POLSKA AKADEMIA NAUK
INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI

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TOM IX WARSZAWA 1957 ZESZYT 3
PAŃSTWOWE WYDAWNICTWO NAUKOWE

A THREE-DIMENSIONAL THERMOELASTIC PROBLEM WITH DISCONTINUOUS BOUNDARY CONDITIONS

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Consider an elastic semi-space in a steady temperature field. Let the plane $z=0$ bounding the elastic half-space be kept at constant temperature $T=T_0$ inside the circle of radius $r=a$, and let the exterior of that circle be thermally insulated. The object of this paper is to determine the stress in the elastic half-space, supposed to be free from stress at the $z=0$ plane and at infinity.

The temperature field is described by the differential equation

$$\nabla^2 T = 0, \tag{1}$$

and the boundary conditions

$$\begin{cases} T=T_0 \ \text{for} \ r \leqq a \ \text{and} \ z=0, \qquad \dfrac{\partial T}{\partial z}=0 \ \text{for} \ r>a \ \text{and} \ z=0, \\ T=0 \ \text{at infinity.} \end{cases} \tag{2}$$

In view of the axial symmetry of the temperature field, the particular integral of the Eq. (1) will be assumed in the form

$$T=\int_0^\infty C(\alpha)\, e^{-\alpha z} J_0(\alpha r)\, d\alpha, \qquad r=(x^2+y^2)^{1/2}. \tag{3}$$

This integral satisfies the Eq. (1) and the condition $T=0$ at infinity. The coefficient $C(\alpha)$, constituting a function of the parameter α, will be so chosen as to satisfy the boundary conditions in the plane $z=0$. The following relations should be satisfied:

$$\begin{cases} \int_0^\infty C(\alpha) J_0(\alpha r)\, d\alpha = T_0 \quad \text{for} \quad r \leqq a, \\ \int_0^\infty C(\alpha) J_0(\alpha r)\, \alpha\, d\alpha = 0 \quad \text{for} \quad r > a. \end{cases} \tag{4}$$

Since

$$\int_0^\infty \frac{\sin \alpha a}{\alpha} J_0(\alpha r)\, d\alpha = \begin{cases} \dfrac{\pi}{2} & \text{for} \quad r \leqq a, \\ \arcsin \dfrac{a}{r} & \text{for} \quad r > a, \end{cases} \tag{5}$$

and

$$\int_0^\infty \sin \alpha a \, J_0(\alpha r)\, d\alpha = \begin{cases} 0 & \text{for} \quad r \leqq a, \\ (a^2 - r^2)^{-1/2} & \text{for} \quad r > a, \end{cases} \tag{6}$$

we have

$$C(\alpha) = \frac{2T_0}{\pi} \frac{\sin \alpha a}{\alpha}. \tag{7}$$

The temperature fields are therefore described by the relation

$$T = \frac{2T_0}{\pi} \int_0^\infty \frac{\sin \alpha a}{\alpha} e^{-\alpha z} J_0(\alpha r)\, d\alpha. \tag{8}$$

To determine the stress σ_{ij} the potential of thermoelastic displacement Φ, [1], will be used. This is related to the displacement components u, v, w by the equations

$$\frac{\partial \Phi}{\partial x} = u, \qquad \frac{\partial \Phi}{\partial y} = v, \qquad \frac{\partial \Phi}{\partial z} = w. \tag{9}$$

Introducing the Eqs. (9) in the three displacement equations of the theory of elasticity, we reduce them to the unique equation, [1],

$$\nabla^2 \Phi = \frac{1+\nu}{1-\nu} a_t T, \tag{10}$$

where ν denotes Poisson's ratio and a_t the coefficient of thermal expansion. Knowledge of the function Φ enables us to determine the stress components $\bar{\sigma}_{ij}$ from the equations, [1],

$$\bar{\sigma}_{ij} = 2G\left(\frac{\partial^2 \Phi}{\partial i\, \partial j} - \nabla^2 \Phi \delta_{ij}\right) \qquad i, j = x, y, z, \tag{11}$$

where δ_{ij} is Kronecker's delta.

Since

$$e^{-\alpha z} = \frac{2}{\pi} \int_0^\infty \frac{\gamma \sin \gamma z}{\alpha^2 + \gamma^2} d\gamma,$$

the Eq. (10) can be represented in the form

$$\nabla^2 \Phi = \frac{1+\nu}{1-\nu} a_t \frac{4T_0}{\pi^2} \int_0^\infty \int_0^\infty \frac{\gamma \sin \alpha a \sin \gamma z}{\alpha(\alpha^2 + \gamma^2)} J_0(\alpha r)\, d\alpha\, d\gamma. \tag{12}$$

