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THERMAL STRESSES DUE TO THE ACTION OF HEAT SOURCES IN A VISCOELASTIC SPACE

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Introduction

In this paper, we shall be concerned with thermal stresses in a viscoelastic space due to the action of heat sources. The problem will be treated as quasi-static, the inertia terms in the equations of equilibrium being rejected. We shall use the correspondence principle introduced by E. Sternberg, [1], thus enabling the use of the solutions of the analogous problems for a perfectly elastic body subjected to the action of a temperature field.

Two ways of solving the problem will be described: first — by means of the potential of thermoelastic displacement; second — by the method of V. Z. Mayzel, [2], generalized to viscoelastic problems.

The problem of thermal stresses in an elastic space caused by the action of a concentrated, linear or plane source of heat will be solved in detail for three simple models of a viscoelastic body.

1. Basic Equations of the Problem

Let us assume that the viscoelastic medium, of linear character, is homogeneous and isotropic. Let us assume also that the deformations are small and that the physical constants (mechanical and thermal) are independent of the coordinates and the temperature.

The relations between stress and strain are, [1],

$$P_1(D)\, s_{ij} = P_2(D)\, e_{ij}, \qquad P_3(D)\, \sigma_{ii} = P_4(D)\, (\varepsilon_{ii} - 3\, a_t T), \tag{1.1}$$

where σ_{ij}, s_{ij}, ε_{ij} and e_{ij} are the component of the stress tensor, stress deviator, strain tensor and strain deviator, respectively. Therefore

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\,\sigma_{kk}, \qquad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\,\varepsilon_{kk}, \tag{1.2}$$

where δ_{ij} denotes Kronecker's delta, α_t — the coefficient of thermal dilatation and T — the temperature. The operators $P_i(D)$ $(i=1, 2, 3, 4)$ are represented by the equations, [1],

$$P_i(D)=\sum_{n=0}^{N_i} a_i^{(n)} D^n, \qquad a_i^{(N_i)} \neq 0, \tag{1.3}$$

where $D^n=\partial^n/\partial t^n$ is the n-th derivative with respect to time t. The coefficients $a_i^{(n)}$ are independent of the coordinates and the temperature, and are constants. In the particular case of a perfectly elastic body, the operators $P_i(D)$ become the first terms of the sum (1.3)

$$a_1^{(0)}=1, \qquad a_2^{(0)}=2\,G, \qquad a_3^{(0)}=1, \qquad a_4^{(0)}=3\,K, \qquad N_i=0,$$

where G is the shear modulus and K — the compressibility modulus of a perfectly elastic body.

We shall consider quasi-static problems, the inertia terms and the mass forces being disregarded in the equations of equilibrium.

Therefore:

$$\frac{\partial \sigma_{ij}}{\partial x_j}=0\,. \tag{1.4}$$

The components of strain are given by the equations:

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{1.5}$$

Substituting the stresses in the equations of equilibrium (1.4), and expressing them by means of the Eqs. (1.1) and (1.5) in terms of displacements, we obtain the following system of three equations:

$$P_2(D)\,P_3(D)\,\nabla^2 u_i+\frac{1}{3}\,[2\,P_4(D)\,P_1(D)+P_2(D)\,P_3(D)]\,\frac{\partial \theta}{\partial x_i}= \tag{1.6}$$
$$=2\,P_4(D)\,P_1(D)\,\alpha_t\,\frac{\partial T}{\partial x_i} \qquad (i=1, 2, 3),$$

where ∇^2 is Laplace operator and $\theta=$ dilatation.

In order to find the particular integral of the system of equations (1.6), we shall introduce the potential of thermoelastic displacement Φ related to the displacement components by the equations:

$$u_i=\frac{\partial \Phi}{\partial x_i} \qquad (i=1, 2, 3). \tag{1.7}$$

Bearing in mind that

$$\nabla^2 u_i=\frac{\partial}{\partial x_i}\nabla^2 \Phi, \qquad \frac{\partial \theta}{\partial x_i}=\frac{\partial}{\partial x_i}\nabla^2 \Phi,$$

we shall reduce the system of equations (1.6), after integrating it with respect to x_i, to the unique equation:

$$(1.8)\qquad [2\,P_2(D)\,P_3(D)+P_4(D)\,P_1(D)]\,\nabla^2\Phi=3\,P_1(D)\,P_4(D)\,\alpha_t T.$$

Expressing the stresses by means of the function Φ and using the Eq. (1.8), we obtain the following equations for stresses:

$$(1.9)\qquad P_1(D)\,\sigma_{ij}=P_2(D)\left(\frac{\partial^2}{\partial x_i\,\partial x_j}-\delta_{ij}\nabla^2\right)\Phi\qquad (i=1,2,3).$$

For a bounded body, the function Φ does not satisfy any boundary conditions because the assumption (1.7) does not lead to the most general solution of the Eqs. (1.6).

