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THERMAL STRESSES IN TRANSVERSALLY ISOTROPIC BODIES

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1. General Statement of the Problem. Basic Relations

This paper aims to determine the state of stress due to the action of a continuous and discontinuous temperature field in an elastic body showing transversal isotropy. In particular, we shall discuss the state of stress in an infinite space, a semi-space and in an elastic layer both for steady and non-steady temperature fields. This problem was solved recently by B. Sharma, [1], for a continuous steady temperature field using the potential of thermoelastic displacement. In the present paper, another method of solution is proposed, based on the displacement function. We shall assume that the temperature field does not influence the mechanical properties of the elastic body. For a non-steady temperature field, the inertia forces will be rejected, whereby the problem becomes treated as a quasi-static one. The system of coordinates will be assumed in such a way that the three planes coincide with those of elastic symmetry. Denote by E, ν Young's modulus and Poisson's ratio in the directions x_1, x_2, and by E', ν' the same quantities for the direction x_3. Let α, λ denote the coefficients of thermal expansion and heat transfer, respectively, in the directions x_1, x_2, and α', λ' the same quantities in the direction x_3.

The stress components σ_{ij} $(i, j = 1, 2, 3)$ are related to the displacement components u_i $(i = 1, 2, 3)$ by the equations:

$$
(1.1) \qquad \left\{
\begin{aligned}
&\sigma_{11} = A_{11}\frac{\partial u_1}{\partial x_1} + A_{12}\frac{\partial u_2}{\partial x_2} + A_{13}\frac{\partial u_3}{\partial x_3} - \beta T,\\
&\sigma_{22} = A_{12}\frac{\partial u_1}{\partial x_1} + A_{11}\frac{\partial u_2}{\partial x_2} + A_{13}\frac{\partial u_3}{\partial x_3} - \beta T,\\
&\sigma_{33} = A_{13}\frac{\partial u_1}{\partial x_1} + A_{13}\frac{\partial u_2}{\partial x_2} + A_{33}\frac{\partial u_3}{\partial x_3} - \beta' T,\\
&\sigma_{12} = A_{66}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad \sigma_{13} = A_{44}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right),\\
&\sigma_{23} = A_{44}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right),
\end{aligned}
\right.
$$

where

$$(1.2)\quad \begin{cases} A_{11}=\dfrac{1}{DE'}\left(\dfrac{1}{E}-\dfrac{\nu'^2}{E'}\right), \qquad A_{12}=\dfrac{1}{DE'}\left(\dfrac{\nu}{E}+\dfrac{\nu'^2}{E'}\right), \qquad A_{13}=\dfrac{\nu^2(1+\nu)}{EE'D}, \\ A_{33}=\dfrac{1-\nu^2}{DE^2}, \qquad A_{44}=\dfrac{E'}{2(1+\nu')}, \qquad A_{66}=\dfrac{E}{2(1+\nu)}, \\ \beta=\dfrac{1+\nu}{DEE'}(\alpha+\alpha'\nu'), \qquad \beta'=\dfrac{1+\nu}{ED}\left(\dfrac{2\nu'}{E'}\alpha+\dfrac{1-\nu}{E}\alpha'\right), \\ D=\dfrac{1+\nu}{EE'}\left(\dfrac{1-\nu}{E}-\dfrac{2\nu'^2}{E'}\right). \end{cases}$$

If we substitute the stresses (1.1) in the equations of equilibrium of an elastic element, the mass forces being absent, we obtain the displacement equations:

$$(1.3)\quad \begin{cases} A_{11}\dfrac{\partial^2 u_1}{\partial x_1^2}+A_{66}\dfrac{\partial^2 u_1}{\partial x_2^2}+A_{44}\dfrac{\partial^2 u_1}{\partial x_3^2}+ \\ \qquad +\dfrac{\partial}{\partial x_1}\left[(A_{12}+A_{66})\dfrac{\partial u_2}{\partial x_2}+(A_{13}+A_{44})\dfrac{\partial u_3}{\partial x_3}\right]-\beta\dfrac{\partial T}{\partial x_1}=0, \\ A_{66}\dfrac{\partial^2 u_2}{\partial x_1^2}+A_{11}\dfrac{\partial^2 u_2}{\partial x_2^2}+A_{44}\dfrac{\partial^2 u_2}{\partial x_3^2}+ \\ \qquad +\dfrac{\partial}{\partial x_2}\left[(A_{12}+A_{66})\dfrac{\partial u_1}{\partial x_1}+(A_{13}+A_{44})\dfrac{\partial u_3}{\partial x_3}\right]-\beta\dfrac{\partial T}{\partial x_2}=0, \\ A_{44}\dfrac{\partial^2 u_3}{\partial x_1^2}+A_{44}\dfrac{\partial^2 u_3}{\partial x_2^2}+A_{33}\dfrac{\partial^2 u_3}{\partial x_3^2}+ \\ \qquad +\dfrac{\partial}{\partial x_3}\left[(A_{13}+A_{44})\dfrac{\partial u_1}{\partial x_1}+(A_{13}+A_{44})\dfrac{\partial u_2}{\partial x_2}\right]-\beta'\dfrac{\partial T}{\partial x_3}=0. \end{cases}$$

The heat equation of a body of transversal isotropy has the form

$$(1.4)\qquad k\left(\frac{\partial^2 T}{\partial x_1^2}+\frac{\partial^2 T}{\partial x_2^2}\right)+k'\frac{\partial^2 T}{\partial x_3^2}-c\varrho\frac{\partial T}{\partial t}=-W.$$

The system of equations (1.3) and (1.4) may be written in the operational form:

$$(1.5)\qquad \sum_{j=1}^{4} L_{ij}u_j=-W\delta_{4i} \qquad (i=1,2,3,4),$$

where

$$L_{11}=A_{11}\frac{\partial^2}{\partial x_1^2}+A_{66}\frac{\partial^2}{\partial x_2^2}+A_{44}\frac{\partial^2}{\partial x_3^2}, \qquad L_{12}=L_{21}=(A_{12}+A_{66})\frac{\partial^2}{\partial x_1\partial x_2},$$

$$L_{22}=A_{66}\frac{\partial^2}{\partial x_1^2}+A_{11}\frac{\partial^2}{\partial x_2^2}+A_{44}\frac{\partial^2}{\partial x_3^2}, \qquad L_{13}=L_{31}=(A_{13}+A_{44})\frac{\partial^2}{\partial x_1\partial x_3},$$

$$L_{33}=A_{44}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+A_{33}\frac{\partial^2}{\partial x_3^2}, \qquad L_{23}=L_{32}=(A_{13}+A_{44})\frac{\partial^2}{\partial x_2\partial x_3},$$

$$L_{44}=k\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+k'\frac{\partial^2}{\partial x_3^2}-c\varrho\frac{\partial}{\partial t}, \qquad L_{14}=-\beta\frac{\partial}{\partial x_1},$$

$$L_{24}=-\beta\frac{\partial}{\partial x_2}, \quad L_{34}=-\beta'\frac{\partial}{\partial x_3}, \quad L_{41}=0, \quad L_{42}=0, \quad L_{43}=0,$$

with the additional notation: $u_4=T$.

Let us express the displacements u_i $(i=1,2,3)$ and the temperature u_4 by means of four functions χ_i $(i=1,2,3,4)$ in the following manner [2], [3]:

$$(1.6)\quad \begin{cases} u_1=\begin{vmatrix}\chi_1 & L_{12} & L_{13} & L_{14}\\ \chi_2 & L_{22} & L_{23} & L_{24}\\ \chi_3 & L_{32} & L_{33} & L_{34}\\ \chi_4 & 0 & 0 & L_{44}\end{vmatrix}, \qquad u_2=\begin{vmatrix}L_{11} & \chi_1 & L_{13} & L_{14}\\ L_{21} & \chi_2 & L_{23} & L_{24}\\ L_{31} & \chi_3 & L_{33} & L_{34}\\ 0 & \chi_4 & 0 & L_{44}\end{vmatrix},\\[2ex] u_3=\begin{vmatrix}L_{11} & L_{12} & \chi_1 & L_{14}\\ L_{21} & L_{22} & \chi_2 & L_{24}\\ L_{31} & L_{32} & \chi_3 & L_{34}\\ 0 & 0 & \chi_4 & L_{44}\end{vmatrix}, \qquad u_4=\begin{vmatrix}L_{11} & L_{12} & L_{13} & \chi_1\\ L_{21} & L_{22} & L_{23} & \chi_2\\ L_{31} & L_{32} & L_{33} & \chi_3\\ 0 & 0 & 0 & \chi_4\end{vmatrix},\end{cases}$$

The functions $\chi_i(i=1,2,3,4)$ satisfy the equations:

$$(1.7)\qquad \begin{vmatrix}L_{11} & L_{12} & L_{13} & L_{14}\\ L_{21} & L_{22} & L_{23} & L_{24}\\ L_{31} & L_{32} & L_{33} & L_{34}\\ 0 & 0 & 0 & L_{44}\end{vmatrix}\chi_i=-W\delta_{4i} \qquad (i=1,2,3,4).$$

It is evident that the function χ_4 satisfies the non-homogeneous equation, and the remaining functions $\chi_i(i=1,2,3)$ — the homogeneous equation. After performing the operations indicated in the Eq. (1.7), we can represent that equation in the form:

$$(1.8)\qquad k'A_{33}A_{44}^2\left(\mu_1^2\nabla^2+\frac{\partial^2}{\partial x_3^2}\right)\left(\mu_3^2\nabla^2+\frac{\partial^2}{\partial x_3^2}\right)\left(\mu_5^2\nabla^2+\frac{\partial^2}{\partial x_3^2}-\sigma^2\frac{\partial}{\partial t}\right)\times$$
$$\times\left(\mu_7^2\nabla^2+\frac{\partial^2}{\partial x_3^2}\right)\chi_i=-W\delta_{4i} \qquad (i=1,2,3,4),$$

where

$$\mu_{1,3}^2=\begin{cases}\varepsilon^2(\varrho\pm\sqrt{\varrho^2-1}), & \varrho>1,\\ \varepsilon^2, & \varrho=1,\\ \varepsilon^2\left(\sqrt{\dfrac{1+\varrho}{2}}\pm i\sqrt{\dfrac{1-\varrho}{2}}\right)^2, & \varrho<1,\end{cases}$$

$$\mu_5^2=\frac{k}{k'},\qquad \mu_7^2=\frac{A_{66}}{A_{44}},\qquad \varepsilon^4=\frac{A_{11}}{A_{33}},$$

$$\varrho=\frac{A_{11}A_{33}-2A_{13}A_{44}-A_{13}^2}{2A_{44}\sqrt{A_{11}A_{33}}},\qquad \sigma^2=\frac{c\varrho}{k'},$$

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}.$$

For the solution of the problems considered in this paper, two functions, χ_3 and χ_4, are sufficient. Since in the expressions for u_i (according to the Eqs. 1.6), the operator $\mu_7^2\nabla^2+\partial^2/\partial x_3^2$ will appear, therefore we assume that:

$$(1.9)\quad \varphi=A_{44}\left(\mu_7^2\nabla^2+\frac{\partial^2}{\partial x_3^2}\right)\left(\mu_3^2\nabla^2+\frac{\partial^2}{\partial x_3^2}-\sigma^2\frac{\partial}{\partial t}\right)\chi_3,\qquad \psi=A_{44}\left(\mu_7^2\nabla^2+\frac{\partial^2}{\partial x_3^2}\right)\chi_4,$$

A similar state is observed due to the Eqs. (1.1) in the expressions for stresses. The Eqs. (1.6) will therefore be represented assuming that $\chi_1=0$, $\chi_2=0$ in the form:

$$(1.10)\quad \begin{cases} u_1=\beta A_{44}\dfrac{\partial}{\partial x_1}\left(\nabla^2+a\dfrac{\partial^2}{\partial x_3^2}\right)\psi-A_{44}s\dfrac{\partial^2\varphi}{\partial x_1\partial x_3},\\ u_2=\beta A_{44}\dfrac{\partial}{\partial x_2}\left(\nabla^2+a\dfrac{\partial^2}{\partial x_3^2}\right)\psi-A_{44}s\dfrac{\partial^2\varphi}{\partial x_2\partial x_3},\\ u_3=\beta A_{44}\dfrac{\partial}{\partial x_3}\left(b\nabla^2+c\dfrac{\partial^2}{\partial x_3^2}\right)\psi+A_{44}\left(t\nabla^2+\dfrac{\partial^2}{\partial x_3^2}\right)\varphi,\\ T=u_4=A_{33}A_{44}\left(\mu_1^2\nabla^2+\dfrac{\partial^2}{\partial x_3^2}\right)\left(\mu_3^2\nabla^2+\dfrac{\partial^2}{\partial x_3^2}\right)\psi, \end{cases}$$

where

$$a=\eta-\varkappa(1+\gamma\eta),\qquad b=\mu_1^2\mu_3^2\varkappa\eta-\gamma\eta-1,\qquad c=\varkappa=\frac{\beta'}{\beta},$$
$$s=(1+\gamma\eta),\qquad t=\mu_1^2\mu_3^2\eta$$

and

$$\eta=\frac{\mu_1^2+\mu_3^2+2\gamma}{\mu_1^2\mu_3^2-\gamma^2}=\frac{A_{33}}{A_{44}},\qquad \gamma=\frac{A_{13}}{A_{33}}.$$

