P O L S K A   A K A D E M I A   N A U K
INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI

ARCHIWUM MECHANIKI STOSOWANEJ

ARCHIVES DE MÉCANIQUE APPLIQUÉE

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PAŃSTWOWE WYDAWNICTWO NAUKOWE

A PLANE DISTORTION PROBLEM

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1. Introduction

Consider a simply connected thin plate free from stresses at the edges. Assume that a given state of strain exists in the region Γ of the plate; let the strain components ε_{ij}^0 be equal to zero outside that region. The introduction of initial strains ε_{ij}^0 will provoke a certain state of strain (ε_{ij}) and stress (σ_{ij}) throughout the entire plate. The object of this paper is to determine the strain components ε_{ij} and the stress components σ_{ij} for given initial deformations ε_{ij}^0.

The problem stated above is of practical interest. It will be illustrated by way of two examples.

(a) Consider an arbitrary plate. Let the strains $\varepsilon_{xx}^0 = \text{const}$, $\varepsilon_{yy}^0 = \text{const}$, and $\varepsilon_{xy}^0 = 0$ (Fig. 1) be prescribed in the region of the rectangle Γ, with the edges a and b. The state of stress due to these initial strains can be interpreted as follows. Let us cut out of the plate the rectangle Γ and replace it by a rectangle of the same material but with the edges $a + \Delta a$, $b + \Delta b$, where Δa and Δb are assumed to be small quantities in relation to the lengths of the edges of the rectangle. The «assemblage» of this rectangle can be only done by force: a state of stress (σ_{ij}) will result in the plate. The field of initial strains is characterized in our case by constant quantities $\varepsilon_{xx}^0 = \Delta a/a$ and $\varepsilon_{yy}^0 = \Delta b/b$. It is clear that we can imagine cases in which the components ε_{ij}^0 are point functions in the region Γ. In the cases considered in this paper, we are concerned with assemblage stresses resulting from errors in the manufacture of plate elements.

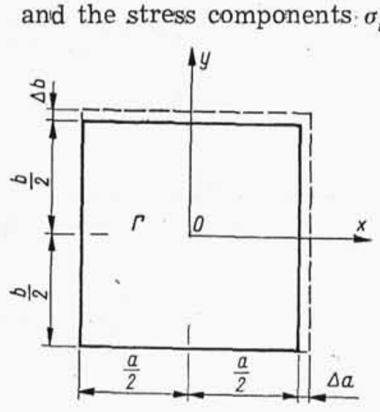

Fig. 1

(b) Let the region Γ of the plate be heated to the temperature T. Let $T = 0$ outside the region Γ. If a region Γ of the plate is removed and heated to the temperature T, it will undergo dilatation. The dilatation

$\Theta^0 = \varepsilon^0_{xx} + \varepsilon^0_{yy} = 2\alpha_t T$ will be constant over the whole region. The «reassemblage» of the heated region in the plate will provoke a state of thermal stress (σ_{ij}) in the plate.

By contrast to the problem of thermal stress due to a discontinuous temperature, such as has been considered in many investigations among which mention should be made of the recent works of M. Hiecke, [1], [2] and [3], the more general problem of assemblage stresses has, as far as the present author knows, not yet been treated in greater detail.

The solution of the problem stated to begin with will be obtained here by using Green function for generalized displacement equations and the generalized Airy equation.

The relations between the state of stress (σ_{ij}) and the state of strain (ε_{ij}) for initial strain have, in a plane state of stress, the form

$$\begin{cases} \sigma_{xx} = \dfrac{E}{1-\nu^2}\left[(\varepsilon_{xx} - \varepsilon^0_{xx}) + \nu(\varepsilon_{yy} - \varepsilon^0_{yy})\right], \\ \sigma_{yy} = \dfrac{E}{1-\nu^2}\left[(\varepsilon_{yy} - \varepsilon^0_{yy}) + \nu(\varepsilon_{xx} - \varepsilon^0_{xx})\right], \\ \sigma_{xy} = G(\varepsilon_{xy} - \varepsilon^0_{xy}), \qquad G = \dfrac{E}{2(1+\nu)}. \end{cases} \tag{1.1}$$

Here E is Young's modulus, G modulus of elasticity in shear and ν Poisson's ratio. Expressing the strains as a function of displacements we have

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{1.2}$$

Introducing the relations (1.1) in the equations of equilibrium of a plate element,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \tag{1.3}$$

we obtain the displacement equations of our problem in the form

$$\begin{cases} \nabla^2 u + \varrho \dfrac{\partial \Theta}{\partial x} = \dfrac{2}{1-\nu}\dfrac{\partial}{\partial x}(\varepsilon^0_{xx} + \nu\varepsilon^0_{yy}) + \dfrac{\partial \varepsilon^0_{xy}}{\partial y}, \\ \nabla^2 v + \varrho \dfrac{\partial \Theta}{\partial y} = \dfrac{2}{1-\nu}\dfrac{\partial}{\partial y}(\varepsilon^0_{yy} + \nu\varepsilon^0_{xx}) + \dfrac{\partial \varepsilon^0_{xy}}{\partial x}, \end{cases} \tag{1.4}$$

where

$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \qquad \varrho = \frac{1+\nu}{1-\nu}.$$

This system may be transformed to the form

$$(1.5)\qquad \begin{cases} \nabla^2\nabla^2 u = \dfrac{\partial}{\partial x}\left[\nabla^2(\varepsilon^0_{xx}+\nu\varepsilon^0_{yy})+(1+\nu)\dfrac{\partial^2}{\partial y^2}(\varepsilon^0_{xx}-\varepsilon^0_{yy})\right]+ \\ \qquad\qquad\qquad\qquad + \dfrac{\partial}{\partial y}\left[\nabla^2-(1+\nu)\dfrac{\partial^2}{\partial x^2}\right]\varepsilon^0_{xy}, \\ \nabla^2\nabla^2 v = \dfrac{\partial}{\partial y}\left[\nabla^2(\varepsilon^0_{xx}\nu+\varepsilon^0_{yy})+(1+\nu)\dfrac{\partial^2}{\partial x^2}(\varepsilon^0_{yy}-\varepsilon^0_{xx})\right]+ \\ \qquad\qquad\qquad\qquad + \dfrac{\partial}{\partial x}\left[\nabla^2-(1+\nu)\dfrac{\partial^2}{\partial y^2}\right]\varepsilon^0_{xy}. \end{cases}$$

For a given state of strain (ε^0_{ij}) the right hand members of the Eqs. (1.5) is known. From the solution of non-homogeneous biharmonic equations for assumed boundary conditions, we obtain the functions u and v, and from the Eqs. (1.1), (1.2), the strains and the stresses.

Note that in the particular case, $\varepsilon^0_{xx}=\varepsilon^0_{yy}=\varepsilon^0$, $\varepsilon^0_{xy}=0$, the Eqs. (1.5) can be reduced to one equation. Then, introducing the function Φ characterized by the relations

$$(1.6)\qquad u=\frac{\partial\Phi}{\partial x},\qquad v=\frac{\partial\Phi}{\partial y},$$

we reduce the system of equations (1.5) to

$$(1.7)\qquad \nabla^2\nabla^2\Phi=(1+\nu)\nabla^2\varepsilon^0.$$

The solution of the problem of state of stress (σ_{ij}) due to a given strain field (ε^0_{ij}) over the region Γ, will be obtained in a most simple manner using the Green function. Consider the case in which the field ε^0_{xx} alone is given over the region Γ. Assume first that ε^0_{xx} exists only over an inifinitely small region $d\Gamma$ surrounding the point $P(\xi,\eta)$ and is equal to zero over the remaining part of Γ. The quantity $(\varepsilon^0_{xx}\,d\Gamma)$ is called the nucleus of elastic strain. This nucleus provokes in the plate displacements the components of which are $u^*(x,y;\xi,\eta)$ and $v^*(x,y;\xi,\eta)$, and the state of stress the components of which are $\sigma^*(x,y;\xi,\eta)$. These functions are the Green functions of our problem. If, now, a state of initial strain $\varepsilon^0_{xx}(\xi,\eta)$ exists in the region Γ of the plate, and if $\varepsilon^0_{xx}=0$ outside that region, the displacement and stress components may be expressed by the integral expressions

$$(1.8.1)\qquad \begin{cases} u(x,y)=\displaystyle\iint_{(\Gamma)}\varepsilon^0_{xx}(\xi,\eta)\,u^*(x,y;\xi,\eta)\,d\xi\,d\eta, \\ v(x,y)=\displaystyle\iint_{(\Gamma)}\varepsilon^0_{xx}(\xi,\eta)\,v^*(x,y;\xi,\eta)\,d\xi\,d\eta, \end{cases}$$

$$(1.8.2)\qquad \sigma_{ij}(x,y)=\iint_{(\Gamma)}\varepsilon^0_{xx}(\xi,\eta)\,\sigma^*_{ij}(x,y;\xi,\eta)\,d\xi\,d\eta.$$

The functions u^* and v^* will be determined from the equations

$$(1.9)\qquad \begin{cases} \nabla^2\nabla^2 u^* = (\varepsilon^0_{xx}\, d\Gamma)\dfrac{\partial}{\partial x}\left[\nabla^2 + (1+\nu)\dfrac{\partial^2}{\partial y^2}\right]\delta(x-\xi)\,\delta(y-\eta), \\ \nabla^2\nabla^2 v^* = (\varepsilon^0_{yy}\, d\Gamma)\dfrac{\partial}{\partial y}\left[\nu\nabla^2 - (1+\nu)\dfrac{\partial^2}{\partial x^2}\right]\delta(x-\xi)\,\delta(y-\eta), \end{cases}$$

where δ denotes a Dirac function.

