$

$$x \gtrless \xi, \qquad \xi > 0. \tag{3.22}$$

Performing the inverse Laplace transformation, we obtain [11]:

$$(3.23)\quad \sigma_{11}(x,t) = -\frac{Q\vartheta_* \varrho}{2\varkappa\gamma\sqrt{\gamma}}\{[H_1[\pm(x-\xi),t] + H_1(x+\xi,t) - 2N(\xi,x,t)]\sqrt{\gamma\varkappa} - \\ -[K[\pm(x-\xi),t] - K(x+\xi,t)]\}, \qquad x \gtrless \xi, \qquad \xi > 0.$$

The following notations have been introduced

$$H_1(z,t) = -\frac{1}{2\sqrt{\beta}} e^{\beta t}\left[e^{z\sqrt{\beta\varkappa}}\operatorname{erfc}\left(\frac{z}{\sqrt{4\varkappa t}} + \sqrt{\beta t}\right) - e^{-z\sqrt{\beta\varkappa}}\operatorname{erfc}\left(\frac{z}{\sqrt{4\varkappa t}} - \sqrt{\beta t}\right)\right],$$

$$N(\xi,x,t) = \int_0^t g(\xi, t-\tau) f(x,\tau)\, d\tau,$$

where

$$g(\xi,t) = -\frac{e^{\beta t}\sqrt{\beta}}{2}\left[e^{\xi\sqrt{\beta/\varkappa}}\operatorname{erfc}\left(\frac{\xi}{\sqrt{4\varkappa t}} + \sqrt{\beta t}\right) - \right. \\ \left. - e^{-\xi\sqrt{\beta\varkappa}}\operatorname{erfc}\left(\frac{\xi}{\sqrt{4\varkappa t}} - \sqrt{\beta t}\right)\right] + \frac{e^{-\xi^2/4\varkappa t}}{\sqrt{\pi t}},$$

$$f(x,t) = \left[e^{-\frac{x\varepsilon\sqrt{\gamma}}{2}} + \frac{\varepsilon x\sqrt{\gamma}}{2}\int_{x\sqrt{\gamma}}^{t} e^{-\frac{\varepsilon v}{2}} \frac{I_1\left(\frac{\varepsilon}{2}\sqrt{v^2 - x^2\gamma}\right)}{\sqrt{v^2 - x^2\gamma}}\, dv\right]\eta(t - x\sqrt{\gamma}),$$

and

$$K(z,t) = e^{-\frac{\varepsilon t}{2}} I_0\left(\frac{\varepsilon}{2}\sqrt{t^2 - z^2\gamma}\right)\eta(t - x\sqrt{\gamma}),$$

and η is the symbol of the Heaviside function.

The stresses $\widetilde{\sigma}_{22}$ and $\widetilde{\sigma}_{33}$ will be obtained from the Eq. (3.21):

$$(3.24)\quad \widetilde{\sigma}_{22} = \widetilde{\sigma}_{33} = -2G(p)\vartheta(p)\widetilde{T} + \widetilde{\sigma}_{11}[1 - h(p)] = \\ = -\frac{2\mu_0 p}{p+\varepsilon}\vartheta_*\widetilde{T} + \frac{2\mu_0}{\lambda_0 + 2\mu_0}\widetilde{\sigma}_{11}.$$

Performing the inverse Laplace transformation, we obtain

$$(3.25)\quad \sigma_{22} = \sigma_{33} = -\frac{Q\vartheta_*\mu_0}{\sqrt{\varkappa}}\{A[\pm(x-\xi),t] + A(x+\xi,t)\} + \frac{2\mu_0}{\lambda_0 + 2\mu_0}\sigma_{11},$$

where

$$A(z,t) = \frac{ie^{-\varepsilon t}}{2\sqrt{\varepsilon}}\left[\exp\left(iz\sqrt{\frac{\varepsilon}{\varkappa}}\right)\operatorname{erfc}\left(\frac{z}{\sqrt{4\varkappa t}} + i\sqrt{\varepsilon t}\right) - \right. \\ \left. - \exp\left(-iz\sqrt{\frac{\varepsilon}{\varkappa}}\right)\operatorname{erfc}\left(\frac{z}{\sqrt{4\varkappa t}} - i\sqrt{\varepsilon t}\right)\right].$$

Let a plane source of heat act in the plane $x=\xi$ of a viscoelastic semi-space. Let us assume that in this case we have $T=0$ in the $x=0$ plane. The function $\widetilde{\Phi}^*$ will be expressed by the equation:

$$(3.26)\quad \widetilde{\Phi}^* = -\frac{Q\vartheta(p)}{2\varkappa p\,[p^2\sigma^2(p)-q]}\left\{\frac{1}{\sqrt{q}}\,[e^{\mp(x-\xi)\sqrt{q}} - e^{-(x+\xi)\sqrt{q}}] - \frac{1}{p\sigma(p)}\,[e^{\mp(x-\xi)p\sigma(p)} - e^{-(x+\xi)p\sigma(p)}]\right\}, \qquad x \gtreqless \xi, \qquad \xi>0.$$