Substituting the Eqs. (1.10) in (1.1) we obtain after some transformations:

$$\begin{cases} \sigma_{11}=\beta A_{33}A_{44}\left[\dfrac{\partial^2}{\partial x_3^2}\left(d\nabla^2+e\dfrac{\partial^2}{\partial x_3^2}\right)-q\dfrac{\partial^2}{\partial x_2^2}\left(\nabla^2+a\dfrac{\partial^2}{\partial x_3^2}\right)\right]\psi-\\ \qquad -A_{33}A_{44}\dfrac{\partial}{\partial x_3}\left(a_1\dfrac{\partial^2}{\partial x_1^2}+b_1\dfrac{\partial^2}{\partial x_2^2}-\gamma\dfrac{\partial^2}{\partial x_3^2}\right)\varphi, \end{cases}$$

$$
(1.11)\quad
\left\{
\begin{aligned}
\sigma_{22} &= \beta A_{33} A_{44} \left[\frac{\partial^2}{\partial x_3^2} \left(d\nabla^2 + e \frac{\partial^2}{\partial x_3^2} \right) - q \frac{\partial^2}{\partial x_1^2} \left(\nabla^2 + a \frac{\partial^2}{\partial x_3^2} \right) \right] \psi - \\
&\qquad - A_{33} A_{44} \frac{\partial}{\partial x_3} \left(b_1 \frac{\partial^2}{\partial x_1^2} + a_1 \frac{\partial^2}{\partial x_2^2} - \gamma \frac{\partial^2}{\partial x_3^2} \right) \varphi , \\
\sigma_{33} &= \beta A_{33} A_{44} \nabla^2 \left(d\nabla^2 + e \frac{\partial^2}{\partial x_3^2} \right) \psi + A_{33} A_{44} \frac{\partial}{\partial x_3} \left(p\nabla^2 + \frac{\partial^2}{\partial x_3^2} \right) \varphi , \\
\sigma_{12} &= \beta A_{33} A_{44} q \frac{\partial^2}{\partial x_1 \partial x_2} \left(\nabla^2 + a \frac{\partial^2}{\partial x_3^2} \right) \psi + A_{33} A_{44} (b_1 - a_1) \frac{\partial^3 \varphi}{\partial x_1 \partial x_2 \partial x_3} , \\
\sigma_{23} &= - \beta A_{33} A_{44} \frac{\partial^2}{\partial x_2 \partial x_3} \left(d\nabla^2 + e \frac{\partial^2}{\partial x_3^2} \right) \psi + \\
&\qquad + A_{33} A_{44} \frac{\partial}{\partial x_2} \left(a_1 \nabla^2 - \gamma \frac{\partial^2}{\partial x_3^2} \right) \varphi , \\
\sigma_{13} &= - \beta A_{33} A_{44} \frac{\partial^2}{\partial x_1 \partial x_3} \left(d\nabla^2 + e \frac{\partial^2}{\partial x_3^2} \right) \psi + \\
&\qquad + A_{33} A_{44} \frac{\partial}{\partial x_1} \left(a_1 \nabla^2 - \gamma \frac{\partial^2}{\partial x_3^2} \right) \varphi ,
\end{aligned}
\right.
$$

where the following further notations are introduced:

$$
d = \gamma - \mu_1^2 \mu_3^2 \varkappa , \qquad e = \varkappa\gamma - 1 , \qquad q = \frac{2\mu_7^2}{\eta} , \qquad p = \mu_1^2 + \mu_3^2 + \gamma ,
$$

$$
a_1 = \mu_1^2 \mu_3^2 , \qquad b_1 = \mu_1^2 \mu_3^2 - 2\mu_7^2 (\gamma + \eta^{-1}) .
$$

The function ψ satisfies the non-homogeneous equation:

$$
(1.12.1)\quad k' A_{33} A_{44} \left(\mu_1^2 \nabla^2 + \frac{\partial^2}{\partial x_3^2} \right) \left(\mu_3^2 \nabla^2 + \frac{\partial^2}{\partial x_3^2} \right) \left(\mu_5^2 \nabla^2 + \frac{\partial^2}{\partial x_3^2} - \sigma^2 \frac{\partial}{\partial t} \right) \psi = - W
$$

and the function φ, the homogeneous equation:

$$
(1.12.2)\qquad \left(\mu_1^2 \nabla^2 + \frac{\partial^2}{\partial x_3^2} \right) \left(\mu_3^2 \nabla^2 + \frac{\partial^2}{\partial x_3^2} \right) \varphi = 0 .
$$

It is easy to see that the function φ is the Galerkin function generalized to the case of transversal isotropy, [4], [5] and [6].

In the case of axially symmetric problems, it will be more convenient to use displacements and stresses expressed in cylindrical coordinates. Introducing the notation $\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$ we obtain the differential equation for the function φ and ψ substituting in the place of ∇^2 in the Eqs.

(1.12.1) and (1.12.2) the operator ∇_r^2. The displacements and the temperature are given by the equations:

$$(1.13)\quad \begin{cases} u_r = \beta A_{44} \dfrac{\partial}{\partial r}\left(\nabla_r^2 + a\dfrac{\partial^2}{\partial z^2}\right)\psi - A_{44}\, s\, \dfrac{\partial^2 \varphi}{\partial r \partial z}, \qquad \mu_\varphi = 0, \\ w = \beta A_{44}\dfrac{\partial}{\partial z}\left(b\nabla_r^2 + c\dfrac{\partial^2}{\partial z^2}\right)\psi + A_{44}\left(t\nabla_r^2 + \dfrac{\partial^2}{\partial z^2}\right)\varphi, \\ T = A_{33}A_{44}\left(\mu_1^2\nabla_r^2 + \dfrac{\partial^2}{\partial z^2}\right)\left(\mu_3^2\nabla_r^2 + \dfrac{\partial^2}{\partial z^2}\right)\psi, \\ r = (x_1^2 + x_2^2)^{1/2}, \qquad z = x_3; \end{cases}$$

and the stresses — by the equations:

$$(1.14)\quad \begin{cases} \sigma_{rr} = \beta A_{33}A_{44}\left[\dfrac{\partial^2}{\partial z^2}\left(d\nabla_r^2 + e\dfrac{\partial^2}{\partial z^2}\right) - q\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(\nabla_r^2 + a\dfrac{\partial^2}{\partial z^2}\right)\right]\psi - \\ \qquad\qquad - A_{33}A_{44}\dfrac{\partial}{\partial z}\left[a_1\nabla_r^2 - (a_1 - b_1)\dfrac{1}{r}\dfrac{\partial}{\partial r} - \gamma\dfrac{\partial^2}{\partial z^2}\right]\varphi, \\ \sigma_{\varphi\varphi} = \beta A_{33}A_{44}\left[\dfrac{\partial^2}{\partial z^2}\left(d\nabla_r^2 + e\dfrac{\partial^2}{\partial z^2}\right) - q\dfrac{\partial^2}{\partial r^2}\left(\nabla_r^2 + a\dfrac{\partial^2}{\partial z^2}\right)\right]\psi - \\ \qquad\qquad - A_{33}A_{44}\dfrac{\partial}{\partial z}\left[a_1\nabla_r^2 - (a_1 - b_1)\dfrac{\partial^2}{\partial r^2} - \gamma\dfrac{\partial^2}{\partial z^2}\right]\varphi, \\ \sigma_{zz} = \beta A_{33}A_{44}\nabla_r^2\left(d\nabla_r^2 + e\dfrac{\partial^2}{\partial z^2}\right)\psi + A_{33}A_{44}\dfrac{\partial}{\partial z}\left(p\nabla_r^2 + \dfrac{\partial^2}{\partial z^2}\right)\varphi, \end{cases}$$

$$\sigma_{rz} = -\beta A_{33}A_{44}\frac{\partial^2}{\partial r \partial z}\left(d\nabla_r^2 + e\frac{\partial^2}{\partial z^2}\right)\psi + $$
$$+ A_{33}A_{44}\frac{\partial}{\partial r}\left(a_1\nabla_r^2 - \gamma\frac{\partial^2}{\partial z^2}\right)\varphi.$$

The assumption of the functions φ and ψ suffices for the determination of the thermal stresses in simple systems: an infinite space, a semi-space and an elastic layer. On the other hand, for the solution of the state of stress in thick circular and rectangular plates, we should, to satisfy all the boundary conditions, take for the solution, besides the functions φ and ψ, also the functions χ_1 and χ_2. In further considerations, we shall be confined to the most general case, that is $\varrho < 1$.

The passage to the case $\varrho = 1$ will be achieved assuming the roots of the characteristic equation μ_j in the form $\mu_j = \varepsilon(\alpha_j + i\beta_j)$ and passing with the results to the limit for $\alpha_j \to 1$, $\beta_j \to 0$; the case of $\varrho > 1$ will be obtained by putting $\beta_j = 0$.

2. Steady-State Thermal Stresses in a Space and a Semi-Space Showing Transversal Isotropy

Let a steady point source of intensity W act at the point $(0, \zeta)$. In this case, we are concerned with an axially symmetric problem. To determine the state of stress in an elastic space, the function ψ will be sufficient. The Eq. (1.12.1) has, in the case of steady temperature, the form:

$$k' A_{33} A_{44} \left(\mu_1^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_5^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\psi = = -W \frac{\delta(r)}{2\pi r}\delta(z-\zeta), \tag{2.1}$$

The function $\bar{\psi}$, a particular integral of the Eq. (2.1), will be assumed in the form of a Fourier-Hankel integral:

$$\bar{\psi} = \int_0^\infty \int_0^\infty A(\alpha, \gamma) J_0(\alpha r) \cos\gamma(z-\zeta)\, d\alpha d\gamma. \tag{2.2}$$

Since

$$\frac{\delta(r)}{2\pi r} = \frac{1}{2\pi}\int_0^\infty \alpha J_0(\alpha r) d\alpha, \qquad \delta(z-\zeta) = \frac{1}{\pi}\int_0^\infty \cos\gamma(z-\zeta)\, d\gamma,$$

the following integral[1] is a solution of the Eq. (12.1):

$$\bar{\psi} = \frac{W}{2\pi^2 A_{33} A_{44} k'} \int_0^\infty \int_0^\infty \frac{\alpha J_0(\alpha r)\cos\gamma(z-\zeta)\, d\alpha d\gamma}{(\mu_1^2\alpha^2+\gamma^2)(\mu_3^2\alpha^2+\gamma^2)(\mu_5^2\alpha^2+\gamma^2)}, \tag{2.3}$$

or

$$\bar{\psi} = \frac{W}{4\pi k' A_{33} A_{44}} \sum_{j=1,3,5} C_{jj} \int_0^\infty \alpha^{-4} J_0(\alpha r)\, e^{-\alpha\mu_j(z-\zeta)}\, d\alpha; \tag{2.4}$$

where

$$\begin{cases} C_{jj} = \dfrac{1}{\mu_j(\mu_j^2-\mu_i^2)(\mu_j^2-\mu_l^2)} & (i, j, l = 1, 3, 5, \qquad i \neq j \neq l), \\ \mathrm{Re}(\mu_{2n-1}) > 0 & (n = 1, 2, 3). \end{cases} \tag{2.5}$$

The Eq. (2.4) is valid only for $z-\zeta > 0$. Introducing the notations $\mu_2 = -\mu_1$, $\mu_4 = -\mu_3$, $\mu_6 = -\mu_5$, we can represent the function $\bar{\psi}$ for $z-\zeta < 0$ in the form:

$$\bar{\psi} = \frac{W}{4\pi k' A_{33} A_{44}} \sum_{j=2,4,6} C_{jj} \int_0^\infty \alpha^{-4} J_0(\alpha r)\, e^{-\alpha\mu_j(z-\zeta)}\, d\alpha, \tag{2.6}$$

[1] See Appendix II.

where

$$C_{jj} = \frac{1}{\mu_j(\mu_j^2 - \mu_i^2)(\mu_j^2 - \mu_l^2)} \qquad (i, j, l = 2, 4, 6, \qquad i \neq j \neq l), \tag{2.7}$$

Substituting $\overline{\psi}$ in the Eqs. (1.13) and (1.14), we obtain for $z-\zeta > 0$

$$\left\{\begin{aligned} \overline{u}_r &= \frac{W\beta}{4\pi k' A_{33}} \sum_{j=1,3,5} C_{jj} g(\mu_j) \frac{1}{r} [R_j - \mu_j(z-\zeta)], \\ \overline{w} &= \frac{-W\beta}{4\pi k' A_{33}} \sum_{j=1,3,5} C_{jj} \mu_j h(\mu_j) \ln [R_j + \mu_j(z-\zeta)], \\ T &= \frac{W}{4\pi k' \mu_5 R_5}, \end{aligned}\right. \tag{2.8}$$

where

$$R_j = [r^2 + \mu_j^2(z-\zeta)^2]^{1/2}, \qquad r^2 = (x_1^2 + x_2^2)^{1/2},$$
$$g(\mu_j) = 1 - a\mu_j^2, \qquad h(\mu_j) = b - c\mu_j^2.$$

Next:

$$\left\{\begin{aligned} \overline{\sigma}_{zz} &= \frac{W\beta}{4\pi k'} \sum_{j=1,3,5} C_{jj} f(\mu_j) R_j^{-1}, \\ \overline{\sigma}_{rr} &= -\frac{W\beta}{4\pi k'} \sum_{j=1,3,5} C_{jj} \left\{ \mu_j^2 f(\mu_j) R_j^{-1} + q g(\mu_j) \frac{1}{r^2} [R_j - \mu_j(z-\zeta)] \right\}, \\ \overline{\sigma}_{\varphi\varphi} &= -\frac{W\beta}{4\pi k'} \sum_{i=1,3,5} C_{jj} \left\{ \mu_j^2 f(\mu_j) R_j^{-1} + q g(\mu_j) \left[R_j^{-1} - \frac{R_j - \mu_j(z-\zeta)}{r^2} \right] \right\}, \\ \overline{\sigma}_{rz} &= \frac{W\beta}{4\pi k'} \sum_{j=1,3,5} C_{jj} \mu_j f(\mu_j) \frac{1}{r} [1 - \mu_j(z-\zeta) R_j^{-1}], \\ f(\mu_j) &= d - \mu_j^2 e. \end{aligned}\right. \tag{2.9}$$

For $z-\zeta<0$ we obtain equations analogous to (2.8) and (2.9), the only difference being that summation will be effected according to even subscripts and C_{jj} will be taken from the Eq. (2.7). It is evident that the stresses and the temperature vanish at infinity.

Knowing the stresses σ_{ij} in the system of cylindrical coordinates, we may pass to the stress components in rectangular coordinates by way of the following transformation formulae:

$$\overline{\sigma}_{11} = \overline{\sigma}_{rr} \cos^2\varphi + \overline{\sigma}_{\varphi\varphi} \sin^2\varphi, \qquad \overline{\sigma}_{12} = \tfrac{1}{2}(\overline{\sigma}_{rr} - \overline{\sigma}_{\varphi\varphi}) \sin 2\varphi,$$
$$\overline{\sigma}_{22} = \overline{\sigma}_{rr} \sin^2\varphi + \overline{\sigma}_{\varphi\varphi} \cos^2\varphi, \qquad \overline{\sigma}_{13} = \overline{\sigma}_{rz} \cos\varphi,$$
$$\overline{\sigma}_{33} = \overline{\sigma}_{zz}, \qquad \overline{\sigma}_{23} = \overline{\sigma}_{rz} \sin\varphi,$$
$$\cos\varphi = \frac{x_1}{r}, \qquad \sin\varphi = \frac{x_2}{r}.$$

Shifting the source from the point $(0,\zeta)$ to the point (ξ_1,ξ_2,ξ_3), we shall obtain the most general expressions for stresses $\bar{\sigma}_{ij}$ in the system of rectangular coordinates.