If a nucleus of elastic strain $(\varepsilon^0_{yy}\, d\Gamma)$ and $(\varepsilon^0_{xy}\, d\Gamma)$ exists in the plate the Green functions are constructed in an analogous way. Thus the Green functions for the nucleus $(\varepsilon^0_{yy}\, d\Gamma)$ will be obtained by solving the system of equations

$$(1.10)\qquad \begin{cases} \nabla^2\nabla^2 u^{**} = (\varepsilon^0_{yy}\, d\Gamma)\dfrac{\partial}{\partial x}\left[\nu\nabla^2 - (1+\nu)\dfrac{\partial^2}{\partial y^2}\right]\delta(x-\xi)\,\delta(y-\eta), \\ \nabla^2\nabla^2 v^{**} = (\varepsilon^0_{yy}\, d\Gamma)\dfrac{\partial}{\partial y}\left[\nabla^2 + (1+\nu)\dfrac{\partial^2}{\partial y^2}\right]\delta(x-\xi)\,\delta(y-\eta). \end{cases}$$

If u^{***} and v^{***} denote the Green functions for the displacement components in the case of the nucleus $(\varepsilon^0_{xy}\, d\Gamma)$, the corresponding system of differential equations will take the form

$$(1.11)\qquad \begin{cases} \nabla^2\nabla^2 u^{***} = (\varepsilon^0_{xy}\, d\Gamma)\dfrac{\partial}{\partial y}\left[\nabla^2 - (1+\nu)\dfrac{\partial^2}{\partial x^2}\right]\delta(x-\xi)\,\delta(y-\eta), \\ \nabla^2\nabla^2 v^{***} = (\varepsilon^0_{xy}\, d\Gamma)\dfrac{\partial}{\partial x}\left[\nabla^2 - (1+\nu)\dfrac{\partial^2}{\partial x^2}\right]\delta(x-\xi)\,\delta(y-\eta). \end{cases}$$

Finally, for the state of initial strain (ε^0_{ij}), in the region Γ, we obtain by superposition

$$(1.12)\qquad \begin{cases} u(x,y) = \displaystyle\iint_{(\Gamma)} [\varepsilon^0_{xx}(\xi,\eta)\, u^*(x,y;\xi,\eta) + \varepsilon^0_{yy}(\xi,\eta)\, u^{**}(x,y;\xi,\eta) + \\ \qquad\qquad + \varepsilon^0_{xy}(\xi,\eta)\, u^{***}(x,y;\xi,\eta)]\, d\xi\, d\eta, \\ v(x,y) = \displaystyle\iint_{(\Gamma)} [\varepsilon^0_{xx}(\xi,\eta)\, v^*(x,y;\xi,\eta) + \varepsilon^0_{yy}(\xi,\eta)\, v^{**}(x,y;\xi,\eta) + \\ \qquad\qquad + \varepsilon^0_{xy}(\xi,\eta)\, v^{***}(x,y;\xi,\eta)]\, d\xi\, d\eta, \\ \sigma_{ij}(x,y) = \displaystyle\iint_{(\Gamma)} [\varepsilon^0_{xx}(\xi,\eta)\, \sigma^*_{ij}(x,y;\xi,\eta) + \varepsilon^0_{yy}(\xi,\eta)\, \sigma^{**}_{ij}(x,y;\xi,\eta) + \\ \qquad\qquad + \varepsilon^0_{xy}(\xi,\eta)\, \sigma^{***}_{ij}(x,y;\xi,\eta)]\, d\xi\, d\eta, \\ \qquad i,\ j = x,\ y, \qquad \sigma_{ij} = \sigma_{ji}. \end{cases}$$

Further considerations will chiefly concern the determination of the Green functions. The knowledge of these functions will enable us to determine, by means of integration, the state of stress in the plate due to the action of the initial strains (ε^0_{ij}).

In many particular cases, the problem considered in this paper can be solved in a much simpler manner by introducing the A i r y function F, characterized by the relations

$$\sigma_{xx}=\frac{\partial^2 F}{\partial y^2}, \qquad \sigma_{yy}=\frac{\partial^2 F}{\partial x^2}, \qquad \sigma_{xy}=-\frac{\partial^2 F}{\partial x \partial y}. \tag{1.13}$$

It satisfies the equilibrium equations (1.3). Using that function to determine the deformations (ε_{ij}) appearing in the Eqs. (1.1) we have:

$$\begin{cases} \varepsilon_{xx}-\varepsilon^0_{xx}=\dfrac{1}{E}\left(\dfrac{\partial^2 F}{\partial y^2}-\nu\dfrac{\partial^2 F}{\partial x^2}\right), \\ \varepsilon_{yy}-\varepsilon^0_{yy}=\dfrac{1}{E}\left(\dfrac{\partial^2 F}{\partial x^2}-\nu\dfrac{\partial^2 F}{\partial y^2}\right), \\ \varepsilon_{xy}-\varepsilon^0_{xy}=-\dfrac{1}{G}\dfrac{\partial^2 F}{\partial x \partial y}. \end{cases} \tag{1.14}$$

Introducing the above relations in the compatibility equation

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2}+\frac{\partial^2 \varepsilon_{yy}}{\partial x^2}=\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \tag{1.15}$$

we have the following non-homogeneous biharmonic equation for the function F:

$$\nabla^2 \nabla^2 F=-2\,G\,(1+\nu)\left(\frac{\partial^2 \varepsilon^0_{xx}}{\partial y^2}+\frac{\partial^2 \varepsilon^0_{yy}}{\partial x^2}-\frac{\partial^2 \varepsilon^0_{xy}}{\partial x \partial y}\right). \tag{1.16}$$

The convenience of using the A i r y stress function F consists above all in that one equation must be solved instead of two.

Denoting by F^* the G r e e n function for the nucleus $(\varepsilon^0_{xx}\,d\Gamma)$ acting at the point (ξ,η), and by F^{**} and F^{***} for the nuclei $(\varepsilon^0_{yy}\,d\Gamma)$ and $(\varepsilon^0_{xy}\,d\Gamma)$, we obtain for the strain field over the region Γ of the plate, the relation

$$F(x,y)=\int\int_{\Gamma}(\varepsilon^0_{xx}F^*+\varepsilon^0_{yy}F^{**}+\varepsilon^0_{xy}F^{***})\,d\xi\,d\eta. \tag{1.17}$$

The G r e e n functions for stresses will be obtained by means of differentiation according to the Eqs. (1.3). The considerations presented here will be completed with certain particular solutions for plates of most simple types: an infinite plate, a semi-inifinite plate and a plate strip.

2. An Infinite Plate

Let us determine first the functions u^*, v^* and F^* for an infinite plate with a nucleus of elastic strain $(\varepsilon^0_{**}\,d\Gamma)$ acting over an infinitely small neighbourhood of the point $P(\xi,\eta)$. The system of equations (1.9) and the Eq. (1.16) will be solved assuming that displacements and stress should vanish at infinity.