To the displacements u_i, expressed by means of the relations (1.7), the displacements u_i should be added, thus satisfying the system of homogeneous equations (1.6) (or, in other words, for $T=0$), chosen in such a way that any boundary conditions are satisfied. In the subsequent considerations we shall confine ourselves to the state of stress in an infinite viscoelastic space; the stresses σ_{ij} expressed by the Eqs. (1.9) will be the stresses in which we are interested.

Let us assume that at the initial time the body was free, in its natural unstressed state. Applying Laplace transformation to the Eq. (1.8) and the relations (1.9) we obtain:

$$(1.10)\qquad \nabla^2\widetilde{\Phi}(x_r,p)=\vartheta(p)\,\widetilde{T}(x_r,p),$$

$$(1.11)\qquad \widetilde{\sigma}_{ij}(x_r,p)=2\,G(p)\left(\frac{\partial^2}{\partial x_i\,\partial x_j}-\delta_{ij}\nabla^2\right)\widetilde{\Phi}(x_r,p),$$

where

$$\widetilde{T}(x_r,p)=\int_0^\infty e^{-pt}\,T(x_r,t)\,dt,\qquad \widetilde{\Phi}(x_r,p)=\int_0^\infty e^{-pt}\,\Phi(x_r,t)\,dt,$$

$$\widetilde{\sigma}_{ij}(x_r,p)=\int_0^\infty e^{-pt}\,\sigma_{ij}(x_r,t)\,dt.$$

The quantities $G(p)$, $\vartheta(p)$ are functions of the parameter p of the transformation:

$$\vartheta(p)=\frac{3\,P_1(p)\,P_4(p)\,\alpha_t}{2\,P_2(p)\,P_3(p)+P_4(p)\,P_1(p)},\qquad 2\,G(p)=\frac{P_2(p)}{P_1(p)}.$$

It was assumed that:

$$\Phi^{(\beta-1)}(x_r,0)=0\qquad \text{for}\quad \beta=1,2,\dots,\max[(N_2+N_3);\ (N_1+N_4)],$$

$$T^{(\gamma-1)}(x_r,0)=0\qquad \text{for}\quad \gamma=1,2,\dots,(N_1+N_4).$$

The initial conditions for the functions Φ are at the same time the initial conditions for the displacements u_i and the stresses σ_{ij}.

Let us observe that the following relations, [3], are valid for a perfectly elastic body

$$\nabla^2 \tilde{\Phi}^0(x_r, p) = \vartheta_0 \tilde{T}(x_r, p), \qquad u_i^0 = \frac{\partial \Phi^0}{\partial x_i}, \tag{1.12}$$

$$\tilde{\sigma}_{ij}^0(x_r, p) = 2G\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\tilde{\Phi}^0, \tag{1.13}$$

where $\vartheta_0 = [(1+\nu)/(1-\nu)]\,\alpha_t$ and G are constants, and ν is Poisson's ratio.

Comparing the Eqs. (1.10) and (1.11) we see that:

$$\tilde{\Phi}(x_r, p) = \frac{\vartheta(p)}{\vartheta_0}\tilde{\Phi}^0(x_r, p), \tag{1.14}$$

$$\tilde{\sigma}_{ij}(x_r, p) = \frac{G(p)\,\vartheta(p)}{G\vartheta_0}\tilde{\sigma}_{ij}^0(x_r, p). \tag{1.15}$$

Introducing the functions $\tilde{F}(p)$ and $\tilde{f}(p)$ [1]

$$\tilde{F}(p) = \frac{G(p)\,\vartheta(p)}{p}, \qquad \tilde{f}(p) = \frac{\vartheta(p)}{p}, \tag{1.16}$$

and performing the inverse Laplace transformation, we obtain from the Eqs. (1.14) and (1.15):

$$\Phi(x_r, t) = \frac{1}{\vartheta_0}\int_0^t f(t-\tau)\frac{\partial}{\partial\tau}\Phi^0(x_r, \tau)\,d\tau, \tag{1.17}$$

$$\sigma_{ij}(x_r, t) = \frac{1}{G\vartheta_0}\int_0^t F(t-\tau)\frac{\partial}{\partial\tau}\sigma_{ij}^0(x_r, \tau)\,d\tau. \tag{1.18}$$

It is seen that, owing to the above equations, we can determine the displacements and the stresses in a viscoelastic body using the solutions for a perfectly elastic body.

In many cases, it will be more convenient to determine first the function

$$\Psi(x_r, t) = \frac{1}{\vartheta_0}\int_0^t F(t-\tau)\frac{\partial}{\partial\tau}\Phi^0(x_r, \tau)\,d\tau, \tag{1.19}$$

and use it to determine the stresses:

$$\sigma_{ij}(x_r, t) = 2\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\Psi(r, t). \tag{1.20}$$

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[1] The functions $\tilde{F}(p)$, $\tilde{f}(p)$ are assumed in a form securing the inverse transformation of these functions.