Let us consider, finally, the particular case of a heat source uniformly distributed over a circular area of radius r_0, in the $z=0$ plane. Then, the right-hand member of the Eq. (2.1) has the form:

$$-\frac{Wr_0}{\pi}\int_0^\infty\int_0^\infty J_1(\alpha r_0)J_0(\alpha r)\cos\gamma z\,d\alpha\,d\gamma = -W\delta(z)\begin{cases}1 & \text{for} \quad 0<r<r_0,\\ \frac{1}{2} & \text{for} \quad r=r_0,\\ 0 & \text{for} \quad r_0<r<\infty.\end{cases}$$

Then

$$\bar{\psi}(r,z)=\frac{Wr_0}{2k'A_{33}A_{44}}\sum_{j=1,3,5}C_{jj}\int_0^\infty \alpha^{-5}J_1(\alpha r_0)J_0(\alpha r)e^{-\alpha\mu_j z}\,d\alpha. \tag{2.10}$$

The stresses are expressed by the equations:

$$\begin{cases}\bar{\sigma}_{rr}=-\frac{W\beta r_0}{2k'}\sum_{j=1,3,5}C_{jj}\Big\{\mu_j^2 f(\mu_j)\int_0^\infty \alpha^{-1}J_1(\alpha r_0)J_0(\alpha r)e^{-\alpha\mu_j z}\,d\alpha+\\ \qquad +\frac{qg(\mu_j)}{r}\int_0^\infty \alpha^{-2}J_1(\alpha r_0)J_0(\alpha r)e^{-\alpha\mu_j z}\,d\alpha\Big\},\\ \bar{\sigma}_{\varphi\varphi}=-\frac{W\beta r_0}{2k'}\sum_{j=1,3,5}C_{jj}\Big\{\mu_j^2 f(\mu_j)\int_0^\infty \alpha^{-1}J_1(\alpha r_0)J_0(\alpha r)e^{-\alpha\mu_j z}\,d\alpha+\\ \qquad +qg(\mu_j)\int_0^\infty \alpha^{-1}J_1(\alpha r_0)\Big[J_0(\alpha r)-\frac{J_1(\alpha r)}{\alpha r}\Big]e^{-\alpha\mu_j z}\,d\alpha\Big\},\\ \bar{\sigma}_{zz}=\frac{W\beta r_0}{2k'}\sum_{j=1,3,5}C_{jj}f(\mu_j)\int_0^\infty \alpha^{-1}J_1(\alpha r_0)J_0(\alpha r)e^{-\alpha\mu_j z}\,d\alpha,\\ \bar{\sigma}_{rz}=\frac{W\beta r_0}{2k'}\sum_{j=1,3,5}C_{jj}\mu_j f(\mu_j)\int_0^\infty \alpha^{-1}J_1(\alpha r_0)J_1(\alpha r)e^{-\alpha\mu_j z}\,d\alpha.\end{cases} \tag{2.11}$$

The integrals appearing here may be expressed by means of elliptic integrals, Legendre functions and hypergeometric series. The corresponding equations are given in Appendix I.

3. Concentrated Steady-State Heat Source in an Elastic Semi-Space

Let a steady concentrated source act at the point $(0,\zeta)$ of an elastic semi-space bounded by the plane $z=0$. The solution of the differential equation

$$k'A_{33}A_{44}\left(\mu_1^2\nabla_r^2+\frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2\nabla_r^2+\frac{\partial^2}{\partial z^2}\right)\left(\mu_5^2\nabla_r^2+\frac{\partial^2}{\partial z^2}\right)\psi = -\frac{W}{2\pi r}\delta(r)\delta(z-\zeta) \tag{3.1}$$

will be represented in the form of the function $\bar{\psi}$, obtained previously for an infinite space and a sum of harmonic functions ψ_{ij} satisfying the equation

$$\left(\mu_i^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right) \psi_{ij} = 0 . \tag{3.2}$$

This will be more convenient than assuming two functions ψ and φ, of which the first gives the solution for an infinite space and the second corrects the boundary conditions in the plane $z = 0$.

Therefore, the general solution of the Eq. (3.1) has the form:

$$\psi(r, z) = \frac{W}{4\pi k' A_{33} A_{44}} \int_0^\infty \alpha^{-4} J_0(\alpha r) \left[\sum_j C_{jj} e^{-\alpha\mu_j(z-\zeta)} + \right. \\ \left. + \sum_{i,j} C_{ij} e^{-\alpha(\mu_i z - \mu_j \zeta)} \right] d\alpha . \tag{3.3}$$

In the first sum the subscripts j take the values $j = 1, 3, 5$ for $z - \zeta > 0$, and $j = 2, 4, 6$ for $z - \zeta < 0$. Since the integrands should be zero at infinity and its second term should not involve singularities in the region considered — which ensures the vanishing of the stresses at infinity — therefore

$$\mathrm{Re}\,(\mu_i z - \mu_j \zeta) > 0 .$$

This will take place if, at the same time, $\mathrm{Re}\,\mu_i > 0$, $\mathrm{Re}\,\mu_j < 0$. This condition limits the number of the constants C_{ij} to nine, where the subscript j takes the values 2, 4, 6 and the subscript i — the values 1, 3, 5. The coefficients $C_{2n-1,2k}$, $n, k = 1, 2, 3$ remain different from zero.

We substitute $\psi(r, z)$ in the expression (1.13) for the temperature T. We have:

$$T = \frac{W}{4\pi k'} \int_0^\infty J_0(\alpha r) \left[\sum_j C_{jj} (\mu_1^2 - \mu_j^2)(\mu_3^2 - \mu_j^2) e^{-\alpha\mu_j(z-\zeta)} + \right. \\ \left. + \sum_{i,j} C_{ij} (\mu_1^2 - \mu_i^2)(\mu_3^2 - \mu_i^2) e^{-\alpha(\mu_i z - \mu_j \zeta)} \right] d\alpha \qquad (j = 2, 4, 6, \qquad i = 1, 3, 5). \tag{3.4}$$

Since $\mu_1^2 = \mu_2^2$, $\mu_3^2 = \mu_4^2$, therefore in the first sum C_{66} alone will appear, and in the second the coefficients $C_{5j} (j = 2, 4, 6)$ alone. Therefore, bearing in mind that $\mu_5^2 = \mu_6^2$, we obtain:

$$T = \frac{W}{4\pi k'} (\mu_1^2 - \mu_5^2)(\mu_3^2 - \mu_5^2) \int_0^\infty J_0(\alpha r) [C_{66} e^{-\alpha\mu_6(z-\zeta)} + C_{52} e^{-\alpha(\mu_5 z - \mu_2 \zeta)} + \\ + C_{54} e^{-\alpha(\mu_5 z - \mu_4 \zeta)} + C_{56} e^{-\alpha(\mu_5 z - \mu_6 \zeta)}] \, d\alpha . \tag{3.5}$$

In the case of the boundary condition $T(r,0)=0$, we obtain:

$$T(r,0)=\frac{W}{4\pi k'}(\mu_1^2-\mu_5^2)(\mu_3^2-\mu_5^2)\int_0^\infty J_0(\alpha r)\,[C_{66}\,e^{\alpha\mu_6\zeta}+C_{52}\,e^{\alpha\mu_2\zeta}+$$
$$+C_{54}\,e^{\alpha\mu_4\zeta}+C_{56}\,e^{\alpha\mu_6\zeta}]\,d\alpha=0,$$

whence

$$C_{66}+C_{56}=0,\qquad C_{52}=0,\qquad C_{54}=0. \tag{3.6}$$

Introducing these values in the expression (3.5), we obtain after integration:

$$T(r,z)=\frac{W}{4\pi k'\mu_5}\left[\frac{1}{\sqrt{r^2+\mu_5^2(z-\zeta)^2}}-\frac{1}{\sqrt{r^2+\mu_5^2(z+\zeta)^2}}\right]. \tag{3.7}$$

The remaining constants C_{ij} will be determined from the kinematic or dynamic boundary conditions at the edge $z=0$. Let us assume that the stresses are lacking at this edge. Then:

$$\sigma_{zz}(r,0)=0,\qquad \sigma_{rz}(r,0)=0. \tag{3.8}$$

Using the Eqs. (1.14) and (3.3), we obtain for $z-\zeta<0$

$$\begin{cases} \sigma_{zz}(r,z)=\dfrac{W\beta}{4\pi k'}\displaystyle\int_0^\infty J_0(\alpha r)\Big[\sum_j C_{jj}f(\mu_j)\,e^{-\alpha\mu_j(z-\zeta)}+ \\ \qquad\qquad +\displaystyle\sum_{i,j} C_{ij}f(\mu_i)\,e^{-\alpha(\mu_i z-\mu_j\zeta)}\Big]\,d\alpha, \\ \sigma_{rz}(r,z)=\dfrac{W}{4\pi k'}\displaystyle\int_0^\infty J_1(\alpha r)\Big[\sum_j C_{jj}\mu_j f(\mu_j)\,e^{-\alpha\mu_j(z-\zeta)}+ \\ \qquad\qquad +\displaystyle\sum_{i,j} C_{ij}\mu_i f(\mu_i)\,e^{-\alpha(\mu_i z-\mu_j\zeta)}\Big]\,d\alpha. \end{cases} \tag{3.9}$$

Bearing in mind the boundary conditions (3.8), and setting the coefficients of the function $e^{\alpha\mu_j\zeta}$ equal to zero, we obtain the following system of equations:

$$\begin{cases} C_{22}f(\mu_2)+C_{12}f(\mu_1)+C_{32}f(\mu_3)=0, \\ C_{44}f(\mu_4)+C_{14}f(\mu_1)+C_{34}f(\mu_3)=0, \\ C_{66}[f(\mu_6)-f(\mu_5)]+C_{16}f(\mu_1)+C_{36}f(\mu_3)=0, \end{cases} \tag{3.10}$$

$$\begin{cases} C_{22}\mu_2 f(\mu_2)+C_{12}\mu_1 f(\mu_1)+C_{32}\mu_3 f(\mu_3)=0, \\ C_{44}\mu_4 f(\mu_4)+C_{14}\mu_1 f(\mu_1)+C_{34}\mu_3 f(\mu_3)=0, \\ C_{66}[\mu_6 f(\mu_6)-\mu_5 f(\mu_5)]+C_{16}\mu_1 f(\mu_1)+C_{36}\mu_3 f(\mu_3)=0, \\ f(\mu_{2n-1})=f(\mu_{2n}) \qquad (n=1,2,3). \end{cases} \tag{3.11}$$

The Eqs. (3.6) were used here. Since the quantities $C_{jj} (j = 2, 4, 6)$ are given by the Eqs. (2.7), we have six non-homogeneous equations, (3.10), (3.11), for the determination of six quantities C_{ij}:

$$(3.12)\quad \begin{cases} C_{12} = \dfrac{\mu_1 + \mu_3}{\mu_1 - \mu_3} C_{22}, & C_{32} = \dfrac{2\mu_1}{\mu_3 - \mu_1} \dfrac{f(\mu_2)}{f(\mu_3)} C_{22}, \quad C_{52} = 0, \\ C_{14} = \dfrac{2\mu_3}{\mu_1 - \mu_3} \dfrac{f(\mu_3)}{f(\mu_1)} C_{44}, & C_{34} = \dfrac{\mu_1 + \mu_3}{\mu_3 - \mu_1} C_{44}, \quad C_{54} = 0, \\ C_{16} = \dfrac{2\mu_5}{\mu_1 - \mu_5} \dfrac{f(\mu_5)}{f(\mu_1)} C_{66}, & C_{36} = \dfrac{2\mu_5}{\mu_3 - \mu_1} \dfrac{f(\mu_5)}{f(\mu_3)} C_{66}, \quad C_{56} = -C_{66}. \end{cases}$$

After determinating the constants C_{ij} we know the function ψ. Thus, by way of the Eqs. (1.13) and (1.14), we can determine the displacements and the stresses.

Let us consider now the case of $\left[\dfrac{\partial T}{\partial z}\right]_{z=0} = 0$, which is a case of thermal isolation in the $z = 0$ plane. We have:

$$(3.13)\quad \left[\frac{\partial T}{\partial z}\right]_{z=0} = -\frac{W}{4\pi k'} (\mu_1^2 - \mu_5^2)(\mu_3^2 - \mu_5^2) \int_0^\infty J_0(\alpha r)\alpha \, [\mu_6 C_{66} e^{\alpha\mu_6\zeta} + \mu_5 C_{52} e^{\alpha\mu_2\zeta} + \mu_5 C_{54} e^{\alpha\mu_4\zeta} + \mu_5 C_{56} e^{\alpha\mu_6\zeta}] \, d\alpha = 0.$$

Hence:

$$C_{66}\mu_6 + C_{56}\mu_5 = 0, \qquad C_{52} = 0, \qquad C_{54} = 0.$$

Since $\mu_6 = -\mu_5$, therefore $C_{66} = C_{56}$. Thus:

$$(3.14)\qquad T = \frac{W}{4\pi k' \mu_5} \left[\frac{1}{\sqrt{r^2 + \mu_5^2 (z - \zeta)^2}} + \frac{1}{\sqrt{r^2 + \mu_5^2 (z + \zeta)^2}} \right].$$

If the boundary $z = 0$ is free from stresses, we obtain from the boundary conditions a system of equations analogous to (3.10) and (3.11). The first two equations of the groups (3.10) and (3.11) remain unchanged. The third equations of these groups take the different form:

$$(3.15)\qquad \begin{cases} 2C_{66} f(\mu_5) + C_{16} f(\mu_1) + C_{36} f(\mu_3) = 0, \\ C_{16} f(\mu_1) + C_{36} f(\mu_3) = 0. \end{cases}$$

Hence:

$$(3.16)\qquad C_{16} = \frac{2\mu_3}{\mu_1 - \mu_3} \frac{f(\mu_5)}{f(\mu_1)} C_{66}, \qquad C_{36} = \frac{2\mu_1}{\mu_3 - \mu_1} \frac{f(\mu_5)}{f(\mu_3)} C_{66}.$$