Moreover, the function u^* should be symmetric with respect to the straight line $y=\eta$ and antisymmetric with respect to the straight line $x=\xi$. On the other hand, the function v^* should be antisymmetric with respect to the axis $y=\eta$, and symmetric with respect to the axis $x=\xi$. The function F^* should be symmetric with respect to the lines $x=\xi$ and $y=\eta$. This follows from the symmetry of normal stresses with respect to these lines. The above conditions will be satisfied assuming that,

$$(2.1)\qquad \begin{cases} u^* = \int\limits_0^\infty \int\limits_0^\infty A(\alpha,\beta) \sin\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta, \\ v^* = \int\limits_0^\infty \int\limits_0^\infty B(\alpha,\beta) \cos\alpha(x-\xi)\sin\beta(y-\eta)\, d\alpha\, d\beta, \\ F^* = \int\limits_0^\infty \int\limits_0^\infty C(\alpha,\beta) \cos\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta. \end{cases}$$

Expressing the Dirac function by means of a Fourier integral, we obtain the solution of the Eqs. (1.9) in the form

$$(2.2)\qquad \begin{cases} u^* = \dfrac{(\varepsilon^0_{xx}\, d\Gamma)}{\pi^2} \int\limits_0^\infty \int\limits_0^\infty \alpha \left[\dfrac{1}{\alpha^2+\beta^2} + (1-\nu)\dfrac{\beta^2}{(\alpha^2+\beta^2)^2}\right] \sin\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta = \\ \qquad = \dfrac{(\varepsilon^0_{xx}\, d\Gamma)}{4\pi}\, \dfrac{(x-\xi)}{r^2}\left[2+(1+\nu)\dfrac{r^2-2(y-\eta)^2}{r^2}\right], \\ v^* = \dfrac{(\varepsilon^0_{xx}\, d\Gamma)}{\pi^2} \int\limits_0^\infty \int\limits_0^\infty \beta \left[\dfrac{\nu}{\alpha^2+\beta^2} - (1+\nu)\dfrac{\alpha^2}{(\alpha^2+\beta^2)^2}\right] \cos\alpha(x-\xi)\sin\beta(y-\eta)\, d\alpha\, d\beta = \\ \qquad = \dfrac{(\varepsilon^0_{xx}\, d\Gamma)}{4\pi}\, \dfrac{(y-\eta)}{r^2}\left[2\nu-(1+\nu)\dfrac{r^2-2(x-\xi)^2}{r^2}\right], \\ r = \sqrt{(x-\xi)^2+(y-\eta)^2}. \end{cases}$$

Integrating the Eq. (1.16), we obtain

$$(2.3)\qquad F^* = 2G(1+\nu)(\varepsilon^0_{xx}\, d\Gamma) \int\limits_0^\infty \int\limits_0^\infty \frac{\beta^2}{(\alpha^2+\beta^2)^2} \cos\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta.$$

From the Eqs. (1.13), we obtain

$$
(2.4)\quad
\begin{cases}
\sigma^{*}_{xx} = -2G(1+\nu)(\varepsilon^{0}_{xx}\,d\Gamma)\int\limits_0^\infty\int\limits_0^\infty \dfrac{\beta^4}{(\alpha^2+\beta^2)^2}\cos\alpha(x-\xi)\cos\beta(y-\\
\qquad -\eta)\,d\alpha\,d\beta = \dfrac{K_1}{r^2}\left\{1-4\dfrac{(x-\xi)^2}{r^2}\left[1-\dfrac{2(y-\eta)^2}{r^2}\right]\right\},\\
\sigma^{*}_{yy} = -\dfrac{2G(1+\nu)(\varepsilon^{0}_{xx}\,d\Gamma)}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \dfrac{\alpha^2\beta^2}{(\alpha^2+\beta^2)^2}\cos\alpha(x-\xi)\cos\beta(y-\\
\qquad -\eta)\,d\alpha\,d\beta = \dfrac{K_1}{r^2}\left[1-\dfrac{8(x-\xi)^2(y-\eta)^2}{r^4}\right],\\
\sigma^{*}_{xy} = -\dfrac{2G(1+\nu)(\varepsilon^{0}_{xx}\,d\Gamma)}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \dfrac{\alpha\beta^3}{(\alpha^2+\beta^2)^2}\sin\alpha(x-\xi)\sin\beta(y-\\
\qquad -\eta)\,d\alpha\,d\beta = \dfrac{2K_1}{r^4}(x-\xi)(y-\eta)\left[1-\dfrac{4(x-\xi)^2}{r^2}\right],
\end{cases}
$$

where

$$K_1 = \frac{G(1+\nu)(\varepsilon^{0}_{xx}\,d\Gamma)}{2\pi}.$$

Knowledge of the G r e e n function σ^{*}_{ij} enables us to determine the state of stress σ_{ij} in a case in which the state of initial strain $\varepsilon^{0}_{xx}(\xi,\eta)$ is given over the region Γ. The stresses σ_{ij} will be obtained by integration according to the Eq. (1.8.2).

In the case of the nucleus of elastic strain $(\varepsilon^{0}_{yy}\,d\Gamma)$ at the point $P(\xi,\eta)$, the functions σ^{*}_{ij} will be obtained in an analogous manner. We have

$$
(2.5)\quad
\begin{cases}
\sigma^{**}_{xx} = \dfrac{K_2}{r^2}\left[1-\dfrac{8(x-\xi)^2(y-\eta)^2}{r^4}\right],\\
\sigma^{**}_{yy} = -\dfrac{K_2}{r^4}\left[3(y-\eta)^2-(x-\xi)^2-\dfrac{8(x-\xi)^2(y-\eta)^2}{r^2}\right] =\\
\qquad = \dfrac{K_2}{r^2}\left\{1-4\dfrac{(y-\eta)^2}{r^2}\left[1-\dfrac{2(x-\xi)^2}{r^2}\right]\right\},\\
\sigma^{**}_{xy} = -\dfrac{2K_2}{r^6}(x-\xi)(y-\eta)\left[3(y-\eta)^2-(x-\xi)^2\right] =\\
\qquad = 2K_1\dfrac{(x-\xi)(y-\eta)}{r^4}\left[1-4\dfrac{(y-\eta)^2}{r^2}\right],
\end{cases}
$$

where

$$K_2 = \frac{G(1+\nu)(\varepsilon^0_{yy}\, d\Gamma)}{2\pi}.$$

Let a nucleus of elastic strain $(\varepsilon^0_{xy}\, d\Gamma)$ act at the point $P(\xi, \eta)$ of an infinite plate.

The function u^{***} should be symmetric with respect to the line $x=\xi$ and antisymmetric with respect to the line $y=\eta$. On the other hand, the function v^{***} should be antisymmetric with respect to the line $x=\xi$ and symmetric with respect to the line $y=\eta$. Moreover, the displacement components should be zero at infinity. The function F^* should be antisymmetric with respect to both the axis $x=\xi$ and $y=\eta$. This follows from the fact that the stress σ^{***}_{xy} is symmetric with respect to the axis $x=\xi$ and $y=\eta$. Let us assume therefore that

$$\text{(2.6)} \qquad \begin{cases} u^{***} = \int\limits_0^\infty\int\limits_0^\infty A(\alpha,\beta)\cos\alpha(x-\xi)\sin\beta(y-\eta)\, d\alpha\, d\beta, \\ v^{***} = \int\limits_0^\infty\int\limits_0^\infty B(\alpha,\beta)\sin\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta, \\ F^{***} = \int\limits_0^\infty\int\limits_0^\infty C(\alpha,\beta)\sin\alpha(x-\xi)\sin\beta(y-\eta)\, d\alpha\, d\beta. \end{cases}$$

Expressing also the Dirac function by a double even Fourier integral, we have

$$\text{(2.7)} \qquad \begin{cases} u^{***} = \dfrac{(\varepsilon^0_{xy}\, d\Gamma)}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \beta\left[\dfrac{1}{\alpha^2+\beta^2} - (1+\nu)\dfrac{\alpha^2}{(\alpha^2+\beta^2)^2}\right]\cos\alpha(x-\xi)\times \\ \qquad \times\sin\beta(y-\eta)\, d\alpha\, d\beta = \dfrac{(\varepsilon^0_{xy}\, d\Gamma)}{4\pi}\dfrac{y-\eta}{r^2}\left[(1-\nu)+2(1+\nu)\dfrac{(x-\xi)^2}{r^2}\right], \\ v^{***} = \dfrac{(\varepsilon^0_{xy}\, d\Gamma)}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \alpha\left[\dfrac{1}{\alpha^2+\beta^2} - (1+\nu)\dfrac{\beta^2}{(\alpha^2+\beta^2)^2}\right]\sin\alpha(x-\xi)\times \\ \qquad \times\cos\beta(y-\eta)\, d\alpha\, d\beta = \dfrac{(\varepsilon^0_{xy}\, d\Gamma)}{4\pi}\dfrac{x-\xi}{r^2}\left[(1-\nu)+2(1+\nu)\dfrac{(y-\eta)^2}{r^2}\right], \\ F^{***} = \dfrac{2G(1+\nu)(\varepsilon^0_{xy}\, d\Gamma)}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \dfrac{\alpha\beta}{(\alpha^2+\beta^2)^2}\sin\alpha(x-\xi)\times \\ \qquad \times\sin\beta(y-\eta)\, d\alpha\, d\beta = \dfrac{G(1+\nu)(\varepsilon^0_{xy}\, d\Gamma)}{\pi}\,\dfrac{(x-\xi)(y-\eta)}{r^2}. \end{cases}$$