Let us consider the following relation between the state of stress and that of strain, different from (1.1):

$$\sigma_{ij}(x_r, t) = \delta_{ij} \int_0^t \lambda(t-\tau) \frac{\partial}{\partial \tau} \theta(x_r, \tau)\, d\tau + 2 \int_0^t \mu(t-\tau) \frac{\partial}{\partial \tau} \varepsilon_{ij}(x_r, t)\, d\tau, \quad t > 0. \tag{1.21}$$

This relation was given by M. A. Biot, [4], and generalized by D. S. Berry, [5], to three-dimensional viscoelastic problems. These relations concern also bodies free up to the initial moment. $\lambda(t), \mu(t)$ are relaxation functions which for a perfectly plastic body reduce to Lamé's constants.

In the case of thermal stresses, the relations (1.21) should be generalized to the form:

$$\sigma_{ij}(x_r, t) = \delta_{ij} \int_0^t \lambda(t-\tau) \frac{\partial}{\partial \tau} [\theta(x_r, \tau) - 3\alpha_t T(x_r, \tau)]\, d\tau + 2 \int_0^t \mu(t-\tau) \frac{\partial}{\partial \tau} [\varepsilon_{ij}(x_r, \tau) - \delta_{ij} \alpha_t T(x_r, \tau)]\, d\tau. \tag{1.22}$$

Let us perform on these relations Laplace's transformation. Then,

$$\widetilde{\sigma}_{ij}(x_r, p) = \delta_{ij}\, p \widetilde{\lambda}(p) [\widetilde{\theta}(x_r, p) - 3\alpha_t \widetilde{T}(x_r, p)] + 2p\widetilde{\mu}(p) [\widetilde{\varepsilon}_{ij}(x_r, p) - \delta_{ij} \alpha_t \widetilde{T}(x_r, p)]. \tag{1.23}$$

Expressing the stresses in terms of displacements, substituting in the equations of equilibrium (1.4) and introducing the potential of thermoelastic strain Φ, we obtain the equation

$$\nabla^2 \widetilde{\Phi}(x_r, p) = \widetilde{g}(p) \widetilde{T}(x_r, p), \tag{1.24}$$

where

$$\widetilde{g}(p) = \frac{3\widetilde{\lambda}(p) + 2\widetilde{\mu}(p)}{\widetilde{\lambda}(p) + 2\widetilde{\mu}(p)}\, \alpha_t .$$

The relations (1.23) will be transformed, using the Eq. (1.24), to obtain:

$$\widetilde{\sigma}_{ij}(x_r, p) = 2\widetilde{\mu}(p)\, p \left(\frac{\partial^2}{\partial x_i\, \partial x_j} - \delta_{ij} \nabla^2 \right) \widetilde{\Phi}(x_r, p). \tag{1.25}$$

Comparing the Eq. (1.24) and (1.25) with the Eq. (1.12) and (1.19), we find that:

$$\widetilde{\Phi}(x_r, p) = \frac{\widetilde{g}(p)}{\vartheta_0} \widetilde{\Phi}^0(x_r, p), \tag{1.26}$$

$$\widetilde{\sigma}_{ij}(x_r, p) = \frac{\widetilde{h}(p)}{\vartheta_0 G} \widetilde{\sigma}^0_{ij}(x_r, p)\, p, \qquad \widetilde{h}(p) = \widetilde{\mu}(p) \widetilde{g}(p). \tag{1.27}$$

Using the convolution theorem we have:

$$\Phi(x_r, t) = \frac{1}{\vartheta_0} \int_0^t g(t-\tau)\, \Phi^0(x_r, \tau)\, d\tau, \tag{1.28}$$

$$\sigma_{ij}(x_r, t) = \frac{1}{\vartheta_0 G} \int_0^t h(t-\tau) \frac{\partial}{\partial \tau} \sigma_{ij}^0(x_r, \tau)\, d\tau. \tag{1.29}$$

Sometimes it will be more convenient to determine the stresses by means of the function Ψ,

$$\sigma_{ij}(x_r, t) = 2\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij} \nabla^2\right) \Psi(x_r, t), \tag{1.30}$$

where

$$\Psi(x_r, t) = \frac{1}{\vartheta_0} \int_0^t h(t-\tau) \frac{\partial}{\partial \tau} \Phi^0(x_r, t)\, d\tau. \tag{1.31}$$

The Eqs. (1.28)-(1.31) represent the correspondence principle between the state of thermal stress in a viscoelastic body and a perfectly elastic body. Another method of finding the stresses and strains in viscoelastic bodies is possible, using solutions for a perfectly elastic body. The method which will be given here will be a generalization to viscoelastic problems of the known method of V. M. Mayzel, [2].

In this method, we find the displacements of the perfectly elastic body from the equation:

$$u_i^0(x_r, t) = \alpha_t \int_{(V)} T(\xi_r, t)\, S_i^0(x_r; \xi_r)\, dV \qquad (i = 1, 2, 3). \tag{1.32}$$

This equation is obtained by V. M. Mayzel as a particular case of E. Betti's theorem of reciprocity of displacements to thermal stresses.