Let us consider the case of clamping in the plane $z = 0$. From the boundary conditions, we have:

$$(3.17)\qquad u_r(r, 0) = 0, \qquad w(r, 0) = 0.$$

From the Eqs. (1.13) and taking the Eq. (3.3) into consideration, we obtain for $z-\zeta<0$

$$
(3.18)\quad \begin{cases} u_r(r,z)=\dfrac{W\beta}{4\pi k' A_{33}}\displaystyle\int_0^\infty a^{-1}J_1(ar)\Big[\sum_j C_{jj}g(\mu_j)e^{-a\mu_j(z-\zeta)}+ \\ \qquad\qquad +\displaystyle\sum_{i,j}C_{ij}g(\mu_i)e^{-a(\mu_i z-\mu_j\zeta)}\Big]da, \\ w(r,z)=\dfrac{W\beta}{4\pi k' A_{33}}\displaystyle\int_0^\infty a^{-1}J_0(ar)\Big[\sum_j C_{jj}h(\mu_j)\mu_j e^{-a\mu_j(z-\zeta)}+ \\ \qquad\qquad +\displaystyle\sum_{i,j}C_{ij}h(\mu_i)\mu_i e^{-a(\mu_i z-\mu_j\zeta)}\Big]da, \end{cases}
$$

$$(j=2,4,6,\qquad i=1,3,5).$$

If we assume in addition that $T(r,0)=0$, or, in other words, that $C_{56}=-C_{66}$, $C_{52}=0$, $C_{54}=0$, we obtain from the boundary conditions (3.17)

$$
(3.19)\quad \begin{cases} C_{12}=\dfrac{\mu_3 g(\mu_1)h(\mu_3)+\mu_1 g(\mu_3)h(\mu_1)}{\Delta}C_{22}, \\ C_{32}=-\dfrac{2\mu_1 g(\mu_1)h(\mu_1)}{\Delta}C_{22}, \\ C_{14}=\dfrac{2\mu_3 g(\mu_3)h(\mu_3)}{\Delta}C_{44}, \\ C_{34}=-\dfrac{\mu_1 g(\mu_3)h(\mu_1)+\mu_3 g(\mu_1)h(\mu_3)}{\Delta}C_{44}, \\ C_{16}=\dfrac{2\mu_5 g(\mu_3)h(\mu_5)}{\Delta}C_{66},\qquad C_{36}=-\dfrac{g(\mu_1)}{g(\mu_3)}C_{66}, \\ \Delta=\mu_1 h(\mu_1)g(\mu_3)-\mu_3 h(\mu_3)g(\mu_1). \end{cases}
$$

If we assume the boundary conditions

$$(3.20)\qquad \left[\frac{\partial T}{\partial z}\right]_{z=0}=0,\qquad u_r(r,0)=0,\qquad w(r,0)=0$$

the constants $C_{12}, C_{32}, C_{14}, C_{34}$ will be taken from the equations (3.19) and C_{16}, C_{36} from the equations

$$(3.21)\qquad C_{36}=-\frac{2\mu_1 g(\mu_5)h(\mu_1)}{\Delta}C_{66},\qquad C_{16}=-\frac{\mu_3 h(\mu_3)}{\mu_1 h(\mu_1)}C_{66}.$$

In the case of the boundary conditions

$$(3.22)\qquad \sigma_{rz}(r,0)=0,\qquad w(r,0)=0,\qquad T(r,0)=0$$

we have

$$
(3.23)\quad \begin{cases} C_{12}=C_{22}, & C_{32}=0, \quad C_{52}=0, \\ C_{14}=0, & C_{34}=C_{44}, \quad C_{54}=0, \\ C_{16}=\dfrac{2\mu_5}{\mu_1}\dfrac{h(\mu_5)f(\mu_3)-h(\mu_3)f(\mu_5)}{h(\mu_1)f(\mu_3)-h(\mu_3)f(\mu_1)}C_{66}, \\ C_{36}=\dfrac{2\mu_5}{\mu_3}\dfrac{h(\mu_5)f(\mu_1)-h(\mu_1)f(\mu_5)}{h(\mu_3)f(\mu_1)-h(\mu_1)f(\mu_3)}C_{66}, & C_{56}=-C_{66}, \end{cases}
$$

Finally, in the case of the boundary conditions

$$
(3.24)\qquad \sigma_{rz}(r,0)=0, \qquad w(r,0)=0, \qquad \left[\frac{\partial T}{\partial z}\right]_{z=0}=0
$$

we obtain

$$
(3.25)\qquad \begin{cases} C_{12}=C_{22}, & C_{32}=0, & C_{52}=0, \\ C_{14}=0, & C_{34}=C_{44}, & C_{54}=0, \\ C_{16}=0, & C_{36}=0, & C_{56}=-C_{66}. \end{cases}
$$

In all the above cases we can obtain the solution by superposition

$$
(3.26)\qquad \psi(r,z)=\bar{\psi}(r,z)+\frac{W}{4\pi k' A_{33}A_{44}}\int_0^\infty \alpha^{-4}J_0(\alpha r)\left(\sum_{i,j} C_{ij}\,e^{-\alpha(\mu_i z-\mu_j\zeta)}\right)d\alpha
$$
$$
(i=1,3,5, \qquad j=2,4,6),
$$

where $\bar{\psi}$ is given by the Eq. (2.4) for $z-\zeta>0$ or (2.6) for $z-\zeta<0$. The displacement and stress components are represented by the equations

$$
(3.27)\quad \begin{cases} u_r=\bar{u}_r+\dfrac{W\beta}{4\pi k' A_{33}}\displaystyle\sum_{i,j} C_{ij}g(\mu_i)\frac{1}{r}\,[R_{ij}-(\mu_i z-\mu_i\zeta)], \\ w=\bar{w}-\dfrac{W\beta}{4\pi k' A_{33}}\displaystyle\sum_{i,j} C_{ij}\mu_i h(\mu_i)\ln[R_{ij}+(\mu_i z-\mu_j\zeta)], \\ \sigma_{zz}=\bar{\sigma}_{zz}+\dfrac{W\beta}{4\pi k'}\displaystyle\sum_{i,j} C_{ij}f(\mu_i)R_{ij}^{-1}, \\ \sigma_{rr}=\bar{\sigma}_{rr}-\dfrac{W\beta}{4\pi k'}\displaystyle\sum_{i,j} C_{ij}\left[\mu_i^2 f(\mu_i)R_{ij}^{-1}+qg(\mu_i)\frac{R_{ij}-(\mu_i z-\mu_j\zeta)}{r^2}\right], \\ \sigma_{\varphi\varphi}=\bar{\sigma}_{\varphi\varphi}-\dfrac{W\beta}{4\pi k'}\displaystyle\sum_{i,j} C_{ij}\left\{\mu_i^2 f(\mu_i)R_{ij}^{-1}+qg(\mu_i)\left[\frac{1}{R_{ij}}-\frac{R_{ij}-(\mu_i z-\mu_j\zeta)}{r^2}\right]\right\}, \\ \sigma_{rz}=\bar{\sigma}_{rz}+\dfrac{W\beta}{4\pi k'}\displaystyle\sum_{i,j} C_{ij}\mu_i f(\mu_i)\frac{1}{r}\left(1-\frac{\mu_i z-\mu_j\zeta}{R_{ij}}\right), \end{cases}
$$
$$
(i=1,3,5, \quad j=2,4,6).
$$

where

$$R_{ij} = [r^2 + (\mu_i z - \mu_j \zeta)^2]^{1/2}.$$

The quantities $\overline{u}_r$, $\overline{w}$ are taken from the Eqs. (2.8), and the stresses $\overline{\sigma}_{zz}, \overline{\sigma}_{rr}, \overline{\sigma}_{\varphi\varphi}, \overline{\sigma}_{rz}$ from (2.9).

It can be shown that in the particular case of isotropy and thermal isolation in the plane $z=0$, the stresses σ_{zz} and σ_{rz} vanish for $\zeta \to 0$, so that the stresses σ_{rr} and $\sigma_{\varphi\varphi}$ remain the only ones different from zero.

4. The Stresses Due to Heat Transfer in the Region Γ of the Plane $z=0$ Bounding the Elastic Semi-Plane

Let the temperature of the region Γ of the plane $z=0$ be $T(x_1, x_2, 0)$ and let $T=0$ outside that region Γ. Our objective is to determine the displacements and the stresses due to such thermal conditions. Consider first the auxiliary problem of determining the displacements and stresses due to the action of thermal exposition on the element $d\Gamma$ at the origin. The thermal boundary condition will be expressed in the form:

$$T(r,0) = (Td\Gamma)\frac{\delta(r)}{2\pi r} = \frac{(Td\Gamma)}{2\pi}\int_0^\infty \alpha J_0(\alpha r)\,d\alpha. \tag{4.1}$$

Since in the elastic semi-space there are no heat sources, our starting point will be the homogeneous equation (2.1). The solution of this equation will be assumed in the form:

$$\psi(r,z) = \frac{(Td\Gamma)}{2\pi A_{33}A_{44}}\int_0^\infty \alpha^{-3}J_0(\alpha r)\left(\sum_{j=1,3,5} D_{jj}e^{-\alpha\mu_j z}\right)d\alpha. \tag{4.2}$$

Using the last of the equations (1.13), we obtain:

$$\begin{aligned} T(r,z) &= \frac{(Td\Gamma)}{2\pi}\int_0^\infty \alpha J_0(\alpha r)\left[\sum_{j=1,3,5} D_{jj}(\mu_1^2-\mu_j^2)(\mu_3^2-\mu_j^2)e^{-\alpha\mu_j z}\right]d\alpha = \\ &= \frac{(Td\Gamma)}{2\pi}D_{55}(\mu_1^2-\mu_5^2)(\mu_3^2-\mu_5^2)\int_0^\infty \alpha J_0(\alpha r)e^{-\alpha\mu_5 z}\,d\alpha. \end{aligned} \tag{4.3}$$

From the boundary condition (4.1), we shall determine the constant:

$$D_{55} = \frac{1}{(\mu_1^2-\mu_5^2)(\mu_3^2-\mu_5^2)}. \tag{4.4}$$

Therefore,

$$T(r,z) = \frac{(Td\Gamma)}{2\pi}\int_3^\infty \alpha e^{-\alpha\mu_5 z}J_0(\alpha r)d\alpha = \frac{(Td\Gamma)}{2\pi}\,\frac{\mu_5 z}{(r^2+\mu_5^2 z^2)^{3/2}}. \tag{4.5}$$

The remaining integration constants D_{11}, D_{33} will be found from the kinematic or dynamic boundary conditions at the boundary $z=0$. Let us assume first that the plane $z=0$ is free from stresses. Therefore, we require that:

$$\sigma_{zz}(r,0)=0\,,\qquad \sigma_{rz}(r,0)=0\,. \tag{4.6}$$

Using the Eqs. (1.14), we obtain:

$$\begin{cases} \sigma_{zz}(r,0)=\dfrac{\beta(Td\Gamma)}{2\pi}\displaystyle\int_0^\infty aJ_0(ar)\sum_{j=1,3,5}D_{jj}f(\mu_j)\,da=0\,,\\[2ex] \sigma_{rz}(r,0)=\dfrac{\beta(Td\Gamma)}{2\pi}\displaystyle\int_0^\infty aJ_1(ar)\sum_{j=1,3,5}D_{jj}\mu_j f(\mu_j)\,da=0. \end{cases} \tag{4.7}$$

From the solution of this system of equations we have:

$$D_{11}=\frac{\mu_3-\mu_5}{\mu_1-\mu_3}\,\frac{f(\mu_5)}{f(\mu_1)}D_{55}\,,\qquad D_{33}=\frac{\mu_5-\mu_1}{\mu_1-\mu_3}\,\frac{f(\mu_5)}{f(\mu_3)}D_{55}\,. \tag{4.8}$$

In a way similar to that of the case of sources in the semi-space, we can determine the constants $D_{jj}\,(j=1,3,5)$ for other boundary conditions in the plane $z=0$. From the Eqs. (1.13), (1.14) we obtain:

$$\begin{cases} u_r=\dfrac{(Td\Gamma)\beta}{2\pi A_{33}}\displaystyle\sum_{j=1,3,5}D_{jj}\,g(\mu_j)\frac{1}{r}\left(1-\frac{\mu_j z}{R_j}\right),\\[2ex] w=\dfrac{(Td\Gamma)\beta}{2\pi A_{33}}\displaystyle\sum_{j=1,3,5}D_{jj}h(\mu_j)R_j^{-1},\\[2ex] \sigma_{zz}=\dfrac{(Td\Gamma)\beta}{2\pi}\displaystyle\sum_{j=1,3,5}D_{jj}f(\mu_j)\mu_j z R_j^{-3},\\[2ex] \sigma_{rr}=-\dfrac{(Td\Gamma)\beta}{2\pi}\displaystyle\sum_{j=1,3,5}D_{jj}\left[\mu_j^2 f(\mu_j)\mu_j zR_j^{-3}+qg(\mu_j)\,\frac{1}{r^2}\left(1-\frac{\mu_j z}{R_j}\right)\right],\\[2ex] \sigma_{\varphi\varphi}=-\dfrac{(Td\Gamma)\beta}{2\pi}\displaystyle\sum_{j=1,3,5}D_{jj}\left\{[\mu_j^2 f(\mu_j)+qg(\mu_j)]\,\mu_j zR_j^{-3}-\right.\\ \qquad\qquad\left.-qg(\mu_j)\dfrac{1}{r^2}\left(1-\dfrac{\mu_j z}{R_j}\right)\right\},\\[2ex] \sigma_{rz}=\dfrac{(Td\Gamma)\beta}{2\pi}\displaystyle\sum_{j=1,3,5}D_{jj}\mu_j f(\mu_j)rR_j^{-3}, \end{cases} \tag{4.9}$$

where $R_j=(r^2+\mu_j^2z^2)^{1/2}$.

The solution presented here constitutes a generalization of the known problem of E. Sternberg, [7], to the semi-space of transverse isotropy.

The knowledge of the displacements and stresses for thermal exposition in the element $d\Gamma$ will enable the determination of the displacement and stress in the case of thermal exposition in the region Γ lying in the plane $z=0$.

Let us consider the particular case in which $T_0 = \text{const}$ over the region of a circle of radius r_0. The condition (4.1) has the form:

$$T(r,0) = T_0 r_0 \int_0^\infty J_1(\alpha r_0) J_0(\alpha r)\, d\alpha = T_0 \begin{cases} 1 & \text{for} \quad 0 < r < r_0, \\ 0 & \text{for} \quad r_0 < r < \infty. \end{cases} \tag{4.10}$$

The stresses take the form:

$$\begin{cases} \sigma_{zz} = T_0 r_0 \beta \sum_{j=1,3,5} D_{jj} f(\mu_j) \int_0^\infty J_1(\alpha r_0) J_0(\alpha r)\, e^{-\alpha\mu_j z}\, d\alpha, \\ \sigma_{rr} = -T_0 r_0 \beta \sum_{j=1,3,5} D_{jj} \Big[\mu_j^2 f(\mu_j) \int_0^\infty J_1(\alpha r_0) J_0(\alpha r)\, e^{-\alpha\mu_j z}\, d\alpha + \\ \qquad + q g(\mu_j) \int_0^\infty (\alpha r)^{-1} J_1(\alpha r_0) J_1(\alpha r)\, e^{-\alpha\mu_j z}\, d\alpha \Big], \\ \sigma_{\varphi\varphi} = -T_0 r_0 \beta \sum_{j=1,3,5} D_{jj} \int_0^\infty \Big\{ [\mu_j^2 f(\mu_j) + q g(\mu_j)]\, J_1(\alpha r_0) J_0(\alpha r) - \\ \qquad - q g(\mu_j) J_1(\alpha r_0) J_1(\alpha r) (\alpha r)^{-1} \Big\} e^{-\alpha\mu_j z}\, d\alpha, \\ \sigma_{rz} = T_0 r_0 \beta \sum_{j=1,3,5} D_{jj} \mu_j f(\mu_j) \int_0^\infty J_1(\alpha r_0) J_1(\alpha r)\, e^{-\alpha\mu_j z}\, d\alpha. \end{cases} \tag{4.11}$$

The temperature field is given by the relation:

$$T = T_0 r_0 \int_0^\infty J_1(\alpha r_0) J_0(\alpha r)\, e^{-\alpha\mu_5 z}\, d\alpha. \tag{4.12}$$

The integrals appearing in the Eqs. (4.11) and (4.12) may be expressed by means of elliptic functions and Legendre's functions (see the Appendix I).