The Green functions for the stress components σ_{ij}^{**} take, according to the Eqs. (1.13), the form

$$
(2.8)\quad \begin{cases}
\sigma_{xx}^{***} = -K_3 \dfrac{(x-\xi)(y-\eta)}{r^4}\left[3 - \dfrac{4(y-\eta)^2}{r^2}\right] = \\
\qquad\qquad = K_3 \dfrac{(x-\xi)(y-\eta)}{r^4}\left[1 - 4\dfrac{(x-\xi)^2}{r^2}\right], \\
\sigma_{yy}^{***} = -K_3 \dfrac{(x-\xi)(y-\eta)}{r^4}\left[3 - \dfrac{4(x-\xi)^2}{r^2}\right] = \\
\qquad\qquad = K_3 \dfrac{(x-\xi)(y-\eta)}{r^4}\left[1 - 4\dfrac{(y-\eta)^2}{r^2}\right], \\
\sigma_{xy}^{***} = -\dfrac{K_3}{2r^2}\left[1 - \dfrac{8(x-\xi)^2(y-\eta)^2}{r^4}\right],
\end{cases}
$$

where

$$K_3 = \frac{G(1+\nu)}{\pi}\,(\varepsilon_{xy}^0\, d\Gamma).$$

3. A Semi-Infinite Plate

Let a nucleus of elastic strain $(\varepsilon_{xx}^0\, d\Gamma)$ act at the point $P(\xi, 0)$ of a semi-infinite plate. The function F^* should satisfy the Eq. (1.16) with the boundary conditions,

$$(3.1)\qquad \sigma_{xx}^* = 0, \qquad \sigma_{xy}^* = 0 \qquad \text{for} \qquad x = 0.$$

Moreover, all stress components σ_{ij}^* should vanish at infinity.

The function F^* will comprise two parts Φ^* and Ψ^*. The first of these functions should satisfy the equation

$$(3.2)\qquad \nabla^2\nabla^2\Phi^* = -2G(1+\nu)(\varepsilon_{xx}^0\, d\Gamma)\frac{\partial^2}{\partial y^2}\delta(x-\xi)\,\delta(y)$$

with the boundary conditions $\Phi^* = 0$ and $\nabla^2\Phi^* = 0$ at the edge $x = 0$.

These conditions may also be expressed in the form

$$(3.3)\qquad \bar{\sigma}_{xx}^* = \frac{\partial^2\Phi^*}{\partial y^2} = 0, \qquad \bar{\sigma}_{yy}^* = \frac{\partial^2\Phi^*}{\partial x^2} = 0$$

on the line $x = 0$.

The function Ψ^* should satisfy the homogeneous biharmonic equation

$$(3.4)\qquad \nabla^2\nabla^2\Psi^* = 0$$

with the boundary conditions

$$(3.5)\qquad \sigma_{xx}^* = \frac{\partial^2\Psi^*}{\partial y^2} = 0, \qquad \bar{\sigma}_{xy} - \frac{\partial^2\Psi^*}{\partial x\,\partial y} = 0 \qquad \text{for} \qquad x = 0,$$

where

$$\bar{\sigma}_{xy} = -\frac{\partial^2 \Phi^*}{\partial x \partial y}.$$

The final form of the Green function is obtained by superposition, thus,

$$(3.6) \qquad \begin{cases} F^* = \Phi^* + \Psi^*, & \sigma^*_{xx} = \bar{\sigma}^*_{xx} + \bar{\bar{\sigma}}^*_{xx}, \\ \sigma^*_{yy} = \bar{\sigma}^*_{yy} + \bar{\bar{\sigma}}^*_{yy}, & \sigma^*_{xy} = \bar{\sigma}^*_{xy} + \bar{\bar{\sigma}}^{**}_{xy}. \end{cases}$$

The function Φ^* is obtained as a result of the action of two nuclei of elastic strain of which one, positive, is located at the point $P(\xi, 0)$ and the other, negative, at the point $P'(-\xi, 0)$ of the infinite plate.

Using the Eqs. (2.4) we obtain

$$(3.7) \quad \begin{cases} \bar{\sigma}^*_{xx} = \dfrac{\partial^2 \Phi^*}{\partial y^2} = K_1 \left\{ \dfrac{1}{r_1^2} \left[1 - 4 \dfrac{(x-\xi)^2}{r_1^2} \left(1 - \dfrac{2y^2}{r_1^2} \right) \right] - \right. \\ \qquad\qquad \left. - \dfrac{1}{r_2^2} \left[1 - 4 \dfrac{(x+\xi)^2}{r_2^2} \left(1 - \dfrac{2y^2}{r_2^2} \right) \right] \right\}, \\ \bar{\sigma}^*_{yy} = \dfrac{\partial^2 \Phi^*}{\partial x^2} = K_1 \left\{ \dfrac{1}{r_1^2} \left[1 - \dfrac{8(x-\xi)^2 y^2}{r_1^4} \right] - \dfrac{1}{r_2^2} \left[1 - \dfrac{8(x+\xi)^2 y^2}{r_2^4} \right] \right\}, \\ \bar{\sigma}^*_{xy} = -\dfrac{\partial^2 \Phi^*}{\partial x \partial y} = 2K_1 y \left\{ \dfrac{(x-\xi)}{r_1^4} \left[1 - 4 \dfrac{(x-\xi)^2}{r_1^2} \right] - \right. \\ \qquad\qquad \left. - \dfrac{(x+\xi)}{r_2^4} \left[1 - 4 \dfrac{(x+\xi)^2}{r_2^2} \right] \right\}, \\ r_{1,2} = \sqrt{(x \mp \xi)^2 + y^2}. \end{cases}$$

It is seen that the stresses $\bar{\sigma}^*_{xx}$ and $\bar{\sigma}^*_{yy}$ vanish at the edge $x = 0$. The stresses $\bar{\sigma}^*_{xy}$ remain different from zero.

It will be convenient for the subsequent considerations to represent the stresses $[\bar{\sigma}^*_{xy}]_{x=0}$ by means of a Fourier integral

$$(3.8) \qquad [\bar{\sigma}^*_{xy}]_{x=0} = -2K_1 \xi \int_0^\infty e^{-\beta\xi} \sin \beta y \, d\beta.$$

The function Ψ^* is assumed in the form

$$\Psi^* = x \int_0^\infty B(\beta) e^{-\beta x} \cos \beta y \, d\beta.$$

This function satisfies the conditions at infinity and the first of the conditions (3.5).

From the second of the conditions (3.5) we have

$$(3.9) \qquad B(\beta) = 2K_1 \xi e^{-\beta\xi} \beta.$$

Hence,

$$\Psi^* = 2\,K_1\,x\,\xi \int_0^\infty \beta\, e^{-\beta(x+\xi)} \cos\,\beta y\, d\beta = 2\,K_1\,x\,\xi\,\frac{(x+\xi)^2 - y^2}{r_2^4}\,.$$

Then, we obtain successively

$$(3.10)\qquad \begin{cases} \bar{\sigma}^*_{xx} = \dfrac{\partial^2 \Psi^*}{\partial y^2} = -\,12\,K_1\,\dfrac{x\xi}{r_2^4}\left[1 - \dfrac{8\,(x+\xi)^2\,y^2}{r_2^4}\right], \\ \bar{\sigma}^*_{yy} = \dfrac{\partial^2 \Psi^*}{\partial x^2} = -\,4\,K_1\,\dfrac{\xi}{r_2^4}\left\{2\,(x+\xi)\left(1 - \dfrac{4\,y^2}{r_2^2}\right) - \right. \\ \qquad\qquad\qquad\qquad\qquad \left. -\,3\,x\left[1 - \dfrac{8\,(x+\xi)^2\,y^2}{r_2^4}\right]\right\}, \\ \bar{\sigma}^*_{xy} = -\dfrac{\partial^2 \Psi^*}{\partial x \partial y} = -\,4\,K_1\,\dfrac{y\xi}{r_2^4}\left\{1 - \dfrac{4\,(x+\xi)^2}{r_2^2} - \right. \\ \qquad\qquad\qquad\qquad\qquad \left. -\,\dfrac{12\,x\,(x+\xi)}{r_2^2}\left[1 - \dfrac{2\,(x+\xi)^2}{r_2^2}\right]\right\}. \end{cases}$$

On the basis of the Eqs. (3.6), we obtain the Green functions σ^*_{ij} for the case of a nucleus of strain $(\varepsilon^0_{xx}\,d\Gamma)$ at the point $P(\xi,\,0)$ of a semi-infinite plate.

The case of a nucleus of elastic strain $(\varepsilon^0_{yy}\,d\,I)$ at the point $P(\xi,\,0)$ of a semi-infinite plate offers nothing essentially new in our considerations.