In the Eq. (1.32) $u_i^0(x_r, t)$ denotes the displacement component of the point x_r in the direction of the x_i-axis, due to the temperature field $T(x_r, t)$. The function $S_i^0(x_r; \xi_r)$ is the Green's function, and represents the sum of normal stresses due to a concentrated unit force at the point ξ_r acting in the direction of the sought for displacement u_i^0, assuming that the body is in the isothermal state ($T = 0$).

Using Lee's correspondence principle, [6], we can represent the transformed displacement $\tilde{u}_i(x_r, p)$ in the viscoelastic body by an analogous integral expression,

$$\tilde{u}_i(x_r, p) = \alpha_t \int_{(V)} \tilde{T}(\xi_r, p)\, \tilde{S}_i(x_r; \xi_r)\, dV \qquad (i = 1, 2, 3), \tag{1.33}$$

where $\tilde{S}_i(x_r; \xi_r)$ is a sum of normal stresses at the point ξ_r of a viscoelastic body in an isothermal state ($T = 0$), due to the action of a concentrated unit force at the point ξ_r in the direction of the x_r-axis.

It is seen that knowing the function $S_i^0(x_r;\xi_r)$, the function $\widetilde{S}_i(x_r;\xi_r)$ may easily be found using the correspondence principle for a perfectly elastic and a viscoelastic body. Both bodies are in the same isothermal state. Integrating over the region of the body, we can determine from the Eq. (1.33) the function $\widetilde{u}_i(x_r, p)$ and, after performing the inverse transformation, the function $u_i(x_r, t)$. The stresses σ_{ij} will be obtained from the Eqs. (1.1) or (1.22).

2. Thermal Stresses Due to a Concentrated Heat Source

Let an instantaneous heat source act at the origin. It will result in a temperature field determined by the equation

$$T(R,t)=\frac{Q}{(4\,\pi\varkappa t)^{3/2}}\,e^{-R^2/4\varkappa t}, \qquad R=(x_1^2+x_2^2+x_3^2)^{1/2}, \tag{2.1}$$

where $\varkappa=\lambda/c\varrho$ and λ is the coefficient of heat conduction, ϱ — density and c — specific heat. Next, $Q=W/\varrho c$ where W is the quantity of heat produced by the heat source per unit time and volume.

Let us perform on the Eq. (2.1) Laplace's transformation. We have:

$$\widetilde{T}(R,p)=\frac{Q}{4\,\pi\varkappa R}\,e^{-R\sqrt{p/\varkappa}}. \tag{2.2}$$

The solutions of the Eq. (1.12) is, [7]:

$$\widetilde{\Phi}^0(R,p)=-\frac{\vartheta_0 Q}{4\pi Rp}\left(1-e^{-R\sqrt{p/\varkappa}}\right). \tag{2.3}$$

After performing the inverse Laplace transformation we have:

$$\Phi^0(R,t)=-\frac{\vartheta_0 Q}{4\,\pi R}\,\operatorname{erf}\left(\frac{R}{\sqrt{4\,\varkappa t}}\right), \qquad \operatorname{erf} z=\frac{2}{\sqrt{\pi}}\int_0^z e^{-\eta^2}\,d\eta. \tag{2.4}$$

According to the Eq. (1.19) we have:

$$\Psi(R,t)=-\frac{Q}{4\pi R}\int_0^t F(t-\tau)\frac{\partial}{\partial\tau}\operatorname{erf}\frac{R}{\sqrt{4\pi\tau}}\,d\tau, \qquad F(t)=L^{-1}\widetilde{F}(p). \tag{2.5}$$

The stresses σ_{ij} will be determined from the Eqs. (1.20):

$$\sigma_{RR}(R,t)=-\frac{4}{R}\frac{\partial\Psi}{\partial R}, \qquad \sigma_{\varphi\varphi}=\sigma_{\theta\theta}=-2\left(\frac{\partial^2\Psi}{\partial R^2}+\frac{1}{R}\frac{\partial\Psi}{\partial R}\right). \tag{2.6}$$

Let us consider two particular models of a viscoelastic body: the Kelvin model and the Maxwell model.

For the Kelvin model we have, [1],

$$\left\{\begin{array}{ll} P_1(D)=1, & P_2(D)=2\,G_K(1+t_* D), \\ P_3(D)=D, & P_4(D)=3\,K_K D, \end{array}\right. \tag{2.7}$$

where $t_*=\eta/G_K$ represents the retardation time and η is the viscosity. The constants G_K, K_K are the shear and compressibility moduli of the Kelvin body. For $t_* \to 0$ the viscoelastic body becomes perfectly elastic, the moduli G_K, K_K become G, K of the perfectly elastic body.