Let us consider also the following problem with mixed boundary conditions, [8]. Let the temperature over the region of a circle of radius r_0, lying in the plane $z=0$, be $T_0 = \text{const}$. In the remaining part of the plane $z=0$, let $\partial T/\partial z = 0$. We should satisfy the heat equation

$$\left(\mu_5^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2} \right) T = 0 \tag{4.13}$$

with the boundary conditions

$$(4.14)\qquad \begin{cases} T(r,0) = T_0 \quad \text{for} \quad 0 < r < r_0, \\ \left[\dfrac{\partial T}{\partial z}\right]_{z=0} = 0 \quad \text{for} \quad r_0 < r < \infty, \end{cases}$$

and $T = 0$ at infinity.

The solution of the Eq. (4.13) with the boundary conditions (4.14) is the function:

$$(4.15)\qquad T = \frac{2T_0}{\pi} \int_0^\infty \frac{\sin \alpha r_0}{\alpha} e^{-\alpha\mu_5 z} J_0(\alpha r)\, d\alpha .$$

The function ψ will be assumed in the form:

$$(4.16)\qquad \psi = \frac{2T_0}{\pi A_{33} A_{44}} \int_0^\infty \alpha^{-5} \sin \alpha r_0 J_0(\alpha r) \sum_{j=1,3,5} D_{jj}\, e^{-\alpha\mu_j z}\, d\alpha .$$

Substituting the function ψ in the expression

$$(4.17)\qquad T = A_{33} A_{44} \left(\mu_1^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\psi ,$$

and setting this equal to (4.15), we obtain:

$$D_{55} = \frac{1}{(\mu_1^2 - \mu_5^2)(\mu_3^2 - \mu_5^2)}$$

The remaining constants D_{11}, D_{33} will be determined from the boundary conditions of the problem. Thus for the edge $z = 0$, free from stresses, we shall find the quantities D_{11}, D_{33} from the Eqs. (4.7). The stresses σ_{ij} will be calculated from the equations (1.14):

$$(4.18)\qquad \begin{cases} \sigma_{rr} = -\dfrac{2T_0\beta}{\pi}\Big\{\displaystyle\sum_{j=1,3,5} D_{jj}\mu_j^2 f(\mu_j) \int_0^\infty J_0(\alpha r)\alpha^{-1} \sin(\alpha r_0)\, e^{-\alpha\mu_j z} d\alpha + \\ \qquad\qquad + \dfrac{q}{r} \displaystyle\sum D_{jj} g(\mu_j) \int_0^\infty \alpha^{-2} J_1(\alpha r) \sin(\alpha r_0)\, e^{-\alpha\mu_j z} d\alpha\Big], \\ \sigma_{\varphi\varphi} = -\dfrac{2T_0\beta}{\pi}\Big\{\displaystyle\sum_{j=1,3,5} D_{jj} \int_0^\infty \alpha^{-1} \sin(\alpha r_0)\, [J_0(\alpha r)\, [\mu_j^2 f(\mu_j) + q g(\mu_j)] - \\ \qquad\qquad - q g(\mu_j)\, (\alpha r)^{-1} J_1(\alpha r)]\, e^{-\alpha\mu_j z} d\alpha\Big\}, \\ \sigma_{zz} = \dfrac{2T_0\beta}{\pi} \displaystyle\sum_{j=1,3,5} D_{jj} f(\mu_j) \int_0^\infty J_0'(\alpha r)\, \alpha^{-1} \sin(\alpha r_0)\, e^{-\alpha\mu_j z} d\alpha , \\ \sigma_{rz} = \dfrac{2T_0\beta}{\pi} \displaystyle\sum_{j=1,3,5} D_{jj}\, \mu_j\, f(\mu_j) \int_0^\infty J_1(\alpha r)\, \alpha^{-1} \sin(\alpha r_0)\, e^{-\alpha\mu_j z} d\alpha . \end{cases}$$

The integrals appearing here are given by the equations:

$$\int_0^\infty \alpha^{-1} J_0(\alpha r)\sin(\alpha r_0)\, e^{-\alpha\mu_j z} d\alpha = \arc\sin\left(\frac{2r_0}{\sqrt{\mu_j^2 z^2+(r_0+r)^2}+\sqrt{\mu_j^2 z^2+(r_0-r)^2}}\right),$$

$$\int_0^\infty \alpha^{-1} J_1(\alpha r)\sin(\alpha r_0)\, e^{-\alpha\mu_j z} d\alpha = \frac{1}{r} r_0(1-\eta),$$

$$r_0^2 = \frac{r^2}{1-\eta_j^2} - \frac{\mu_j^2 z^2}{\eta_j^2}, \qquad \eta_j = \frac{1}{r_0\sqrt{2}}\sqrt{r_0^2 - R_j^2 + \sqrt{\Delta}},$$

$$\Delta = R_j^4 + r_0^4 - 2r_0^2 R_j^2 + 4r_0^2\mu_j^2 z^2 > 0.$$

5. The Stresses Due to a Discontinuous Temperature Field

Our basic problem will be to determine the Green's function for stresses for a temperature nucleus. By «temperature nucleus» we mean a state in which the temperature in an infinitesimal neighbourhood of a point P is T, while the temperature outside that neighbourhood is zero. If the point P is the point $(0, \zeta)$ in cylindrical coordinates, then:

$$(5.1)\qquad T = (TdV)\frac{\delta(r)}{2\pi r}\delta(z-\zeta) = \frac{(TdV)}{2\pi^2}\int_0^\infty\int_0^\infty \alpha J_0(\alpha r)\cos\gamma(z-\zeta)\, d\alpha\, d\gamma.$$

On the other hand, we have:

$$(5.2)\qquad T(r,z) = A_{33}A_{44}\left(\mu_1^2\nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2\nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\overline{\psi}.$$

Assuming the function $\overline{\psi}$ in the form

$$(5.3)\qquad \overline{\psi}(r,z) = \frac{1}{2\pi^2}\int_0^\infty\int_0^\infty C(\alpha,\beta)\, J_0(\alpha r)\cos\gamma(z-\zeta)\, d\alpha\, d\gamma,$$

we shall determine, by comparing the Eqs. (5.2) and (5.1), the quantity:

$$(5.4)\qquad C(\alpha,\beta) = \frac{(TdV)}{A_{33}A_{44}}\,\frac{\alpha}{(\alpha^2\mu_1^2+\gamma^2)(\alpha^2\mu_3^2+\gamma^2)}.$$

Introducing (5.4) in (5.3) and integrating with respect to γ, we obtain:

$$(5.5)\qquad \overline{\psi}(r,z) = \frac{(TdV)}{4\pi A_{33}A_{44}}\sum_{j=1,3} C_{jj}\int_0^\infty \alpha^{-2} e^{-\alpha\mu_j(z-\zeta)} J_0(\alpha r)\, d\alpha \qquad (z-\zeta)\geqslant 0,$$

where

$$C_{jj} = \frac{1}{\mu_j(\mu_k^2-\mu_j^2)} \qquad (j,k = 1,3, \quad j\neq k).$$

Introducing $\bar{\psi}$ in the Eqs. (1.13) and (1.14), we obtain:

$$
(5.6) \quad \begin{cases}
\bar{u}_r = \dfrac{(TdV)\beta}{4\pi A_{33}} \sum\limits_{j=1,3} C_{jj} g(\mu_j) r R_j^{-3}, \\
\bar{w} = \dfrac{(TdV)\beta}{4\pi A_{33}} \sum\limits_{j=1,3} C_{jj} \mu_j^2 h(\mu_j)(z-\zeta) R_j^{-3}, \\
\bar{\sigma}_{zz} = -\dfrac{(TdV)\beta}{4\pi} \sum\limits_{j=1,3} C_{jj} f(\mu_j) R_j^{-3} \left[1 - \dfrac{3\mu_j^2 (z-\zeta)^2}{R_j^2}\right], \\
\bar{\sigma}_{rr} = \dfrac{(TdV)\beta}{4\pi} \sum\limits_{j=1,3} C_{jj} \left[\mu_j^2 f(\mu_j)\left(1 - \dfrac{3\mu_j^2 (z-\zeta)^2}{R_j^2}\right) - q g(\mu_j)\right] R_j^{-3}, \\
\bar{\sigma}_{\varphi\varphi} = \dfrac{(TdV)\beta}{4\pi} \sum\limits_{j=1,3} C_{jj} \left\{\left[[\mu_j^2 f(\mu_j) + q g(\mu_j)]\left(1 - \dfrac{3\mu_j^2 (z-\zeta)^2}{R_j^2}\right) + \right.\right. \\
\qquad\qquad \left.\left. + q g(\mu_j)\right] R_j^{-3}, \right. \\
\bar{\sigma}_{rz} = \dfrac{3(TdV)\beta}{4\pi} \sum\limits_{j=1,3} C_{jj} \mu_j^2 f(\mu_j) r (z-\zeta) R_j^{-5}.
\end{cases}
$$

For $z-\zeta<0$, the summation subscripts in (5.6) and (5.5) are $j=2,4$. In addition we have:

$$
(5.7) \qquad C_{jj} = \frac{-1}{\mu_j(\mu_k^2 - \mu_j^2)} \qquad (j,k=2,4, \quad j \neq k).
$$

If the temperature is given over the region V, then, treating the quantities $\bar{\psi}$, $\bar{u}_r$, $\bar{w}$, $\bar{\sigma}_{ij}$ as Green's functions, we obtain the displacements and the stresses by integrating over V.

Let us consider finally the action of a temperature nucleus in an elastic semi-space bounded by the plane $z=0$. Let the temperature nucleus act at the point $(0, \zeta)$. The procedure will be analogous to that used in the case of a thermal source in a semi-space.

The function ψ will be assumed in the form:

$$
(5.8) \quad \psi(r,z) = \frac{(TdV)}{4\pi A_{33} A_{44}} \int_0^\infty \alpha^{-2} J_0(\alpha r) \left(\sum_j C_{jj} e^{-\alpha\mu_j(z-\zeta)} + \right.
$$
$$
\left. + \sum_{i,j} C_{ij} e^{-\alpha(\mu_i z - \mu_j \zeta)} \right) d\alpha \qquad (j=2,4, \qquad i=1,3, \qquad z-\zeta<0).
$$

The constants C_{jj} are given by the Eq. (5.7), and the constants C_{ij} will be determined from the boundary conditions. In the case of lack of stress in the plane $z=0$, we have:

$$
(5.9) \qquad \sigma_{zz}(r,0) = 0, \qquad \sigma_{rz}(r,0) = 0.
$$

Using the Eqs. (1.14), we shall represent the boundary conditions (5.9) in the form:

$$(5.10)\quad \begin{cases} \int\limits_0^\infty J_0(\alpha r)\alpha^2\Big[\sum_j C_{jj}f(\mu_j)e^{\alpha\mu_j\zeta}+\sum_{i,j}C_{ij}f(\mu_i)e^{\alpha\mu_i\zeta}\Big]d\alpha=0,\\ \int\limits_0^\infty J_1(\alpha r)\alpha^2\Big[\sum_j C_{jj}\mu_j f(\mu_j)e^{\alpha\mu_j\zeta}+\sum_{i,j}C_{ij}\mu_i f(\mu_i)e^{\alpha\mu_i\zeta}\Big]d\alpha=0.\end{cases}$$

From these equations we shall determine the following four integration constants:

$$(5.11)\quad \begin{cases} C_{12}=\dfrac{\mu_1+\mu_3}{\mu_1-\mu_3}C_{22}, & C_{32}=-\dfrac{2\mu_1}{\mu_1-\mu_3}\dfrac{f(\mu_1)}{f(\mu_3)}C_{22},\\ C_{14}=\dfrac{2\mu_3}{\mu_1-\mu_3}\dfrac{f(\mu_3)}{f(\mu_1)}C_{44}, & C_{34}=-\dfrac{\mu_1+\mu_3}{\mu_1-\mu_3}C_{44}.\end{cases}$$

Thus the function ψ is determined. Using the Eqs. (1.13), (1.14) and (5.6) we have:

$$(5.12)\quad \begin{cases} u_r=\bar{u}_r+\dfrac{(TdV)\beta}{4\pi A_{33}}\sum_{i,j}C_{ij}g(\mu_i)rR_{ij}^{-3},\\ w=\bar{w}+\dfrac{(TdV)\beta}{4\pi A_{33}}\sum_{i,j}C_{ij}\mu_i h(\mu_i)(\mu_i z-\mu_j\zeta)R_{ij}^{-3},\\ \sigma_{zz}=\bar{\sigma}_{zz}-\dfrac{(TdV)\beta}{4\pi}\sum_{i,j}C_{ij}f(\mu_i)\Big[1-\dfrac{3(\mu_i z-\mu_j\zeta)^2}{R_{ij}^2}\Big]R_{ij}^{-3},\\ \sigma_{rr}=\bar{\sigma}_{rr}+\dfrac{(TdV)\beta}{4\pi}\sum_{i,j}C_{ij}\Big\{\mu_i^2 f(\mu_i)\Big[1-\dfrac{3(\mu_i z-\mu_j\zeta)^2}{R_{ij}^2}\Big]R_{ij}^{-3}-\\ \qquad\qquad -qg(\mu_i)R_{ij}^{-3}\Big\},\\ \sigma_{\varphi\varphi}=\bar{\sigma}_{\varphi\varphi}+\dfrac{(TdV)\beta}{4\pi}\sum_{i,j}C_{ij}\Big\{[\mu_i^2 f(\mu_i)+qg(\mu_i)]\Big[1-\dfrac{3(\mu_i z-\mu_j\zeta)^2}{R_{ij}^2}\Big]+\\ \qquad\qquad +qg(\mu_i)\Big\}R_{ij}^{-3},\\ \sigma_{rz}=\bar{\sigma}_{rz}+\dfrac{3(TdV)\beta}{4\pi}\sum_{i,j}C_{ij}\mu_i f(\mu_i)r(\mu_i z-\mu_j\zeta)R_{ij}^{-5},\\ R_{ij}=[r^2+(\mu_i z-\mu_j\zeta)^2]^{1/2}.\end{cases}$$

The constants C_{ij} for other boundary conditions in the plane $z=0$ will be determined easily. Knowing the state of displacement and stress due to the temperature nucleus, we can determine the displacement and stresses due to a discontinuous temperature field.