Consider, finally, the case of a nucleus of elastic strain $(\varepsilon^0_{xy}\,d\Gamma)$ acting at the point $P(\xi, 0)$ of the semi-infinite plate. The function F^{***} will be assumed as before in the form

$$(3.11)\qquad F^{***} = \Phi^{***} + \Psi^{***},$$

in which the function Φ^{***} satisfies the equation

$$(3.12)\qquad \nabla^2\,\nabla^2\Phi^{***} = 2\,G\,(1+\nu)\,(\varepsilon^0_{xy}\,d\Gamma)\,\frac{\partial^2}{\partial x\,\partial y}\,\delta\,(x-\xi)\,\delta\,(y)$$

with the conditions

$$(3.13)\qquad \frac{\partial\,\Phi^{***}}{\partial\,x} = 0, \qquad \frac{\partial^2\,\Phi^{***}}{\partial x\,\partial y} = 0$$

at the edge $x = 0$.

These conditions are identical with

$$(3.14)\qquad \bar{u}^{***} = 0, \qquad \bar{\sigma}^{***}_{xy} = 0$$

for $x = 0$.

The function Ψ^{***} should satisfy the equation

$$\nabla^2 \nabla^2 \Psi^{***} = 0 \tag{3.15}$$

with the boundary conditions

$$\bar{\sigma}_{xy}^{***} = -\frac{\partial^2 \Psi^{***}}{\partial x \, \partial y}, \qquad \bar{\sigma}_{xx}^{***} + \frac{\partial^2 \Psi^{***}}{\partial y^2} = 0 \tag{3.16}$$

for $x = 0$.

The Eq. (3.11), together with the conditions (3.13), will be satisfied if a positive nucleus is located at the point $P(\xi, 0)$ of the infinite plate, and a negative nucleus at the point $P'(-\xi, 0)$ of that plate.

Using the Eqs. (2.8) we have

$$\tag{3.17} \left\{ \begin{aligned} \bar{\sigma}_{xx}^{***} &= + K_3 y \left\{ \frac{(x-\xi)}{r_1^4} \left[1 - \frac{4(x-\xi)^2}{r_1^2} \right] - \right. \\ &\qquad \left. - \frac{(x+\xi)}{r_2^4} \left[1 - \frac{4(x+\xi)^2}{r_2^2} \right] \right\} = \frac{\partial^2 \Phi^{***}}{\partial y^2}, \\ \bar{\sigma}_{yy}^{***} &= + K_3 y \left\{ \frac{(x-\xi)}{r_1^4} \left[1 - \frac{4y^2}{r_1^2} \right] - \frac{(x+\xi)}{r_2^4} \left[1 - \frac{4y^2}{r_2^2} \right] \right\} = \frac{\partial^2 \Phi^{***}}{\partial x^2}, \\ \bar{\sigma}_{xy}^{***} &= - \frac{K_3}{2} \left\{ \frac{1}{r_1^2} \left[1 - \frac{8(x-\xi)^2 y^2}{r_1^4} \right] - \right. \\ &\qquad \left. - \frac{1}{r_2^2} \left[1 - \frac{8(x+\xi)^2 y^2}{r_2^4} \right] \right\} = - \frac{\partial^2 \Phi^{***}}{\partial x \, \partial y}. \end{aligned} \right.$$

The function

$$\Psi^{***} = \int_0^\infty \frac{1}{\beta^2} (A + B \beta x) e^{-\beta x} \sin \beta y \, d\beta \tag{3.18}$$

satisfies the Eq. (3.14) and the conditions at infinity. Since $[\bar{\sigma}_{xx}^{***}]_{x=0}$ can be represented in the form

$$[\bar{\sigma}_{xx}^{***}]_{x=0} = K_3 \xi \int_0^\infty \beta^2 e^{-\beta \xi} \sin \beta y \, d\beta. \tag{3.19}$$

It follows from the boundary conditions (3.15) that

$$A = B, \qquad A = K_3 \xi \beta^2 e^{-\beta \xi}. \tag{3.20}$$

Thus we have

$$\Psi^{***} = K_3 \xi \int_0^\infty e^{-\beta (x+\xi)} (1 + \beta x) \sin \beta y \, d\beta = K_3 \frac{\xi y}{r_2^2} \left[1 + \frac{2x(x+\xi)}{r_2^2} \right]. \tag{3.21}$$

We obtain successively

$$(3.22)\quad \begin{cases} \bar{\bar{\sigma}}_{xx}^{***} = \dfrac{\partial^2 \Psi^{***}}{\partial y^2} = 2\,K_3\,\dfrac{\xi y}{r_2^4}\left\{1 - 4\,\dfrac{(x+\xi)^2}{r_2^2} + \dfrac{12\,x\,(x+\xi)}{r_2^2}\left[1 - \dfrac{2\,(x+\xi)^2}{r_2^2}\right]\right\}, \\ \bar{\bar{\sigma}}_{yy}^{***} = \dfrac{\partial^2 \Psi^{***}}{\partial x^2} = 2\,K_3\,\dfrac{\xi y}{r_2^4}\left\{1 - 4\,\dfrac{(x+\xi)^2}{r_2^2} - \dfrac{12\,x(x+\xi)}{r_2^2}\left[1 - \dfrac{2\,(x+\xi)^2}{r_2^2}\right]\right\}, \\ \bar{\bar{\sigma}}_{xy}^{***} = -\dfrac{\partial^2 \Psi^{***}}{\partial x\,\partial y} = -\,6\,K_3\,\dfrac{x\,\xi}{r_2^4}\left[1 - 8\,\dfrac{(x+\xi)^2}{r_2^4}\right]. \end{cases}$$

Adding the stresses of the group (3.16) and (3.21), we obtain the G r e e n function σ_{ij}^{***} of our problem.

4. A Plate Strip

Consider a nucleus of elastic strain $(\varepsilon_{xx}^0\,d\Gamma)$, acting at the point $P(\xi,\,0)$. We assume that the edges $x=0$, $x=a$ are free from stresses, that is $\sigma_{xx}=0$, $\sigma_{xy}=0$ at those edges.

In order to determine the G r e e n function for stresses σ_{ij}^{*}, we shall determine first the function F*, which should satisfy the equation

$$(4.1)\qquad \nabla^2\nabla^2 F^* = -\,2\,G\,(1+\nu)\,(\varepsilon_{xx}^0\,d\Gamma)\,\frac{\partial^2}{\partial y^2}\,[\delta\,(x-\xi)\,\delta\,(y)]$$

with the boundary conditions

$$(4.2)\qquad \frac{\partial^2 F^*}{\partial y^2} = 0,\qquad \frac{\partial^2 F^*}{\partial x\,\partial y} = 0$$

at the edges $x=0$ and $x=a$.

We assume that

$$(4.3)\qquad F^* = \Phi^* + \Psi^*,$$

where the function Φ^* should satisfy the equation

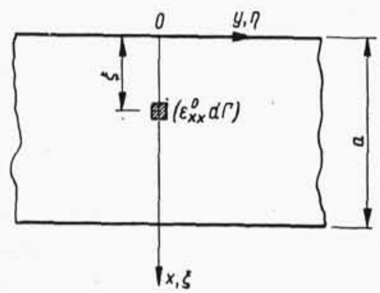

Fig. 2

$$(4.4)\qquad \nabla^2\nabla^2\,\Phi^* = -\,2\,G\,(1+\nu)(\varepsilon_{xx}^0\,d\Gamma)\,\frac{\partial^2}{\partial y^2}\left[\delta\,(x-\xi)\,\delta(y)\right]$$

with the conditions

$$(4.5)\qquad \bar{\sigma}_{xx}^* = \frac{\partial^2 \Phi^*}{\partial y^2} = 0,\qquad \Phi^* = 0$$

for $x=0$ and $x=a$.

The function Ψ^* should, on the other hand, satisfy the biharmonic equation

$$(4.6)\qquad \nabla^2\nabla^2\,\Psi^* = 0$$

with the conditions

$$\overline{\overline{\sigma}}^*_{xx} = \frac{\partial^2 \Psi^*}{\partial y^2} = 0, \qquad -\frac{\partial^2 \Phi^*}{\partial x \partial y} - \frac{\partial^2 \Psi^*}{\partial x \partial y} = 0 \tag{4.7}$$

for $x = 0$ and $x = a$.