Since in this case we have $\widetilde{\varphi}^*=0$, therefore

$$(3.27)\quad \widetilde{\sigma}_{11} = \varrho p^2\widetilde{\Phi}^*, \qquad \widetilde{\sigma}_{22}=\widetilde{\sigma}_{33} = -2G(p)\vartheta(p)\widetilde{T} + [1-h(p)]\,\widetilde{\sigma}_{11},$$

where

$$(3.28)\quad \widetilde{T} = \frac{Q}{2\varkappa p\sqrt{q}}\,[e^{\mp(x-\xi)\sqrt{q}} - e^{-(x+\xi)\sqrt{q}}], \qquad x\gtreqless\xi, \qquad \xi>0.$$

For the model of the viscoelastic body assumed, we obtain:

$$(3.29)\quad \sigma_{11} = -\frac{Q\varrho\vartheta_*}{2\varkappa\gamma\sqrt{\gamma}}\,\{\sqrt{\gamma\varkappa}\,[H_1[\pm(x-\xi),t] - H_1(x+\xi,t)] - [K(\pm[x-\xi),t] - K(x+\xi,t)]\},$$

$$(3.30)\quad \sigma_{22}=\sigma_{33} = -\frac{Q\vartheta_*\mu_0}{\sqrt{\varkappa}}\,[A[\pm(x-\xi),t] - A(x+\xi,t)] + \frac{2\mu_0}{\lambda_0+2\mu_0}\,\sigma_{11}.$$

References

[1] В. И. Даниловская, *Температурные напряжения в упругом полупространстве, возникающие вследствие внезапного нагрева его границы*, Прикл. Мат. Мех. 3, **14** (1950), 316-318.

[2] В. И. Даниловская, *Об одной динамической задаче термоупругости*, Прикл. Мат. Мех., 3, **16** (1952), 341-344.

[3] H. Parcus, *Stress in Centrally Heated Disc.*, Proc. Sec. U. S. Nat. Congr. Appl. Mech., New York 1954.

[4] W. Nowacki, *A Dynamical Problem of Thermoelasticity*, Arch. Mech. stos., 3, **9** (1957), 325-334.

[5] А. М. Катасонов, *Распространение сферических термо-вязко-упругих возмущений*, Вестн. Моск. Ун-та, 2, **12** (1957), 39-49.

[6] P. Chadwick, I. N. Sneddon, *Plane Waves in an Elastic Solid Conducting Heat*, J. Mech. Phys. Solids, 3, **6** (1958), 223-230.

[7] F. J. Lockett, *Effect of Thermal Properties of a Solid on the Velocity of Rayleigh Waves*, J. Mech. Phys. Solids, 1, **7** (1958).

[8] M. A. Biot, *Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena*, J. appl. Phys. 11, 25 (1954), 1385-1391.

[9] E. Sternberg, *On Transient Thermal Stress in Linear Viscoelasticity*, Tech. Rep. No. 3, Brown University 1957.

[10] D. S. Berry, *Stress Propagation in Visco-Elastic Bodies*, J. Mech. Phys. Solids, 3, **6** (1958), 177-185.

[11] W. Nowacki, *Thermal Stress Propagation in Visco-Elastic Bodies*, Bull. Acad. Pol. Sci., cl. IV, 1, **7** (1959).

Streszczenie

DWA JEDNOWYMIAROWE ZAGADNIENIA TERMOSPRĘŻYSTOŚCI

W pracy przedstawiono rozwiązanie dwu zagadnień dynamicznych, odnoszących się do rozprzestrzeniania się naprężeń cieplnych, wywołanych działaniem płaskiego źródła ciepła, umieszczonego w płaszczyźnie $x = \xi$ półprzestrzeni sprężystej i półprzestrzeni lepko-sprężystej.

Rozpatrzono dwa warunki brzegowe dla temperatury: $T = 0$ oraz $\partial T/\partial x = 0$ w płaszczyźnie $x = 0$.

Dla ciała doskonale sprężystego uzyskano rozwiązanie w postaci zamkniętej, dla ciała lepko-sprężystego dla modelu M. A. Biota w postaci całek znanych funkcji.

Резюме

ДВЕ ОДНОМЕРНЫЕ ЗАДАЧИ ТЕРМОУПРУГОСТИ

Дается решение двух динамических задач, касающихся распространения термических напряжений, вызванных действием плоского источника тепла, расположенного в плоскости $x = \xi$ упругого полупространства. Рассматриваются два краевые условия для температуры $T = 0$, а также $\partial T/\partial x = 0$ плоскости $x = 0$.

Для абсолютно упругого тела получается решение в замкнутой форме, тогда как для вязко-упругого тела, для модели М. А. Биота — в виде интегралов известных функций.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA
IBTP POLISH ACADEMY OF SCIENCES

Received March 2, 1959