6. Heat Sources and Temperature Nuclei in an Elastic Layer of Transverse Isotropy

Let a concentrated heat source act at the point $(0, \zeta)$ of an elastic layer of thickness h, Fig. 1. To solve this problem a procedure different than that of the Art. 2 will be chosen. We shall select such a particular integral of the Eq. (1.12.1) that the thermal boundary conditions and some of the mechanical or geometrical boundary conditions are satisfied in the planes $z' = 0, h$. Using the function ψ constituting the solution of the Eq. (1.12.1), we shall determine the stresses $\bar{\sigma}_{ij}$. The final stresses σ_{ij} will be obtained by adding to the stresses $\bar{\sigma}_{ij}$, the stresses $\bar{\bar{\sigma}}_{ij}$ chosen in such a manner that all the boundary conditions in the planes $z' = 0$, h are satisfied.

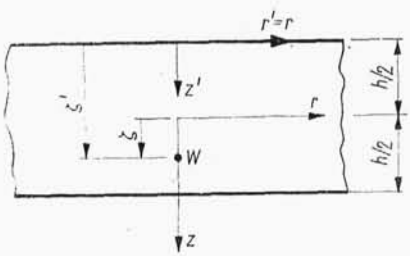

Fig. 1

We are concerned with an axially symmetric temperature field and, in consequence, an axially symmetric state of stress. Assuming that the temperature is equal to zero in the planes $z' = 0$, h, we shall assume the function ψ in the form:

$$\psi = \sum_{n=1}^{\infty} \sin a_n z' \int_0^{\infty} A_n(a)\, J_0(ar)\, da, \qquad a_n = \frac{n\pi}{h}. \tag{6.1}$$

We have assumed that $\psi = 0$ for $z' = 0, h$. The right-hand member of the Eq. (1.12.1) will be expressed by:

$$- \frac{W}{\pi h} \sum_{n=1}^{\infty} \sin a_n \zeta' \sin a_n z' \int_0^{\infty} a\, J_0(ar)\, da. \tag{6.2}$$

Substituting (6.1) and (6.2) in the Eq. (1.12.1), we find the quantity $A_n(a)$ and then the function:

$$\psi = \frac{W}{\pi h k' A_{33} A_{44}} \sum_{n=1}^{\infty} \sin a_n \zeta' \sin a_n z' \int_0^{\infty} \frac{a J_0(ar)\, da}{(\mu_1^2 a^2 + a_n^2)(\mu_3^2 a^2 + a_n^2)(\mu_5^2 a^2 + a_n^2)}. \tag{6.3}$$

Using the last equation of the group (1.13), we obtain the temperature field:

$$T(r, z') = A_{33} A_{44} \left(\mu_1^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\psi = \tag{6.4}$$

$$= \frac{W}{\pi h k'} \sum_{n=1}^{\infty} \sin a_n \zeta' \sin a_n z' \int_0^{\infty} \frac{a J_0(ar)\, da}{\mu_5^2 a^2 + a_n^2},$$

or

$$T(r, z') = \frac{W}{\pi h k' \mu_5^2} \sum_{n=1}^{\infty} K_0\left(\frac{\alpha_n r}{\mu_5}\right) \sin \alpha_n \zeta^1 \sin \alpha_n z', \tag{6.5}$$

where $K_0(\alpha_n r/\mu_5)$ is a modified Bessel function of the third kind. Finally, the temperature field may be expressed by the relation:

$$\begin{aligned} T(r, z') = \frac{W}{4\pi k' \mu_5} \Bigg\{ \frac{1}{R_1} - \frac{1}{R_2} + \sum_{n=1}^{\infty} \Bigg(\frac{1}{\sqrt{R_1^2 + 4nh\mu_5^2[nh - (z' - \zeta')]}} - \\ - \frac{1}{\sqrt{R_2^2 + 4nh\mu_5^2[nh - (z' + \zeta')]}} \Bigg) + \\ + \sum_{n=1}^{\infty} \Bigg(\frac{1}{\sqrt{R_1^2 + 4nh\mu_5^2[nh + (z' - \zeta')]}} - \frac{1}{\sqrt{R_2^2 + 4nh\mu_5^2[nh + (z' + \zeta')]}} \Bigg) \Bigg\}. \\ R_{1,2} = [r^2 + \mu_5^2(z' \mp \zeta')^2]^{1/2} \end{aligned} \tag{6.6}$$

It is evident that the singularity of the temperature field is thus separated. It is contained in the first term R_1^{-1}. For $r = 0$ and $z' \to \zeta', R_1^{-1}$ tends to infinity. By means of the function ψ, we shall determine the stress components $\bar{\sigma}_{ij}$, using the Eqs. (1.14).

Thus:

$$\left\{ \begin{aligned} \bar{\sigma}_{z'z'} &= -\frac{W\beta}{\pi h k'} \sum_{n=1}^{\infty} \sin \alpha_n \zeta' \sin \alpha_n z' \sum_{j=1,3,5} D_{jj} K_0\left(\frac{\alpha_n r}{\mu_j}\right), \\ \bar{\sigma}_{r'r'} &= -\frac{W\beta}{\pi h k'} \sum_{n=1}^{\infty} \sin \alpha_n \zeta' \sin \alpha_n z' \sum_{j=1,3,5} \Bigg[-D_{jj}\mu_j^2 K_0\left(\frac{\alpha_n r}{\mu_j}\right) - \\ &\qquad - \frac{q}{r} H_{jj} L_j\left(\frac{\alpha_n r}{\mu_j}\right) \Bigg], \\ \bar{\sigma}_{\varphi'\varphi'} &= \frac{W\beta}{\pi h k'} \sum_{n=1}^{\infty} \sin \alpha_n \zeta' \sin \alpha_n z' \sum_{j=1,3,5} \Bigg\{ -D_{jj}\mu_j^2 K_0\left(\frac{\alpha_n r}{\mu_j}\right) + \\ &\qquad + q H_{jj} \left[K_0\left(\frac{\alpha_n r}{\mu_j}\right) - \frac{1}{r} L_j\left(\frac{\alpha_n r}{\mu_j}\right) \right] \Bigg\}, \\ \bar{\sigma}_{r'z'} &= \frac{-W\beta}{\pi h k'} \sum_{n=1}^{\infty} \alpha_n \sin \alpha_n \zeta' \cos \alpha_n z' \sum_{j=1,3,5} D_{jj} L_j\left(\frac{\alpha_n r}{\mu_j}\right), \end{aligned} \right. \tag{6.7}$$

where

$$\left\{ \begin{aligned} & D_{jj} = \frac{f(\mu_j)}{\mu_j^2(\mu_j^2 - \mu_l^2)(\mu_j^2 - \mu_k^2)}, \qquad H_{jj} = \frac{g(\mu_j)}{\mu_j^2(\mu_l^2 - \mu_l^2)(\mu_j^2 - \mu_k^2)}, \\ & \qquad (j, k, l = 1, 3, 5, \qquad j \neq k \neq l) \\ & L_j\left(\frac{\alpha_n r}{\mu_j}\right) = \int_0^{\infty} \left[\alpha^2 + \left(\frac{\alpha_n}{\mu_j}\right)^2\right]^{-1} J_1(\alpha r)\, d\alpha = \frac{r}{4}\left[{}_1F_2\left(1; 1, 2; \frac{r^2\alpha_n^2}{4\mu_j^2}\right) + \right. \\ & \qquad \left. + {}_1F_2\left(1; 2, 1; \frac{r^2\alpha_n^2}{4\mu_j^2}\right) \right]. \end{aligned} \right. \tag{6.8}$$

The function

$$ {}_1F_2(\beta_1; \gamma_1, \gamma_2; \varrho) = \sum_{m=0}^{\infty} \frac{(\beta_1)_m \varrho^m}{m!\,(\gamma_1)_m (\gamma_2)_m} $$

is a hypergeometric series. It is evident that on the edges $z' = 0, h$ we have $T = 0$ and $\bar{\sigma}_{z'z'} = 0$. In the planes $z' = 0, h$, the stress $\bar{\sigma}_{rz'}$ is different from zero. It can be represented in the form:

$$ (6.9) \quad \begin{cases} \bar{\sigma}_{r'z'}(r,0) = \dfrac{-W\beta}{\pi h k'} \displaystyle\int_0^{\infty} J_1(\alpha r) \left(\sum_{j=1,3,5} D_{jj}\, \mu_j^2 \sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n \zeta'}{\alpha_n^2 + \alpha^2 \mu_j^2} \right) d\alpha = \\ \qquad = \dfrac{-W\beta}{2\pi k'} \displaystyle\int_0^{\infty} J_1(\alpha r) \left(\sum_{j=1,3,5} D_{jj}\, \mu_j^2\, \vartheta_1^{(j)}(\alpha, \zeta') \right) d\alpha, \\ \bar{\sigma}_{r'z'}(r,h) = \dfrac{-W\beta}{\pi h k'} \displaystyle\int_0^{\infty} J_1(\alpha r) \left(\sum_{j=1,3,5} D_{jj}\, \mu_j^2 \sum_{n=1}^{\infty} \frac{\alpha_n (-1)^n \sin \alpha_n \zeta'}{\alpha_n^2 + \alpha^2 \mu_j^2} \right) d\alpha = \\ \qquad = \dfrac{W\beta}{2\pi k'} \displaystyle\int_0^{\infty} J_1(\alpha r) \left(\sum_{j=1,3,5} D_{jj}\, \mu_j^2\, \vartheta_2^{(j)}(\alpha, \zeta') \right) d\alpha, \end{cases} $$

where we used the relation

$$ (6.10) \quad \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n \zeta'}{\alpha_n^2 + \alpha^2 \mu_j^2} = \frac{h}{2}\, \vartheta_1^{(j)}(\alpha, \zeta'), \\ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \alpha_n \sin \alpha_n \zeta'}{\alpha_n^2 + \alpha^2 \mu_j^2} = \frac{h}{2}\, \vartheta_2^{(j)}(\alpha, \zeta'), \end{cases} $$

in which

$$ \vartheta_1^{(j)}(\alpha, \zeta') = \frac{\operatorname{sh} \alpha \mu_j (h - \zeta')}{\operatorname{sh} \lambda_j}, \qquad \vartheta_2^{(j)}(\alpha, \zeta') = \frac{\operatorname{sh} \alpha \mu_j \zeta'}{\operatorname{sh} \lambda_j}, \qquad \lambda_j = \alpha \mu_j h. $$

In further considerations, the action of the source will be replaced by that of two sources of half the intensity acting symmetrically and anti-symmetrically in relation to the plane $z' = h/2$ (see Fig. 2*a*, 2*b*).

Consider first a system of sources represented at Fig. 2*a*. In the new system of coordinates (r, z), for heat sources symmetric in relation to the plane $z = 0$, we have:

$$ (6.11) \quad \begin{cases} \bar{\sigma}_{zz}^{(s)}\left(r, \pm \dfrac{h}{2}\right) = 0, \\ \bar{\sigma}_{rz}^{(s)}\left(r, \pm \dfrac{h}{2}\right) = \pm \dfrac{W\beta}{4\pi k'} \displaystyle\int_0^{\infty} J_1(\alpha r) \sum_{j=1,3,5} D_{jj}\, \mu_j^2\, \varrho_j^{(s)}(\alpha, \zeta)\, d\alpha, \end{cases} $$

where

$$\varrho_j^{(s)}(\alpha,\zeta)=\frac{\operatorname{ch}\alpha\mu_j\zeta}{\operatorname{ch}\gamma_j},\qquad \gamma_j=\frac{\mu_j\alpha h}{2}.$$

To annul the shear stresses in the planes $z=\pm h/2$, we shall add to the state $\bar{\sigma}_{ij}^{(s)}$ the state $\bar{\bar{\sigma}}_{ij}^{(s)}$ selected in such a manner that in the planes $z=\pm h/2$ we have:

$$(6.12)\qquad \bar{\sigma}_{zz}^{(s)}+\bar{\bar{\sigma}}_{zz}^{(s)}=0,\qquad \bar{\sigma}_{rz}^{(s)}+\bar{\bar{\sigma}}_{rz}^{(s)}=0.$$

The stresses $\bar{\bar{\sigma}}_{ij}^{(s)}$ are expressed by means of the Lekhnitsky function, [9], χ which satisfies the equation

$$(6.13)\qquad \left(\mu_1^2\nabla_r^2+\frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2\nabla_r^2+\frac{\partial^2}{\partial z^2}\right)\chi=0.$$

The stresses $\bar{\bar{\sigma}}_{ij}$ are expressed by means of the Lekhnitsky function thus:

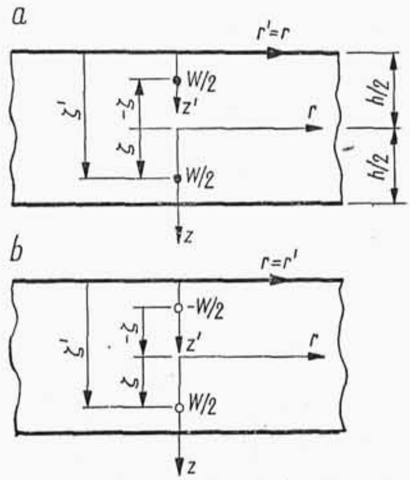

Fig. 2

$$(6.14)\qquad \begin{cases} \bar{\bar{\sigma}}_{rr}=-\dfrac{\partial}{\partial z}\left(\dfrac{\partial^2}{\partial r^2}+b_0\dfrac{1}{r}\dfrac{\partial}{\partial r}+a_0\dfrac{\partial^2}{\partial z^2}\right)\chi,\\ \bar{\bar{\sigma}}_{\varphi\varphi}=-\dfrac{\partial}{\partial z}\left(b_0\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+a_0\dfrac{\partial^2}{\partial z^2}\right)\chi\\ \bar{\bar{\sigma}}_{zz}=\dfrac{\partial}{\partial z}\left(c_0\nabla_r^2+d_0\dfrac{\partial^2}{\partial z^2}\right)\chi,\\ \bar{\bar{\sigma}}_{rz}=\dfrac{\partial}{\partial r}\left(\nabla_r^2+a_0\dfrac{\partial^2}{\partial z^2}\right)\chi, \end{cases}$$

where

$$a_0=-\frac{\gamma}{a_1}=-\frac{A_{13}}{A_{11}},\qquad d_0=\frac{1}{a_1}=\frac{A_{33}}{A_{11}},$$

$$c_0=\frac{p}{a_1}=\frac{A_{11}A_{33}-A_{13}^2-A_{13}A_{44}}{A_{11}A_{44}},$$

$$b_0=\frac{b_1}{a_1}=\frac{A_{12}(A_{13}+A_{44})-A_{11}A_{13}}{A_{11}A_{44}},$$

and the quantities a_1, b_1, p, γ as introduced in Art. 1.