The function Φ^* will be assumed in the form[1]

$$\Phi^* = \int_0^\infty \sum_{n=1}^\infty A_n(\beta) \sin \alpha_n x \cos \beta y \, d\beta. \tag{4.8}$$

Expressing the Dirac function by means of a Fourier series and integral, we have the following solution of the Eq. (4.4):

$$\left\{ \begin{aligned} \Phi^* &= \frac{4K}{a} \int_0^\infty \sum_{n=1}^\infty \frac{\beta^2}{(\alpha_n^2 + \beta^2)^2} \sin \alpha_n \xi \sin \alpha_n x \cos \beta y \, d\beta, \\ K &= \frac{G(1+\nu)(\varepsilon^0_{xx} \, d\Gamma)}{\pi}. \end{aligned} \right. \tag{4.9}$$

This function can be expressed in the closed form

$$\Phi^* = -\frac{K}{4}\left(\psi + y \frac{\partial \psi}{\partial y}\right), \tag{4.10}$$

where

$$\psi = \ln \frac{\operatorname{ch} \frac{\pi y}{a} - \cos \frac{\pi}{a}(x - \xi)}{\operatorname{ch} \frac{\pi y}{a} - \cos \frac{\pi}{a}(x + \xi)}.$$

It can easily be verified that the function Φ^* satisfies the conditions (4.5) and those for $y \to \infty$. For $x \to \xi$ the function Φ^* increases indefinitely showing a singularity of the logarithmic type. The knowledge of Φ^* enables us to determine the stresses $\overline{\sigma}^*_{ij}$ in the closed form

$$\left\{ \begin{aligned} \overline{\sigma}^*_{xx} &= \frac{\partial^2 \Phi^*}{\partial y^2} = -\frac{K}{4}\left(3 \frac{\partial^2 \psi}{\partial y^2} + y \frac{\partial^3 \psi}{\partial y^3}\right), \\ \overline{\sigma}^*_{yy} &= \frac{\partial^2 \Phi}{\partial x^2} = -\frac{K}{4}\left(\frac{\partial^2 \psi}{\partial x^2} + y \frac{\partial^3 \psi}{\partial x^2 \partial y}\right), \\ \overline{\sigma}^*_{xy} &= -\frac{\partial^2 \Phi}{\partial x \partial y} = \frac{K}{4}\left(2 \frac{\partial^2 \psi}{\partial x \partial y} + y \frac{\partial^3 \psi}{\partial x \partial y^2}\right). \end{aligned} \right. \tag{4.11}$$

[1] It is assumed that alternating positive and negative nuclei are periodically distributed along the x-axis (see also [4] and [5]).

It will be convenient for further considerations to represent the stress $\overline{\sigma}^*_{xy}$ at the edge $x=0$ and $x=a$ in the form of the Fourier integral:

$$(4.12)\qquad \begin{cases} [\overline{\sigma}^*_{xy}]_{x=0} = \left[-\dfrac{\partial^2 \Phi^*}{\partial x \partial y}\right]_{x=0} = K \displaystyle\int_0^\infty \beta \eta_1(\xi,\beta) \sin \beta y d\beta, \\ [\overline{\sigma}^*_{xy}]_{x=a} = \left[-\dfrac{\partial^2 \Phi^*}{\partial x \partial y}\right]_{x=0} = K \displaystyle\int_0^\infty \beta \eta_2(\xi,\beta) \sin \beta y d\beta, \end{cases}$$

where

$$\eta_1(\xi,\beta) = \frac{4}{a}\beta^2 \sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n \xi}{(\alpha_n^2+\beta^2)^2}, \qquad \eta_2(\xi,\beta) = \frac{4}{a}\beta^2 \sum_{n=1}^{\infty} \frac{\alpha_n (-1)^n \sin \alpha_n \xi}{(\alpha_n^2+\beta^2)^2};$$

$$(4.13)\qquad \begin{cases} \eta_1(\xi,\beta) = \dfrac{\beta\xi \,\mathrm{sh}\, \lambda \,\mathrm{ch}\, \beta(a-\xi) - \lambda \,\mathrm{sh}\, \beta\xi}{\sinh^2 \lambda}, \\ \eta_2(\xi,\beta) = \dfrac{\beta\xi \,\mathrm{sh}\, \lambda \,\mathrm{ch}\, \beta\xi - \lambda \,\mathrm{ch}\, \lambda \,\mathrm{sh}\, \beta\xi}{\sinh^2 \lambda}, \end{cases} \qquad \lambda = \beta a.$$

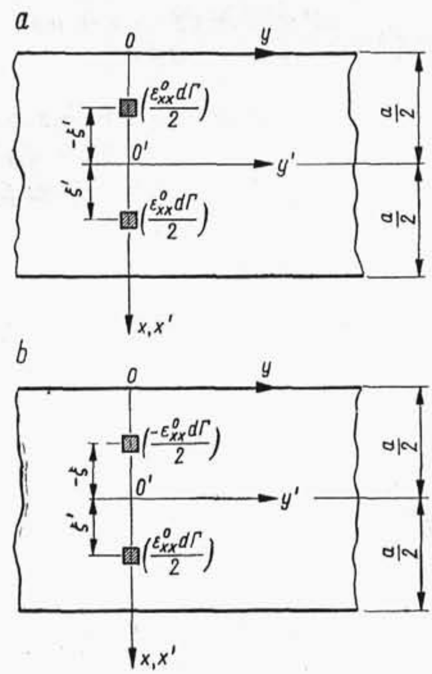

Fig. 3

The action of the nucleus of thermoelastic strain at the point $P(\xi,\eta)$ can be replaced by the action of a system of nuclei represented in Figs. 3a and 3b.

For symmetric nuclei $(\varepsilon_{xx}^{0}\, d\Gamma)/2$ (with respect to the y-axis), we have

$$[\bar{\sigma}_{x'y'}^{*(s)}]_{x'=a/2}=\frac{K}{2}\int_0^\infty \beta\left[\eta_2\left(\frac{a}{2}+\xi',\beta\right)+\eta_2\left(\frac{a}{2}-\xi',\beta\right)\right]\sin\beta y'\,d\beta,$$

or

$$[\bar{\sigma}_{x'y'}^{*(s)}]_{x'=a/2}=\frac{K}{2}\int_0^\infty \beta\,\varrho^{(s)}(\mu,\xi')\sin\beta y'\,d\beta, \tag{4.14}$$

where

$$\varrho^{(s)}(\mu,\xi')=\frac{\beta\xi'\operatorname{ch}\mu\operatorname{sh}\beta\xi'-\mu\operatorname{sh}\mu\operatorname{ch}\beta\xi'}{\operatorname{ch}^2\mu}, \qquad \mu=\frac{\beta a}{2}.$$

For antisymmetric nuclei $(\varepsilon_{xx}^{0}\, d\Gamma)/2$ we have

$$[\bar{\sigma}_{x'y'}^{*(a)}]_{x'=a/2}=\frac{K}{2}\int_0^\infty \beta\,\varrho^{(a)}(\mu,\xi')\sin\beta y'\,d\beta, \tag{4.15}$$

where

$$\varrho^{(a)}(\mu,\xi')=\frac{\beta\xi'\operatorname{sh}\mu\operatorname{ch}\beta\xi'-\mu\operatorname{ch}\mu\operatorname{sh}\beta\xi'}{\operatorname{sh}^2\mu}.$$

For symmetric nuclei, the function $\Psi^{*(s)}$ will be assumed in the form

$$\Psi^{*(s)}=\int_0^\infty \frac{1}{\beta^2}(A\operatorname{ch}\beta x'+B\beta x'\operatorname{sh}\beta x')\cos\beta y'\,d\beta. \tag{4.16}$$

From the condition (4.7), we obtain

$$B=-A\frac{\operatorname{ctg}\mu}{\mu}, \qquad A=\frac{K}{2}\;\frac{\beta\mu\operatorname{sh}\mu\,\varrho^{(s)}(\mu,\xi')}{\operatorname{sh}\mu\operatorname{ch}\mu+\mu}. \tag{4.17}$$

Knowing the function $\Psi^{*(s)}$, we calculate the stresses

$$\begin{cases} \bar{\bar{\sigma}}_{x'x'}^{*(s)}=\dfrac{\partial^2\Psi^{*(s)}}{\partial y'^2}=-\dfrac{K}{2}\displaystyle\int_0^\infty \dfrac{\beta\,\varrho^{(s)}(\mu,\xi')}{\operatorname{ch}\mu\operatorname{sh}\mu+\mu}(\mu\operatorname{sh}\mu\operatorname{ch}\beta x'- \\ \qquad\qquad -\beta x'\operatorname{ch}\mu\operatorname{sh}\beta x')\cos\beta y'\,d\beta, \\ \bar{\sigma}_{y'y'}^{*(s)}=\dfrac{\partial^2\Psi^{*(s)}}{\partial x'^2}=\dfrac{K}{2}\displaystyle\int_0^\infty \dfrac{\beta\,\varrho^{(s)}(\mu,\xi')}{\operatorname{ch}\mu\operatorname{sh}\mu+\mu}[(\mu\operatorname{sh}\mu- \\ \qquad\qquad -2\operatorname{ch}\mu)\operatorname{ch}\beta x'-\beta x'\operatorname{ch}\mu\operatorname{sh}\beta x']\cos\beta y'\,d\beta. \\ \bar{\bar{\sigma}}_{x'y'}^{*(s)}=-\dfrac{\partial^2\Psi^{*(s)}}{\partial x'\partial y'}=\dfrac{K}{2}\displaystyle\int_0^\infty \dfrac{\beta\,\varrho^{(s)}(\mu,\xi')}{\operatorname{ch}\mu\operatorname{sh}\mu+\mu}[(\mu\operatorname{sh}\mu- \\ \qquad\qquad -\operatorname{ch}\mu)\operatorname{sh}\beta x'-\beta x'\operatorname{ch}\mu\operatorname{ch}\beta x']\sin\beta y'\,d\beta. \end{cases} \tag{4.18}$$