Bearing in mind that the Eqs. (2.7) will be obtained from the Eq. (1.16),

$$\widetilde{F}(p)=9\,G_K K_K a_t \frac{pt_*+1}{p\,[3\,K_K+4\,G_K(1+pt_*)]}=a\left(\frac{1}{p}-\frac{\beta}{p+\varkappa_1}\right), \tag{2.8.1}$$

where

$$a=\frac{9\,K_K G_K a_t}{4G_K+3\,K_K}, \qquad \varkappa_1=\frac{4\,G_K+3\,K_K}{4\,G_K t_*}, \qquad \beta=1-\varkappa_1 t_*.$$

We perform the inverse Laplace transformation

$$F(t)=a\,(1-\beta e^{-\varkappa_1 t}) \tag{2.8.2}$$

and substitute the function $F(t-\tau)$ in the Eq. (2.5). After performing the operations required we obtain

$$\Psi(R,t)=-\frac{Qa}{4\pi R}\left[1-\beta e^{-\varkappa_1 t}-\operatorname{erfc}\frac{R}{\sqrt{4\varkappa t}}+\beta N(R,t,\varkappa_1)\right], \tag{2.9}$$

where

$$N(R,t,\varkappa_1)=\frac{1}{2}e^{-\varkappa_1 t}\left[e^{-iR\sqrt{\varkappa_1/\varkappa}}\operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}}-i\sqrt{\varkappa_1 t}\right)+\right.$$

$$\left.+\,e^{iR\sqrt{\varkappa_1/\varkappa}}\operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}}+i\sqrt{\varkappa_1 t}\right)\right],$$

$$\operatorname{erfc} z=1-\operatorname{erf} z.$$

It may be shown that the function $N(R,t,\varkappa_1)$ is real for any real $\varkappa_1$.

For the Maxwell model of a viscoelastic body we have:

$$\left\{\begin{array}{ll} P_1(D)=t_0^{-1}+D, & P_2(D)=2\,G_M D, \\ P_3(D)=D, & P_4(D)=3\,K_M D, \end{array}\right. \tag{2.10}$$

where $t_0=\eta/G_M$ is the relaxation time and η — the viscosity.

The constants G_M, K_M are the shear modulus and the compressibility modulus for the Maxwell body. For $t_0 \to \infty$ these moduli become G, K of the perfectly elastic body.

For the Maxwell model we have:

$$\widetilde{F}(p)=\frac{b}{p+\varkappa_2}, \qquad b=3\,G_M a_t t_0 \varkappa_2, \qquad \varkappa_2=\frac{3\,K_M t_0^{-1}}{4\,G_M+3\,K_M}. \tag{2.11.1}$$

Introducing in (2.5) the function

$$F(t-\tau) = be^{-\varkappa_2(t-\tau)}, \tag{2.11.2}$$

we obtain after performing the operations required

$$\Psi(R, t) = -b\,|e^{-\varkappa_2 t} - N(R, t, \varkappa_2)|, \tag{2.12}$$

where

$$N(R, t, \varkappa_2) = \frac{1}{2} e^{-\varkappa_2 t}\left[e^{-iR\sqrt{\varkappa_2/\varkappa}} \operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}} - i\sqrt{\varkappa_2 t}\right) + \right.$$
$$\left. + e^{iR\sqrt{\varkappa_2/\varkappa}} \operatorname{erfc}\left(\frac{R}{\sqrt{4\varkappa t}} + i\sqrt{\varkappa_2 t}\right)\right].$$

The stresses σ_{ij} will be determined from the Eq. (2.6) by performing the prescribed differentiations on the function Ψ. If a continuous heat source acts at the origin, then, [7],

$$\widetilde{\Phi}^0(R, p) = -\frac{\vartheta_0 Q}{4\pi R p^2}(1 - e^{-R\sqrt{p/\varkappa}})$$

and

$$\Phi^0(R, t) = \frac{\vartheta_0 QR}{8\pi\varkappa}\left[1 - \left(1 + \frac{\vartheta}{2R^2}\right) \operatorname{erf}\frac{R}{\sqrt{\vartheta}} - \frac{1}{R}\sqrt{\frac{\vartheta}{\pi}} e^{-R^2/\vartheta}\right], \quad \vartheta = 4\varkappa t. \tag{2.13}$$

The function Ψ for a continuous heat source will be found from the equation:

$$\Psi(R, t) = \frac{QR}{8\pi\varkappa}\int_0^t F(t-\tau)\frac{\partial}{\partial\tau}\left[1 - \left(1 + \frac{\vartheta}{2R^2}\right) \operatorname{erf}\frac{R}{\sqrt{\vartheta}} - \frac{1}{R}\sqrt{\frac{\vartheta}{\pi}} e^{-R^2/\vartheta}\right], \ \vartheta = 4\varkappa t.$$

Bearing in mind the Eq. (2.8.2), we obtain for the Kelvin model:

$$\Psi(R, t) = -\frac{Qa}{4\pi R}\left[\left(t + \frac{R^2}{2\varkappa}\right) \operatorname{erf}\frac{R}{\vartheta} - \frac{R^2}{2\varkappa} + \frac{R}{2\varkappa}\sqrt{\frac{\vartheta}{\pi}} e^{-R^2/\vartheta} + \right.$$
$$\left. + \frac{\beta}{\varkappa_1} e^{-\varkappa_1 t} - \frac{\beta}{\varkappa_1} N(R, t, \varkappa_1) - \frac{\beta}{\varkappa_1} \operatorname{erf}\frac{R}{\sqrt{\vartheta}}\right], \qquad \vartheta = 4\varkappa t. \tag{2.14}$$

For the Maxwell model we obtain:

$$\Psi(R, t) = \frac{b}{4\pi R}\left[e^{-\varkappa_2 t} - N(R, t, \varkappa_2) - \operatorname{erf}\sqrt{\frac{R}{\sqrt{\vartheta}}}\right], \qquad \vartheta = 4\varkappa t. \tag{2.15}$$

Let an instantaneous heat source act at the origin and let the relations between the state of stress and that of strain be described by the Eqs. (1.22).