It can easily be verified that the Lekhnitsky function is proportional to the displacement function φ introduced in Art. 1.

$$A_{33}A_{44}a_1\varphi=\chi.$$

The Eqs. (6.14) are identical to the corresponding members of the Eqs. (1.14) containing the function φ.

For the state $\bar{\bar{\sigma}}_{ij}^{(s)}$, the Lekhnitsky function will be assumed in the form of the Hankel integral:

$$\chi^{(s)} = \int_0^\infty a^{-3} J_0(ar)\left[A(a)\,\mathrm{sh}\, a\mu_1 z + B(a)\,\mathrm{sh}\, a\mu_3 z\right] da. \tag{6.15}$$

The function $\chi^{(s)}$ is symmetric in relation to the plane $z=0$, and satisfies the Eq. (6.13). The quantities $A(a)$, $B(a)$ will be determined from the boundary conditions (6.12). We have:

$$\begin{cases} A(a) = \dfrac{\mu_1\mu_3^2}{\mu_3^2+\gamma}\,\dfrac{\mathrm{ch}\,\gamma_3}{\mu_1\,\mathrm{sh}\,\gamma_3\,\mathrm{ch}\,\gamma_1 - \mu_3\,\mathrm{sh}\,\gamma_3\,\mathrm{ch}\,\gamma_1}\displaystyle\sum_{j=1,3,5} D_{jj}\mu_j^2\varrho^{(s)}(a,\zeta), \\ B(a) = -\dfrac{\mu_1(\mu_3^2+\gamma)}{\mu_3(\mu_1^2+\gamma)}\,\dfrac{\mathrm{ch}\,\gamma_1}{\mathrm{ch}\,\gamma_3}A(a). \end{cases} \tag{6 16}$$

Knowing the function $\chi^{(s)}$ it is easy to determine from the Eq. (6.14) the stress components $\bar{\bar{\sigma}}_{ij}^{(s)}$.

Consider a system of sources represented at Fig. 2b, and located anti-symmetrically in relation to the plane $z=0$.

In the system of coordinates, we obtain:

$$\begin{cases} \bar{\sigma}_{zz}^{(a)}\left(r, \pm\dfrac{h}{2}\right) = 0, \\ \bar{\sigma}_{rz}^{(a)}\left(r, \pm\dfrac{h}{2}\right) = \pm\dfrac{W\beta}{4\pi k'}\displaystyle\int_0^\infty J_1(ar)\sum_{j=1,3,5} D_{jj}\mu_j^2\varrho_j^{(a)}(a,\zeta)\,da, \end{cases} \tag{6.17}$$

where

$$\varrho_j^{(a)}(a,\zeta) = \frac{\mathrm{sh}\, a\mu_j\zeta}{\mathrm{sh}\,\gamma_j}.$$

To the state $\bar{\sigma}_{ij}^{(a)}$ we add the state $\bar{\bar{\sigma}}_{ij}^{(a)}$, so selected that in the planes $z=\pm h/2$ the following boundary conditions are satisfied:

$$\bar{\sigma}_{rz}^{(a)} + \bar{\bar{\sigma}}_{rz}^{(a)} = 0, \qquad \bar{\sigma}_{zz}^{(a)} + \bar{\bar{\sigma}}_{zz}^{(a)} = 0. \tag{6.18}$$

The stress components $\bar{\bar{\sigma}}_{ij}^{(a)}$ will be determined in terms of the Lekhnitsky function by means of the Eqs. (6.14). The function $\chi^{(a)}$ will be assumed in the form:

$$\chi^{(a)} = \int_0^\infty a^{-3} J_0(ar)\left[C(a)\,\mathrm{ch}\, a\mu_1 z + D(a)\,\mathrm{ch}\, a\mu_3 z\right] da. \tag{6.19}$$

The quantities $C_{(\alpha)}$, $D_{(\alpha)}$ will be determined from the conditions (6.18). As a result, we obtain:

$$(6.20)\quad \begin{cases} C(\alpha) = \dfrac{\mu_1\,\mu_3^2}{\mu_3^2+\gamma}\,\dfrac{\operatorname{sh}\gamma_3}{\mu_1\operatorname{sh}\gamma_3\operatorname{ch}\gamma_1 - \mu_3\operatorname{sh}\gamma_1\operatorname{ch}\gamma_3}\sum_{j=1,3,5} D_{jj}\,\mu_j^2\,\varrho_j^{(a)}(\alpha,\zeta), \\ D(\alpha) = -\dfrac{\mu_1(\mu_3^2+\gamma)}{\mu_3(\mu_1^2+\gamma)}\,\dfrac{\operatorname{sh}\gamma_1}{\operatorname{sh}\gamma_3}\,C(\alpha). \end{cases}$$

The final stresses will be obtained by superposition:

$$(6.21)\qquad \sigma_{ij} = \bar{\sigma}_{ij} + \bar{\sigma}_{ij}^{(s)} + \bar{\sigma}_{ij}^{(a)}.$$

Consider, finally, the action of a temperature nucleus at the point $(0,\zeta)$ of the elastic layer. This nucleus will be determined in the following way. Let the temperature in a volume element dV with the centre at $(0,\zeta)$ be T, the temperature outside that element being zero. This nucleus may be represented by the equation:

$$(6.22)\qquad T(r,z) = \frac{(TdV)}{\pi h}\sum_{n=1}^{\infty}\sin\alpha_n\zeta'\,\sin\alpha_n z'\int_0^{\infty}\alpha J_0(\alpha r)\,d\alpha.$$

The state of stress due to the action of that nucleus will be described by means of the function ψ and χ. Let us assume that:

$$(6.23)\qquad \psi = \sum_{n=1}^{\infty}\sin\alpha_n z'\int_0^{\infty}A_n(\alpha)\,\alpha J_0(\alpha r)\,d\alpha,$$

where $A_n(\alpha)$ is a quantity unknown for the time being. Bearing in mind the last equation of the group (1.13), we have:

$$(6.24)\quad T(r,z) = A_{33}\,A_{44}\left(\mu_1^2\nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2\nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\psi =$$

$$= A_{33}\,A_{44}\sum_{n=1}^{\infty}\sin\alpha_n z'\int_0^{\infty}A_n(\alpha)\,\alpha J_0(\alpha r)\,(\mu_1^2\alpha^2+\alpha_n^2)(\mu_3^2\alpha^2+\alpha_n^2)\,d\alpha.$$

Comparing the relations (6.22) and (6.24), we determine the quantity $A_n(\alpha)$. Then,

$$(6.25)\quad \psi = \frac{(TdV)}{\pi h A_{33}\,A_{44}}\sum_{n=1}^{\infty}\sin\alpha_n\zeta'\,\sin\alpha_n z'\int_0^{\infty}\frac{\alpha J_0(\alpha r)\,d\alpha}{(\mu_1^2\alpha^2+\alpha_n^2)(\mu_3^2\alpha^2+\alpha_n^2)}.$$

For the stresses $\bar{\sigma}_{ij}$, we shall obtain the following equations:

$$
(6.26)\quad \left\{
\begin{aligned}
\bar{\sigma}_{r'r'} &= \frac{(TdV)\beta}{\pi h}\sum_{n=1}^{\infty}\sin\alpha_n\zeta'\sin\alpha_n z'\sum_{j=1,3}\left[B_{jj}\alpha_n^2 K_0\left(\frac{\alpha_n r}{\mu_j}\right)-\right.\\
&\qquad\left.-\frac{\alpha_n}{\mu_j r}M_{jj}K_1\left(\frac{\alpha_n r}{\mu_j}\right)\right],\\
\bar{\sigma}_{\varphi'\varphi'} &= \frac{(TdV)\beta}{\pi h}\sum_{n=1}^{\infty}\sin\alpha_n\zeta'\sin\alpha_n z'\sum_{j=1,3}\left\{B_{jj}\alpha_n^2 K_0\left(\frac{\alpha_n r}{\mu_j}\right)+\right.\\
&\qquad\left.+\frac{\alpha_n^2}{\mu_j^2}M_{jj}\left[K_0\left(\frac{\alpha_n r}{\mu_j}\right)+\frac{\mu_j}{\alpha_n r}K_1\left(\frac{\alpha_n r}{\mu_j}\right)\right]\right\},\\
\bar{\sigma}_{z'z'} &= -\frac{(TdV)\beta}{\pi h}\sum_{n=1}^{\infty}\sin\alpha_n\zeta'\sin\alpha_n z'\sum_{j=1,3}\frac{\alpha_n^2}{\mu_j^2}B_{jj}K_0\left(\frac{\alpha_n r}{\mu_j}\right),\\
\bar{\sigma}_{r'z'} &= -\frac{(TdV)\beta}{\pi h}\sum_{n=1}^{\infty}\sin\alpha_n\zeta'\cos\alpha_n z'\sum_{j=1,3}\frac{\alpha_n^2}{\mu_j}B_{jj}K_1\left(\frac{\alpha_n r}{\mu_j}\right),
\end{aligned}\right.
$$

where

$$
B_{jj}=\frac{f(\mu_j)}{\mu_j^2(\mu_k^2-\mu_j^2)},\qquad M_{jj}=q\,\frac{g(\mu_j)}{\mu_j^2(\mu_k^2-\mu_j^2)}\qquad (j\neq k,\quad j,k=1,3).
$$

On the boundaries $z'=0,h$, the normal stresses $\bar{\sigma}_{z'z'}$ vanish. The stresses $\bar{\sigma}_{r'z'}$ remain different from zero. They will be expressed by the equations:

$$
(6.27)\quad \left\{
\begin{aligned}
\bar{\sigma}_{r'z'}(r,0) &= \frac{(TdV)\beta}{\pi h}\sum_{j=1,3}B_{jj}\mu_j^2\int_0^{\infty}\alpha^2 J_1(\alpha r)\,\vartheta_1^{(j)}(\alpha,\zeta)\,d\alpha,\\
\bar{\sigma}_{r'z'}(r,h) &= -\frac{(TdV)\beta}{\pi h}\sum_{j=1,3}B_{jj}\mu_j^2\int_0^{\infty}\alpha^2 J_1(\alpha r)\,\vartheta_2^{(j)}(\alpha,\zeta)\,d\alpha,
\end{aligned}\right.
$$

where $\vartheta_1^{(j)}$, $\vartheta_2^{(j)}$ are given by the Eqs. (6.10).

The further procedure — that of cancelling the stresses $\bar{\sigma}_{r'z'}$ on the boundaries $z'=0,h$ by adding the state $\bar{\bar{\sigma}}_{ij}$ to the state $\bar{\sigma}_{ij}$ — introduces nothing new. It is identical with that used in the case of a heat source in an elastic layer.

7. Non-Steady Thermal Stresses in an Elastic Space and Semi-Space

Consider the action of a transient heat source at the origin. Let us determine first the function $\bar{\psi}$ satisfying the differential equation (1.12.1):

(7.1)

$$
k' A_{33}A_{44}\left(\mu_1^2\nabla_r^2+\frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2\nabla_r^2+\frac{\partial^2}{\partial z^2}\right)\left(\mu_5^2\nabla_r^2+\frac{\partial^2}{\partial z^2}-\sigma^2\frac{\partial}{\partial t}\right)\bar{\psi}=-\frac{W\delta(r)}{2\pi r}\,\delta(z)\,\delta(t).
$$

Using L a p l a c e's transformation $\psi^* = \int_0^\infty e^{-pt}\,\bar{\psi}(r, z, t)\,dt$, and bearing in mind the initial condition $T(r, z, 0) = 0$, we shall transform the Eq. (7.1) to obtain:

$$(7.2)\quad k' A_{33} A_{44}\left(\mu_1^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_3^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2}\right)\left(\mu_5^2 \nabla_r^2 + \frac{\partial^2}{\partial z^2} - \sigma^2 p\right)\bar{\psi}^* = -\frac{W\delta(r)\delta(z)}{2\pi r}.$$

The solution of this equation is:

$$(7.3)\quad \bar{\psi}^* = \frac{W}{2\pi^2 A_{33} A_{44} k'} \int_0^\infty \int_0^\infty \frac{a J_0(ar) \cos\gamma z\, da\, d\gamma}{(\mu_1^2 a^2 + \gamma^2)(\mu_3^2 a^2 + \gamma^2)(\mu_5^2 a^2 + \gamma^2 + \sigma^2 p)}.$$

Retransforming this, we have:

$$(7.4)\quad \bar{\psi} = \frac{W}{2\pi^2 A_{33} A_{44} \sigma^2 k'} \int_0^\infty \int_0^\infty \frac{a J_0(ar)\, e^{-(\mu_5^2 a^2 + \gamma^2)\frac{t}{\sigma^2}} \cos\gamma z\, da\, d\gamma}{(\mu_1^2 a^2 + \gamma^2)(\mu_3^2 a^2 + \gamma^2)}.$$

Bearing in mind the last equation of the group (1.13), we have:

$$(7.5)\quad T(r, z, t) = \frac{W}{2\pi^2 \sigma^2 k'} \int_0^\infty \int_0^\infty a J_0(ar)\, e^{-(\mu_5^2 a^2 + \gamma^2)\frac{t}{\sigma^2}} \cos\gamma z\, da\, d\gamma = $$

$$= \frac{W \mu_5}{\sigma^2 k' (\pi\vartheta)^{3/2}}\, e^{-\frac{r^2 + \mu_5^2 z^2}{\vartheta}},$$

where

$$\vartheta = \frac{4\mu_5^2 t}{\sigma^2}.$$

Knowing the function $\bar{\psi}$ we can determine the stresses from the Eqs. (1.14). Thus, for instance:

$$(7.6)\quad \sigma_{zz} = \frac{W\beta}{2\pi^2 k' \sigma^2} \int_0^\infty \int_0^\infty \frac{a^3 J_0(ar)(da^2 + e\gamma^2)}{(\mu_1^2 a^2 + \gamma^2)(\mu_3^2 a^2 + \gamma^2)}\, e^{-(\mu_5^2 a^2 + \gamma^2)\frac{t}{\sigma^2}} \cos\gamma z\, da\, d\gamma =$$

$$= \frac{W\beta}{8\pi\sigma^2 k'} \int_0^\infty a^2 J_0(ar) \left[\sum_{j=1,3} N_{jj}\, f(\mu_j)\, r(\mu_j; z, t, a)\, e^{(\mu_j^2 - \mu_5^2)\frac{a^2 t}{\sigma^2}}\right] da,$$

with the notation

$$r(\mu_j; z, t, a) = e^{-\mu_j a z} \operatorname{erfc}\left(\frac{\mu_j a}{\sigma}\sqrt{t} - \frac{\sigma z}{2\sqrt{t}}\right) + e^{\mu_j a z} \operatorname{erfc}\left(\frac{\mu_j a}{\sigma}\sqrt{t} + \frac{\sigma z}{2\sqrt{t}}\right)$$

$$N_{jj} = \frac{1}{\mu_j(\mu_k^2 - \mu_j^2)} \qquad (j = 1, 3, \quad j \neq k).$$

Similarly,

$$(7.7)\quad \sigma_{rz}=\frac{W\beta}{2\pi^2\sigma^2 k'}\int_0^\infty\int_0^\infty \frac{\alpha^3 J_1(\alpha r)(d\alpha^2+e\gamma^2)e^{-(\mu_5^2\alpha^2+\gamma^2)\frac{t}{\sigma^2}}}{(\mu_1^2\alpha^2+\gamma^2)(\mu_3^2\alpha^2+\gamma^2)}\gamma\sin\gamma z\,d\alpha\,d\gamma=$$

$$=\frac{W\beta}{8\pi\sigma^2 k'}\int_0^\infty \alpha^2 J_0(\alpha r)\left[\sum_{j=1,3} N_{jj} f(\mu_j)e^{(\mu_j^2-\mu_5^2)\frac{\alpha^2 t}{\sigma^2}} r(\mu_j; r,t,\alpha)\right]d\alpha,$$

where

$$r(\mu_j; z,t,\alpha)=e^{-\mu_j\alpha z}\operatorname{erfc}\left(\frac{\mu_j\alpha}{\sigma}\sqrt{t}-\frac{\sigma z}{2\sqrt{t}}\right)-e^{\mu_j\alpha z}\operatorname{erfc}\left(\frac{\mu_j\alpha}{\sigma}\sqrt{t}+\frac{\sigma z}{2\sqrt{t}}\right).$$

The solution for a transient source of heat makes possible the solution for sources varying with time in any manner.