For nuclei antisymmetric with respect to the y'-axis (Fig. 3b), the function $\Psi^{*(a)}$ will be assumed in the form

$$(4.19)\qquad \Psi^{*(a)} = \int_0^\infty \frac{1}{\beta^2}\left[A \operatorname{sh} \beta x' + B\beta x' \operatorname{ch} \beta x'\right] \cos \beta y' \, d\beta.$$

From the boundary conditions (4.7), and bearing in mind (4.15), we have

$$(4.20)\qquad B = -A \frac{\operatorname{sh}\mu}{\mu \operatorname{ch}\mu}, \qquad A = \frac{K}{2}\beta \varrho^{(a)}(\mu, \xi) \frac{\mu \operatorname{ch}\mu}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu}.$$

Then,

$$(4.21)\qquad \begin{cases} \overline{\overline{\sigma}}{}^{*(a)}_{x'x'} = \dfrac{\partial^2 \Psi^{*(a)}}{\partial y'^2} = -\dfrac{K}{2}\displaystyle\int_0^\infty \dfrac{\beta \varrho^{(a)}(\mu, \xi')}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu}(\mu \operatorname{ch}\mu \operatorname{sh}\beta x' - \\ \qquad\qquad - \operatorname{sh}\mu \beta x' \operatorname{ch}\beta x') \cos\beta y' \, d\beta, \\ \overline{\overline{\sigma}}{}^{*(a)}_{y'y'} = \dfrac{\partial^2 \Psi^{*(a)}}{\partial x'^2} = \dfrac{K}{2}\displaystyle\int_0^\infty \dfrac{\beta \varrho^{(a)}(\mu, \xi')}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu}\left[(\mu \operatorname{ch}\mu - 2\operatorname{sh}\mu)\times\right. \\ \qquad\qquad \left.\times \operatorname{sh}\beta x' - \beta x' \operatorname{sh}\mu \operatorname{ch}\beta x'\right] \cos\beta y' \, d\beta, \\ \overline{\overline{\sigma}}{}^{*(a)}_{x'y'} = -\dfrac{\partial^2 \Psi^{*(a)}}{\partial x' \partial y'} = \dfrac{K}{2}\displaystyle\int_0^\infty \dfrac{\beta \varrho^{(a)}(\mu, \xi')}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu}\left[(\mu \operatorname{ch}\mu - \operatorname{sh}\mu)\times\right. \\ \qquad\qquad \left.\times \operatorname{sh}\beta x' - \beta x' \operatorname{sh}\mu \operatorname{sh}\beta x'\right] \sin\beta y' \, d\beta. \end{cases}$$

It should be noted that for nuclei of elastic strain symmetric with respect to the y'-axis, the stresses $\overline{\overline{\sigma}}{}^{*(s)}_{x'y'}$ on the straight lines $y' = 0$ and $x' = 0$ vanish. We obtain a normal stress distribution symmetric with respect to the x'- and y'-axis, and a shear stress distribution antisymmetric with respect to these axes. In the case of antisymmetric nuclei, we obtain just the contrary.

The final form of the solution of the Eq. (4.1) is

$$(4.22)\qquad L^* = \Phi^* + \Psi^{*(s)} + \Psi^{*(a)},$$

and the Green functions for stresses σ^*_{ij} will be obtained from

$$(4.23)\qquad \sigma^*_{ij} = \overline{\sigma}^*_{ij} + \overline{\overline{\sigma}}{}^{*(s)}_{ij} + \overline{\overline{\sigma}}{}^{*(a)}_{ij}.$$

Consider, finally, the action of a nucleus of elastic strain $(\varepsilon^0_{xy}\, d\Gamma)$ at the point $P(\xi, 0)$ of a plate strip. Let the edges of the strip be free from stresses.

The G r e e n function F^{***} should satisfy the equation

$$\nabla^2 \nabla^2 F^{***} = 2\,G\,(1+\nu)\,(\varepsilon^0_{xy}\,d\,\Gamma)\,\frac{\partial^2}{\partial x \partial y}\,[\delta\,(x-\xi)\,\delta\,(y)] \tag{4.24}$$

with the boundary conditions

$$\frac{\partial^2 F^{***}}{\partial y^2} = 0, \qquad \frac{\partial^2 F^{***}}{\partial x\,\partial y} = 0 \tag{4.25}$$

at the edges $x = 0$ and $x = a$.

Assume that

$$F^{***} = \Phi^{***} + \Psi^{***}, \tag{4.26}$$

where the function Φ^{***} should satisfy the equation

$$\nabla^2 \nabla^2 \Phi^{***} = 2\,G\,(1+\nu)\,(\varepsilon^0_{xy}\,d\,\Gamma)\,\frac{\partial^2}{\partial x \partial y}\,[\delta\,(x-\xi)\,\delta\,(y)] \tag{4.27}$$

with the boundary conditions

$$\sigma^{***}_{xy} = -\frac{\partial^2 \Phi^{***}}{\partial x\,\partial y} = 0, \qquad \frac{\partial \Phi^{***}}{\partial x} = 0. \tag{4.28}$$

The function Ψ^{***} should satisfy the equation

$$\nabla^2 \nabla^2 \Psi^{***} = 0 \tag{4.29}$$

with the boundary conditions

$$\bar{\sigma}^{***}_{xx} + \frac{\partial^2 \Psi^{***}}{\partial y^2} = 0, \qquad \sigma^{***}_{xy} = -\frac{\partial^2 \Psi^{***}}{\partial x\,\partial y} = 0 \tag{4.30}$$

for $x = 0$ and $x = a$.

Assume the function Φ^{***} in the form

$$\Phi^{***} = \int_0^\infty \sum_{n=1}^\infty A_n(\beta)\cos \alpha_n x \sin \beta y\, d\beta. \tag{4.31}$$

Expressing the D i r a c function by means of a F o u r i e r series and integral, we have the following solution of the Eq. (4.27):

$$\Phi^{***} = -\frac{4\,G\,(1+\nu)\,(\varepsilon^0_{xy}\,d\,\Gamma)}{\pi\,a} \sum_{n=1}^\infty \int_0^\infty \frac{\alpha_n \beta}{(\alpha_n^2+\beta^2)^2} \sin \alpha_n \xi \cos \alpha_n x \sin \beta y\, d\beta. \tag{4.32}$$

This function can be represented in the closed form

$$\Phi^{***} = \frac{K_0}{2}\, y\, \frac{\partial \psi}{\partial x}, \qquad K_0 = \frac{G\,(1+\nu)\,(\varepsilon^0_{xy}\,d\,\Gamma)}{2\,\pi}, \tag{4.33}$$

where

$$\psi = \ln \frac{\operatorname{ch} \frac{\pi}{a} y - \cos \frac{\pi}{a}(x-\xi)}{\operatorname{ch} \frac{\pi}{a} y - \cos \frac{\pi}{a}(x+\xi)}.$$

The stress components $(\bar{\sigma}_{ij}^{***})$ are easily found:

$$(4.34) \qquad \begin{cases} \bar{\sigma}_{xx}^{***} = \dfrac{K_0}{2}\left(2\dfrac{\partial^2 \psi}{\partial x \partial y} + y \dfrac{\partial^3 \psi}{\partial x \partial y^2}\right), \\ \bar{\sigma}_{yy}^{***} = \dfrac{K_0}{2} y \dfrac{\partial^3 \psi}{\partial x^3}, \\ \bar{\sigma}_{xy}^{***} = -\dfrac{K_0}{2}\left(\dfrac{\partial^2 \psi}{\partial x^2} + y \dfrac{\partial^3 \psi}{\partial x^2 \partial y}\right). \end{cases}$$

The Ψ^{***} function will be assumed as composed of two:

$$(4.35) \qquad \Psi^{***} = \Psi_{(s)}^{***} + \Psi_{(a)}^{***},$$

of which the first will correspond to the action of two nuclei $(\varepsilon_{xy}^0 \, d\Gamma)/2$ located symmetrically with respect to the y'-axis, and the second to the action of nuclei, antisymmetric with respect to the same axis.