Let us assume that the relaxation functions $\lambda(t)$ and $\mu(t)$ have the same relaxation time α^{-1}

$$\lambda(t) = \lambda_0 e^{-\alpha t}, \qquad \mu(t) = \mu_0 e^{-\alpha t}. \tag{2.16}$$

Since

$$\widetilde{\lambda}(p)=\frac{\lambda_0}{p+a}, \qquad \widetilde{\mu}(p)=\frac{\mu_0}{p+a},$$

we have

$$\widetilde{h}(p)=\widetilde{\mu}(p)\,\widetilde{g}(p)=\frac{\gamma}{p+a} \qquad \text{where} \qquad \gamma=\mu_0\,\frac{(3\lambda_0+2\mu_0)}{\lambda_0+2\mu_0}.$$

In view of the Eq. (1.31) we have:

$$\Psi(R,t)=-\frac{Q\gamma}{4\pi R}\int_0^t e^{-a(t-\tau)}\frac{\partial}{\partial\tau}\operatorname{erf}\frac{R}{\sqrt{4\varkappa\tau}}\,d\tau; \tag{2.17}$$

or:

$$\Psi(R,t)=-\frac{Q\gamma}{3\pi R}\left[e^{-at}-N(R,t,a)\right]. \tag{2.18}$$

For a continuous source we have:

$$\Psi(R,t)=-\frac{Q\gamma}{4\pi R}\left[1-e^{-at}\operatorname{erfc}\frac{R}{\sqrt{4\varkappa t}}+N(R,t,a)\right]. \tag{2.19}$$

The stresses σ_{ij} will be obtained from the Eq. (1.30).

Let us determine also the displacements u_R due to the action of an instantaneous heat source, using the Eqs. (1.33).

Let us determine first the function $S^0(R,\varrho)$ for a perfectly elastic body. In the particular case under consideration, characterized by the spherical symmetry, $S^0(R,\varrho)$ will be treated as a sum of normal stresses $\sigma^{0,*}_{RR}+\sigma^{0,*}_{\varphi\varphi}+\sigma^{0,*}_{\theta\theta}$ due to the action of a unit load, uniformly distributed over the sphere of radius ϱ. Let us denote by U_R the displacement corresponding to this state. This displacement should satisfy the displacement equation:

$$\frac{d}{dR}\left(\frac{1}{R^2}\frac{d}{dR}(R^2U_R)\right)=0. \tag{2.20}$$

The solution of this equation is $U_R=AR+B/R^2$. Since the displacement U_R should be finite for $R=0$ and equal to zero for $R\to\infty$, therefore:

$$U^{\mathrm{I}}_R=AR, \qquad 0<R\leqslant\varrho, \qquad U^{\mathrm{II}}_R=\frac{B}{R^2}, \qquad \varrho\leqslant R<\infty.$$

In view of the continuity of displacement on the sphere $R=\varrho$ we have $A=B$.

The quantity A will be determined from the relation

$$\left[\sigma^{0,*\mathrm{I}}_{RR}-\sigma^{0,*\mathrm{II}}_{RR}\right]_{R=\varrho}=1. \tag{2.21}$$

Bearing in mind that

$$\sigma_{RR}^{0,*} = \frac{2G}{1-2\nu}\left[2\nu\frac{U_R}{R} + (1-\nu)\frac{dU_R}{dR}\right], \tag{2.22}$$

we obtain from the Eq. (2.21) the relation $A=B=[(1-2\nu)/3E][(1+\nu)/(1-\nu)]$.

The dilatation $\theta^{0,*}$ is expressed by the equation:

$$\theta^{0,*} = \frac{1-2\nu}{E}S^0 = \frac{dU_R}{dR} + 2\frac{U}{R}. \tag{2.23}$$

It can easily be seen that S^0 is given by the following equations:

$$\begin{cases} S^0(R,\varrho) = \dfrac{1+\nu}{1-\nu} & \text{for} \quad 0 < R < \varrho, \\ S^0(R,\varrho) = 0 & \text{for} \quad \varrho \leqslant R < \infty. \end{cases} \tag{2.24}$$

The function $S(R, \varrho)$ for a viscoelastic body will be found by substituting in place of $(1+\nu)/(1-\nu) = 9K/(4G+3K)$ the quantity

$$\frac{3P_4(p)P_1(p)}{2P_2(p)P_3(p)+P_4(p)P_1(p)} = \tilde{k}_1(p)p$$

in the case of a viscoelastic body in which the equations (1.1) are valid, or the quantity

$$\tilde{k}_2(p)p = \frac{3\tilde{\lambda}(p)+2\tilde{\mu}(p)}{\tilde{\lambda}(p)+2\tilde{\mu}(p)}$$

for a body where the relations (1.22) are valid.