From among these cases, the most simple is that of continuous source. The solution may be obtained either from the Eq. (7.4) by integration with respect to time from 0 to ∞, or from the direct solution of the equation (7.1). In the latter case, the right-hand member of the Eq. (7.1) has the form $-W[\delta(r)/2\pi r]\,\delta(z)\eta(t)$, where $\eta(t)$ is Heaviside's function. For the right-hand member of the Eq. (7.2), we have $-W[\delta(r)/2\pi r]\,\delta(z)p^{-1}$. Therefore,

$$\bar{\psi}^*=\frac{W}{2\pi^2 k' A_{33}A_{44}}p^{-1}\int_0^\infty\int_0^\infty\frac{\alpha J_0(\alpha r)\cos\gamma z\,d\alpha\,d\gamma}{(\mu_1^2\alpha^2+\gamma^2)(\mu_3^2\alpha^2+\gamma^2)(\mu_5^2\alpha^2+\gamma^2+\sigma^2 p)}.$$

After performing the inverse Laplace transformation, we obtain:

$$(7.8)\quad \bar{\psi}=\frac{W}{2\pi^2 A_{33}A_{44}k'}\int_0^\infty\int_0^\infty\frac{\alpha J_0(\alpha r)\left(1-e^{-(\alpha^2\mu_5^2+\gamma^2)\frac{t}{\sigma^2}}\right)}{(\mu_1^2\alpha^2+\gamma^2)(\mu_3^2\alpha^2+\gamma^2)(\mu_5^2\alpha^2+\gamma^2)}\cos\gamma z\,d\alpha\,d\gamma.$$

Bearing in mind the Eq. (2.3), we can represent the expression (7.8) in the form:

$$(7.9)\qquad \bar{\psi}(r,z,t)=\bar{\psi}(r,z)+\bar{\psi}_1(r,z,t).$$

The first term of the right-hand member of the Eq. (7.9) does not depend on time and gives the function $\bar{\psi}$ for the case of a steady heat source. The function $\bar{\psi}_1$ may be expressed by the simple integral:

$$\bar{\psi}_1(r,z,t)=-\frac{W}{8\pi A_{33}A_{44}k'}\int_0^\infty \alpha^{-4}J_0(\alpha r)\sum_{j=1,3,5}C_{jj}e^{-(\mu_5^2-\mu_j^2)\frac{\alpha^2 t}{\sigma^2}}r(\mu_j; z,t,\alpha)\,d\alpha,$$

where

$$C_{jj}=\frac{1}{\mu_j(\mu_j^2-\mu_i^2)(\mu_j^2-\mu_l^2)}\qquad (i,j,l=1,3,5,\ i\neq j\neq l).$$

The temperature field is described by the relation:

$$(7.10)\quad T(r,z,t)=T_0(r,z)-\frac{W}{8\pi\mu_5 k'}\int_0^\infty J_0(\alpha r)\,r(\mu_5;z,t,\alpha)\,d\alpha=T_0(r,z)+T_1(r,z,t).$$

The function T_0 is independent of time and given by the Eq. (2.8). The function T_1 depends on time and tends to zero for $t\to\infty$. Using the Eqs. (1.14) and (2.8), we shall determine the stress components. Thus, for instance,

$$(7.11)\qquad \bar{\sigma}_{zz}(r,z,t)=\bar{\sigma}_{zz}(r,z)+\sigma_{zz}^{(1)}(r,z,t),$$

where the function $\bar{\sigma}_{zz}$ is given by the Eq. (2.8) and does not depend on t, while the function $\sigma_{zz}^{(1)}$ is given by the integral:

$$(7.12)\quad \sigma_{zz}^{(1)}(r,z,t)=-\frac{W\beta}{8\pi k'}\int_0^\infty J_0(\alpha r)\sum_{j=1,3,5}C_{jj}f(\mu_j)e^{-(\mu_5^2-\mu_j^2)\frac{\alpha^2 t}{a^2}}\,r(\mu_j;z,t,\alpha)\,d\alpha.$$

To determine the thermal stresses in an elastic semi-space, the most convenient method will be that of reflections.

If we are concerned with a semi-space with a concentrated heat source at the point $(0,\zeta)$, and if $T=0$ in the plane $z=0$, the stresses σ_{ij} will be composed of two parts: σ_{ij} and $\bar{\bar{\sigma}}_{ij}$. The stresses $\bar{\sigma}_{ij}$ concern the infinite space in which two heat sources act: positive at the point $(0,\zeta)$ and negative at the point $(0,-\zeta)$. Such a location of the sources satisfies the boundary condition $T=0$, and gives $\bar{\sigma}_{zz}=0$ at every moment t. The state $\bar{\bar{\sigma}}_{ij}$ is the state of stress in an elastic semi-space $z\geqslant 0$, so selected that the remaining boundary conditions are satisfied in the plane $z=0$.

If the boundary $z=0$ is thermally isolated, $[\partial T/\partial z]_{z=0}=0$, positive concentrated heat source should be located at the points $(0,\zeta)$, and negative at the point $(0,-\zeta)$. In the plane $z=0$, we obtain the stress $\bar{\sigma}_{rz}$ equal to zero. Adding to the stress $\bar{\sigma}_{ij}$, a stress $\bar{\bar{\sigma}}_{ij}$ chosen so that the normal stresses become zero at the boundary, we obtain the final result σ_{ij}.

In the second stage of solution, we shall use the Lekhnitsky function constituting a generalization to transversally isotropic bodies of the familiar Love's function.

Appendix I

Some of the Hankel integrals appearing in the Eq. (2.11), (4.12) and (4.13) may be determined by means of Legendre's functions and elliptic integrals.

Thus:

$$(8.1)\quad I_1(r_0,r)=\int_0^\infty e^{-\alpha\mu_j z}J_1(\alpha r_0)J_1(\alpha r)\,d\alpha=\pi^{-1}(r_0 r)^{-1/2}\,Q_{1/2}\left(\frac{r^2+r_0^2+\mu_j^2 z^2}{2r_0 r}\right),$$

(8.2) $$I_2(r_0, r) = \int_0^\infty e^{-a\mu_j z} J_1(ar_0) J_0(ar)\, da = (\pi r_0)^{-1} [K' E(k,\Theta) - (E' - K') F(k,\Theta) - \mu_j z [(r_0+r)^2 + \mu_j^2 z^2]^{-1/2} K'],$$

where Q_ν denotes Legendre's function of the second kind with superscript $\nu = 1/2$.

$F(k,\Theta)$ and $E(k,\Theta)$ are non-complete elliptic integrals for the completing modulus

$$k = [(r_0 - r)^2 + \mu_j^2 z^2]^{1/2} [(r_0 + r)^2 + \mu_j^2 z^2]^{-1/2}$$

and the argument

$$\Theta = \sin^{-1} \mu_j z [(r_0 - r)^2 + \mu_j^2 z^2]^{-1/2} \qquad (0 \leqslant \Theta \leqslant \pi).$$

Using the integrals I_1 and I_2, we can determine the following integral:

(8.3) $$I_3 = \int_0^\infty a^{-1} e^{-a\mu_j z} J_1(ar_0) J_1(ar)\, da = \frac{1}{2} [r_0 I_2(r, r_0) + r I_2(r_0, r) - \mu_j z I_1].$$

The remaining integrals may be expressed by a series of hypergeometric functions, [11]:

(8.4) $$\int_0^\infty a^{\lambda-3/2} e^{-a\mu_j z} J_\mu(ar_0) J_\nu(ar)\, da = \sum_{m=0}^\infty \frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\,\Gamma(\mu+m+1)} \left(\frac{r}{2\mu_j z}\right)^{2m} {}_2F_1\left(-m, -\mu-m; \nu+1; \frac{r^2}{r_0^2}\right),$$

$$\mathrm{Re}(\lambda+\mu+\nu) > 0.$$

Appendix II

All the integrals appearing in this paper, for instance (2.4), (2.10), (3.3) etc. should either be treated formally as integrals subjected to the differentiation law with respect to the parameter (which is valid for convergent integrals), or we should take only a finite part of a given divergent integral.

The finite part of a divergent integral is defined in the following way.

Let $\int_a^b f(a)\, da$ be divergent and $\int_{a+\varepsilon}^b f(a)\, da$ convergent, $\varepsilon > 0$, and $f(a) = g(a) + h(a)$ where $\int_a^b g(a)\, da$ is divergent and $\int_a^b h(a)\, da$ convergent and $G'(a) = g(a)$. We have:

$$\mathrm{pf} \int_a^b f(a)\, da = G(b) + \int_a^b h(a)\, da.$$

In our case,

$$a = 0, \qquad b = \infty.$$

Thus, for instance, for the integral (2.4) we have:

$$f(a)=a^{-4}J_0(ar)e^{-az},$$

$$h(a)=a^{-4}\left\{J_0(ar)e^{-az}-\left[1-az+a^2\left(\frac{z^2}{2}-\frac{r^2}{4}\right)-z\left(\frac{r^2}{4}-\frac{z^2}{6}\right)a^2\sin a\right]\right\},$$

$$g(a)=a^{-4}\left[1-az+a^2\left(\frac{z^2}{2}-\frac{r^2}{4}\right)-z\left(\frac{r^2}{4}-\frac{z^2}{6}\right)a^2\sin a\right],$$

$$G(a)=-\frac{1}{3}a^{-3}+\frac{z}{2}a^{-2}-\left(\frac{z^2}{2}-\frac{r^2}{4}\right)a^{-1}+z\left(\frac{r^2}{4}-\frac{z^2}{6}\right)\left[\frac{\sin a}{a}-ci(a)\right],$$

$$G(\infty)=0;$$

or:

$$pf\int_0^\infty a^{-4}J_0(ar)e^{-az}\,da=\int_0^\infty h(a)\,da.$$

The latter integral is convergent and differentiation and integration with respect to the parameter under the integration sign is permissible. It should be borne in mind that the part in brackets does not influence the stresses, and the displacements are determined, according to Kirchhoff's theorem, with the degree of indeterminacy to the extent of rigid displacement.

Since we are interested in displacements and stresses only, all the integrals appearing throughout the paper may be differentiated and integrated formally with respect to the parameter under the integration sign, the values of the stresses remaining unchanged. At most, various rigid displacements will be obtained.

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Streszczenie

NAPRĘŻENIA CIEPLNE W CIAŁACH O IZOTROPII POPRZECZNEJ

Rozważa się ośrodek sprężysty poprzecznie izotropowy zarówno sprężyście, jak i termicznie. Wyznaczono stan naprężenia i pole temperatury dla następujących zagadnień: skupione ustalone źródło ciepła lub jądro termosprężystego odkształcenia w przestrzeni i półprzestrzeni oraz w warstwie sprężystej; rozpatrzono kilka rodzajów warunków brzegowych przy założeniu, że na brzegu jest albo izolacja termiczna, albo temperatura $T = 0$. Przykładowo podano także rozwiązanie dla półprzestrzeni poddanej na powierzchni ograniczającej działaniu pola temperatury na obszarze $d\Gamma$ oraz na powierzchni koła.

Z zagadnień nieustalonych rozważono przestrzeń nieograniczoną poddaną działaniu chwilowego lub ciągłego, skupionego źródła ciepła, podając także drogę otrzymania rozwiązań dla półprzestrzeni.

Rozwiązania w zależności od zagadnienia dane są w postaci zamkniętej, całek eliptycznych, pojedynczych szeregów nieskończonych bądź całek pojedynczych.

Dodatkowo podano sposób wyliczenia pewnych całek oraz uwagi dotyczące występujących w pracy całek rozbieżnych.

Резюме

ТЕРМИЧЕСКИЕ НАПРЯЖЕНИЯ В ТЕЛАХ, ОБЛАДАЮЩИХ ПОПЕРЕЧНОЙ ИЗОТРОПИЕЙ

Рассматривается упругая среда поперечно изотропная, как упруго так и термически. Дается решение, т.е. определяется напряженное состояние и температурное поле для следующих вопросов: сосредоточенный, стационарный источник тепла или термоупругое ядро деформации в пространстве и полупространстве и в упругом слое. Рассматривается несколько видов краевых условий при предположении, что на краю существует или термическая изоляция или температура $T = 0$. В качестве примера дается также решение для полупространства, под-

вергнутого на поверхности ограничивающей полупространство, действию температурного поля в области $d\Gamma$, а также на поверхности круга.

Из нестационарных вопросов обсуждается бесконечное пространство, находящееся под действием временного или постоянного сосредоточенного источника тепла, приводя также способ получения решений для полупространства.

Решения, в зависимости от задачи, даются в замкнутом виде, в виде элиптических интегралов, бесконечных рядов или же одинарных интегралов.

В приложениях дается способ исчисления некоторых интегралов, а также замечания, касающиеся выступающих в работе расходящихся интегралов.

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