We assume that

$$(4.36) \qquad \begin{cases} \Psi_{(s)}^{***} = \displaystyle\int_0^\infty \frac{1}{\beta^2} (A \operatorname{ch} \beta x' + B \beta x' \operatorname{sh} \beta x') \sin \beta y' \, d\beta, \\ \Psi_{(a)}^{***} = \displaystyle\int_0^\infty \frac{1}{\beta^2} (C \operatorname{sh} \beta x' + D \beta x' \operatorname{ch} \beta x') \sin \beta y' \, d\beta. \end{cases}$$

The following conditions should be satisfied at the edge $x' = a/2$:

$$(4.37) \qquad -\frac{\partial^2 \Psi_{(s)}^{***}}{\partial x' \partial y'} = 0, \qquad \frac{\partial^2 \Psi_{(s)}^{***}}{\partial y'^2} + K_0 \int_0^\infty \beta \varrho^{(s)}(\xi', \mu) \sin \beta y' \, d\beta = 0,$$

and

$$(4.38) \qquad -\frac{\partial^2 \Psi_{(a)}^{***}}{\partial x' \partial y'} = 0, \qquad \frac{\partial^2 \Psi_{(a)}^{***}}{\partial y'^2} + K_0 \int_0^\infty \beta \varrho^{(a)}(\xi', \mu) \sin \beta y' \, d\beta = 0,$$

where the functions $\varrho^{(s)}(u, \xi')$ and $\varrho^{(a)}(u, \xi')$ are expressed by the Eqs. (4.14) and (4.15).

From the boundary conditions, we have

$$
(4.39)\quad \begin{cases} A = K_0 \beta \varrho^{(s)}(\mu,\xi') \dfrac{\mu \operatorname{ch}\mu + \operatorname{sh}\mu}{\operatorname{ch}\mu \operatorname{sh}\mu + \mu}, & B = -A \dfrac{\operatorname{sh}\mu}{\mu \operatorname{ch}\mu + \operatorname{sh}\mu}, \\ C = K_0 \varrho^{(a)}(\mu,\xi') \dfrac{\operatorname{ch}\mu + \mu \operatorname{sh}\mu}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu}, & D = -C \dfrac{\operatorname{ch}\mu}{\operatorname{ch}\mu + \mu \operatorname{sh}\mu}. \end{cases}
$$

Next, we determine the stress components:

$$
(4.40)\quad \begin{cases} \overline{\overline{\sigma}}^{***}_{x'x'(s)} = -K_0 \displaystyle\int_0^\infty \frac{\beta \varrho^{(s)}(\mu,\xi')}{\operatorname{ch}\mu \operatorname{sh}\mu + \mu} [(\mu \operatorname{ch}\mu + \operatorname{sh}\mu) \operatorname{ch}\beta x' - \\ \qquad\qquad - \beta x' \operatorname{sh}\mu \operatorname{sh}\beta x'] \sin\beta y' \, d\beta, \\ \overline{\overline{\sigma}}^{***}_{y'y'(s)} = K_0 \displaystyle\int_0^\infty \frac{\beta \varrho^{(s)}(\mu,\xi')}{\operatorname{ch}\mu \operatorname{sh}\mu + \mu} [(\mu \operatorname{ch}\mu - \operatorname{sh}\mu) \cosh\beta x' - \\ \qquad\qquad - \beta x' \operatorname{sh}\mu \operatorname{sh}\beta x'] \sin\beta y' \, d\beta, \\ \overline{\overline{\sigma}}^{***}_{x'y'(s)} = K_0 \displaystyle\int_0^\infty \frac{\beta \varrho^{(s)}(\mu,\xi')}{\operatorname{ch}\mu \operatorname{sh}\mu + \mu} [\mu \operatorname{ch}\mu \operatorname{sh}\beta x' - \\ \qquad\qquad - \beta x' \operatorname{sh}\mu \operatorname{ch}\beta x'] \operatorname{ch}\beta y' \, d\beta, \\ \overline{\overline{\sigma}}^{***}_{x'x'(a)} = -K_0 \displaystyle\int_0^\infty \frac{\beta \varrho^{(a)}(\mu,\xi')}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu} [(\operatorname{ch}\mu + \mu \operatorname{sh}\mu) \operatorname{sh}\beta x' - \\ \qquad\qquad - \beta x' \operatorname{ch}\mu \operatorname{ch}\beta x'] \sin\beta y' \, d\beta, \\ \overline{\overline{\sigma}}^{***}_{y'y'(a)} = K_0 \displaystyle\int_0^\infty \frac{\beta \varrho^{(a)}(\mu,\xi')}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu} [(\mu \operatorname{sh}\mu - \operatorname{ch}\mu) \operatorname{sh}\beta x' - \\ \qquad\qquad - \beta x' \operatorname{ch}\mu \operatorname{ch}\beta x'] \sin\beta y' \, d\beta, \\ \overline{\overline{\sigma}}^{***}_{x'y'(a)} = K_0 \displaystyle\int_0^\infty \frac{\beta \varrho^{(a)}(\mu,\xi')}{\operatorname{sh}\mu \operatorname{ch}\mu - \mu} [\mu \operatorname{sh}\mu \operatorname{ch}\beta x' - \\ \qquad\qquad - \beta x' \operatorname{ch}\mu \operatorname{sh}\beta x'] \cos\beta y' \, d\beta. \end{cases}
$$

The final form of the Green function for stresses is obtained by superposition:

$$
(4.41)\quad \sigma^{***}_{ij} = \overline{\sigma}^{***}_{ij} + \overline{\overline{\sigma}}^{***}_{ij(s)} + \overline{\overline{\sigma}}^{***}_{ij(a)}, \qquad i,j = x', y'.
$$

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Streszczenie

O PEWNYM PŁASKIM ZAGADNIENIU DYSTORSYJNYM

Celem pracy jest wyznaczenie składowych stanu naprężenia σ_{ij} i stanu odkształcenia ε_{ij}, wywołanych w tarczy działaniem odkształceń początkowych ε_{ij}^0, występujących na obszarze Γ tarczy. Równania przemieszczeniowe teorii sprężystości przyjmuje w rozpatrywanym zagadnieniu postać (1.5). Rozwiązano je w założeniu, że w punkcie ξ, η działa jądro sprężystego odkształcenia $\varepsilon_{ij}^0 \, d\Gamma$. Otrzymuje się w ten sposób funkcje Greena dla przemieszczeń u, v. Użycie ich zezwala na wyznaczenie przemieszczeń u, v i składowych stanu naprężenia σ_{ij} wywołanych działaniem początkowych odkształceń ε_{ij}^0, występujących na obszarze Γ tarczy [patrz wzory (1.12)].

W wielu przypadkach wygodniej jest posługiwać się funkcją Airy'ego. Zagadnienie sprowadzone zostaje do rozwiązania równania różniczkowego (1.16). Znajomość funkcji Airy'ego F pozwala na wyznaczenie składowych stanu naprężenia σ_{ij} [wzór (1.14)]. Sposób wyznaczenia stanu naprężenia σ_{ij}, wywołanego w tarczy działaniem początkowych odkształceń ε_{ij}^0, objaśniono trzema prostymi przykładami. Dotyczą one tarczy nieograniczonej, półpłaszczyzny tarczowej oraz pasma tarczowego.

Резюме

О НЕКОТОРОЙ ПЛОСКОЙ ДИСТОРСИОННОЙ ЗАДАЧЕ

Работа имеет целью определить компоненты напряженного состояния σ_{ij}, и деформированного состояния ε_{ij}, вызванных в плоском теле действием первоначальных деформаций ε_{ij}^0, которые выступают в области Γ тела. Уравнения перемещений теории упругости в рассматриваемой проблеме принимают вид (1.5). Эти уравнения решаются при предположении, что в точке ξ, η действует упругое ядро деформации $\varepsilon_{ij}^0 \, d\Gamma$. Таким образом получаем функции Грина для перемещений u, v. Использование их позволяет определить перемещения u, v и компонентны напряженного состояния σ_{ij}, вызванные

действием первоначальных деформаций ε_{ij}^{0}, которые выступают в области Γ тела [см. формулы (1.12)].

Во многих случаях удобнее использовать функцию Э р и. Тогда задача сводится к решению дифференциального уравнения (1.16). Значение функции Э р и дает возможность определить, по формулам (1.14), компоненты напряженного состояния σ_{ij}. Способ определения напряженного состояния σ_{ij}, вызванного в теле действием первоначальных деформаций ε_{ij}^{0}, объясняется тремя несложными примерами. Они относятся к бесконечной плоскости, полуплоскости и полосе.

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Received January 28, 1957.