Performing the inverse Laplace transformation on the Eq. (1.33) we obtain:

$$U_R^{(j)}(R,t) = \frac{a_t}{R^2}\int_0^R \varrho^2\left[\int_0^t k_j(t-\tau)\frac{\partial}{\partial\tau}T(\varrho,\tau)\,d\tau\right]d\varrho \qquad (j=1,2). \tag{2.25}$$

For a viscoelastic body for which the relations (1.22) are valid, and assuming that $\lambda(t) = \lambda_0 e^{-\alpha t}$, $\mu(t) = \mu_0 e^{-\alpha t}$, we obtain from the Eq. (2.25):

$$U_R(R,t) = \frac{Q a_t (3\lambda_0 + 2\mu_0)}{4\pi R^2(\lambda_0+2\mu_0)}\left(\operatorname{erf}\frac{R}{\sqrt{4\varkappa t}} - \frac{R}{\sqrt{\pi\varkappa t}}e^{-R^2/4\varkappa t}\right). \tag{2.26}$$

3. Thermal Stresses Due to the Action of an Instantaneous Linear Heat Source

Let an instantaneous linear heat source act at the origin. This will result in a temperature field determined by the equation:

$$T(r,t) = \frac{Q}{4\pi\varkappa t}e^{-r^2/4\varkappa t}, \qquad r = (x_1^2+x_2^2)^{1/2}. \tag{3.1}$$

The Eq. (1.12) takes the form

$$\frac{\partial^2 \widetilde{\Phi}^0}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{\Phi}^0}{\partial r} = \vartheta_0 \widetilde{T}, \qquad \widetilde{T} = \frac{Q}{2\pi\varkappa} K_0\left(r\sqrt{p/\varkappa}\right), \tag{3.2}$$

where $K_0(z)$ is a modified Bessel's function of the third kind. Solving the Eq. (3.2) we obtain:

$$\widetilde{\Phi}^0(r, p) = \frac{Q\vartheta_0}{2\pi p}\left(\ln r + K_0\left(r\sqrt{p/\varkappa}\right)\right). \tag{3.3}$$

Performing the inverse Laplace transformation we obtain

$$\Phi(r, t) = \frac{Q\vartheta_0}{2\pi}\left[\ln r - \frac{1}{2}\,\mathrm{Ei}\left(\frac{-r^2}{4\varkappa t}\right)\right], \tag{3.4}$$

where

$$-\,\mathrm{Ei}(-x) = \int_x^\infty \frac{e^{-u}}{u}\,du.$$

Let us determine the function $\Psi(r, t)$ from the Eq. (1.19):

$$\Psi(r, t) = \frac{Q}{2\pi}\int_0^t F(t-\tau)\left[\delta(\tau)\ln r + \frac{e^{-r^2/4\varkappa t}}{2\tau}\right]d\tau. \tag{3.5}$$

The stresses σ_{ij} will be found from the equation:

$$\begin{cases} \sigma_{rr} = -\dfrac{2}{r}\dfrac{\partial \Psi}{\partial r}, \qquad \sigma_{\varphi\varphi} = -2\dfrac{\partial^2 \Psi}{\partial r^2}, \\ \sigma_{33} = -2\nabla_r^2 \Psi = -2\displaystyle\int_0^t F(t-\tau)\frac{\partial}{\partial \tau} T(r, \tau)\,d\tau. \end{cases} \tag{3.6}$$

For the Kelvin model of a viscoelastic body we obtain, bearing in mind (2.8):

$$\Psi(r, t) = \frac{aQ}{2\pi}\left[\ln r - \frac{1}{2}\,\mathrm{Ei}\left(\frac{-r^2}{4\varkappa t}\right) - \beta e^{-\varkappa_1 t}\ln r - \frac{\beta}{2}\int_0^t e^{-\varkappa_1(t-\tau)}\frac{e^{-r^2/4\varkappa t}}{\tau}\,d\tau\right]. \tag{3.7}$$

For the Maxwell model we have:

$$\Psi(r, t) = \frac{Qb}{2\pi}\left[e^{-\varkappa_2 t}\ln r + \frac{1}{2}\int_0^t e^{-\varkappa_2(t-\tau)}\frac{e^{-r^2/4\varkappa t}}{\tau}\,d\tau\right]. \tag{3.8}$$

4. Thermal Stresses Due to the Action of an Instantaneous Plane Heat Source

Let a plane instantaneous source of heat act in the x_2, x_3 plane. The temperature field is determined by the equation:

$$T(x_1, t) = \frac{Q\, e^{-x_1^2/\vartheta}}{\sqrt{\pi\vartheta}}, \qquad \vartheta = 4\varkappa t. \tag{4.1}$$

The stresses and strains depend only on the variable x_1. Thus, according to the Eq. (1.11) we have:

$$\tilde{\sigma}_{11} = 0, \qquad \tilde{\sigma}_{22} = \tilde{\sigma}_{33} = -2\,G(p)\frac{\partial^2 \tilde{\Phi}}{\partial x_1^2}. \tag{4.2}$$

On the other hand, bearing in mind (1.10),

$$\frac{d^2\tilde{\Phi}}{dx_1^2} = \vartheta(p)\,\tilde{T}, \tag{4.3}$$

we find that the stresses $\tilde{\sigma}_{22}$, $\tilde{\sigma}_{33}$ may be expressed by the simple relation:

$$\tilde{\sigma}_{22} = \tilde{\sigma}_{33} = 2\tilde{F}(p)\,p\,\tilde{T}. \tag{4.4}$$

Performing the inverse Laplace transformation we obtain:

$$\sigma_{22} = \sigma_{33} = -2\int_0^t F(t-\tau)\frac{\partial}{\partial\tau}T(x_1, \tau)\,d\tau. \tag{4.5}$$

For the Kelvin model we have:

$$\sigma_{22} = \sigma_{33} = -\frac{Qa}{\sqrt{\varkappa}}\left[\frac{1}{\sqrt{\pi t}}\,e^{-x_1^2/\vartheta} - \beta M(x_1, t, \varkappa_1) - \beta\frac{e^{-x_1^2/\vartheta}}{\sqrt{\pi t}}\right]. \tag{4.6}$$

For the Maxwell model we have

$$\sigma_{22} = \sigma_{33} = -\frac{Qb}{\sqrt{\varkappa}}\left(\frac{e^{-x_1^2/\vartheta}}{\sqrt{\pi t}} + M(x_1, t, \varkappa_2)\right), \tag{4.7}$$

where

$$M(x_1, t; \varkappa_{1,2}) = \left[\frac{1}{2i}e^{ix_1\sqrt{\varkappa_{1,2}/\varkappa}}\operatorname{erfc}\left(\frac{x_1}{\sqrt{\vartheta}} + i\sqrt{\varkappa_{1,2}\,t}\right) - e^{-ix_1\sqrt{\varkappa_{1,2}/\varkappa}}\operatorname{erfc}\left(\frac{x_1}{\sqrt{\vartheta}} - i\sqrt{\varkappa_{1,2}\,t}\right)\right].$$

It can be shown that the stresses σ_{22}, σ_{33} are real functions if $\varkappa_1, \varkappa_2$ are real, positive or negative. In our case $\varkappa_1 > 0$, $\varkappa_2 > 0$.

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Streszczenie

NAPRĘŻENIA CIEPLNE WYWOŁANE DZIAŁANIEM ŹRÓDEŁ CIEPŁA W PRZESTRZENI LEPKO-SPRĘŻYSTEJ

W pracy rozpatrzono stan naprężenia wywołany w ciele lepko-sprężystym działaniem skupionego, liniowego i płaskiego źródła ciepła. Zagadnienie potraktowano jako quasi-statyczne nie uwzględniając członów inercyjnych w równaniach równowagi.

Wykorzystano zasadę odpowiedniości obmyśloną przez E. Sternberga dla naprężeń cieplnych w ciałach lepko-sprężystych, a pozwalającą na wykorzystanie wyników uzyskanych dla ciał doskonale sprężystych.

Przedstawiono dwie drogi rozwiązania postawionego zagadnienia: za pomocą potencjału termosprężystego przemieszczenia oraz za pomocą rozszerzonej na zagadnienia lepko-sprężyste metody W. Z. Majziela.

Dla trzech prostych modeli ciała lepko-sprężystego wyznaczono stan naprężenia. Wyniki uzyskano bądź w postaci zamkniętej, bądź w postaci całek oznaczonych.

Резюме

ТЕРМИЧЕСКИЕ НАПРЯЖЕНИЯ, ВЫЗВАННЫЕ ДЕЙСТВИЕМ ИСТОЧНИКОВ ТЕПЛА В ВЯЗКО-УПРУГОМ ПРОСТРАНСТВЕ

Рассматривается напряженное состояние, вызванное в вязко-упругом пространстве действием линейного и плоского источника тепла. Вопрос обсуждается, как квази-статическую задачу, не учитывая в уравнениях равновесия, инерционных выражений.

Применяется принцип соответственности Е. Штернберга для термических напряжений в вязко-упругих телах, дающий возможность использовать результаты, полученные для абсолютно упругих тел.

Представлены два метода решения этого вопроса при помощи термоупругого потенциала перемещения и при помощи расширенного на вязко-упругие проблемы метода В. З. Майзеля.

Для трех простых моделей вязко-упругого тела определяется напряженное состояние. Получаются результаты в замкнутом виде или в виде определенных интегралов.

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