The particular integral of the Eq. (12) is

$$\Phi = -\frac{1+\nu}{1-\nu} a_t \frac{4T_0}{\pi^2} \int_0^\infty \int_0^\infty \frac{\gamma \sin \alpha a \sin \gamma z}{\alpha(\alpha^2 + \gamma^2)^2} J_0(\alpha r)\, d\alpha\, d\gamma, \tag{13}$$

or

$$\Phi = -\frac{Az}{2} \int_0^\infty e^{-\alpha z} \alpha^{-2} \sin \alpha a \, J_0(\alpha r) \, d\alpha, \qquad A = \frac{1+\nu}{1-\nu} \alpha_t \frac{2T_0}{\pi}. \tag{14}$$

Using the Eqs. (11), we determine the stress components $\bar{\sigma}_{ij}$ from the Eqs. (11):

$$\begin{cases} \bar{\sigma}_{rr} = 2G\left(\frac{\partial^2 \Phi}{\partial r^2} - \nabla^2 \Phi\right) = -K \int_0^\infty \frac{e^{-\alpha z}}{\alpha} \sin \alpha a \left[(2-\alpha z) J_0(\alpha r) + \right. \\ \qquad \left. + \frac{z}{r} J_1(\alpha r)\right] d\alpha, \\ \bar{\sigma}_{\varphi\varphi} = 2G\left(\frac{1}{r}\frac{\partial \Phi}{\partial r} - \nabla^2 \Phi\right) = -K \int_0^\infty \frac{e^{-\alpha z}}{\alpha} \sin \alpha a \left[2 J_0(\alpha r) - \right. \\ \qquad \left. - \frac{z}{r} J_1(\alpha r)\right] d\alpha, \\ \bar{\sigma}_{zz} = 2G\left(\frac{\partial^2 \Phi}{\partial z^2} - \nabla^2 \Phi\right) = -Kz \int_0^\infty e^{-\alpha z} \sin \alpha a \, J_0(\alpha r) \, d\alpha, \\ \bar{\sigma}_{rz} = 2G \frac{\partial^2 \Phi}{\partial r \partial z} = K \int_0^\infty \frac{e^{-\alpha z}}{\alpha} (1-\alpha z) J_1(\alpha r) \sin \alpha a \, d\alpha, \\ K = AG. \end{cases} \tag{15}$$

Observe that the normal stress $\bar{\sigma}_{zz}$ vanishes for $z = 0$, the stress $\bar{\sigma}_{rz}$, however, does not vanish. In order to suppress it, the state of stress $(\bar{\bar{\sigma}}_{ij})$ should be superposed over $(\bar{\sigma}_{ij})$. This will be obtained by solving the three-dimensional isothermal problem consisting in determining in an elastic half-space the state of stress $(\bar{\bar{\sigma}}_{ij})$ due to the action of the stress $-[\bar{\sigma}_{rz}]_{z=0}$ acting at the $z = 0$ plane. In order to determine $(\bar{\bar{\sigma}}_{ij})$ we shall use Love's function φ, satisfying the biharmonic equation, [2],

$$\nabla^2 \nabla^2 \varphi = 0 \tag{16}$$

with the boundary conditions

$$[\bar{\bar{\sigma}}_{zz}]_{z=0} = 0, \qquad [\bar{\sigma}_{rz} + \bar{\bar{\sigma}}_{rz}]_{z=0} = 0, \tag{17}$$

and $\varphi = 0$ at infinity.

After determining the function φ, the stress components (σ_{ij}) will be determined from

$$\begin{cases} \bar{\bar{\sigma}}_{rr} = \frac{2G}{1-2\nu} \frac{\partial}{\partial z}\left(\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial r^2}\right), \qquad \bar{\bar{\sigma}}_{\varphi\varphi} = \frac{2G}{1-2\nu} \frac{\partial}{\partial z}\left(\nu \nabla^2 \varphi - \frac{1}{r}\frac{\partial \varphi}{\partial r}\right), \\ \bar{\bar{\sigma}}_{zz} = \frac{2G}{1-2\nu} \frac{\partial}{\partial z}\left[(2-\nu) \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2}\right], \\ \bar{\bar{\sigma}}_{rz} = \frac{2G}{1-2\nu} \frac{\partial}{\partial r}\left[(1-\nu) \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2}\right]. \end{cases} \tag{18}$$

The function φ will be assumed in the form

$$\varphi = \int_0^\infty Z(a, z)\, J_0(a r)\, d a, \tag{19}$$

where

$$Z(a, z) = (C + D\, a z)\, e^{-az}.$$

In view of the first of the conditions (17) we have $C = -D(1-2\nu)$.

The stress components $(\bar{\bar{\sigma}}_{ij})$ are determined by the integrals

$$(20)\quad \begin{cases} \bar{\bar{\sigma}}_{rr} = \dfrac{2G}{1-2\nu} \displaystyle\int_0^\infty D(a)\, a^3 e^{-az} \left[(2-az) J_0(ar) + (2\nu - 2 + az) \dfrac{J_1(ar)}{ar}\right] d a, \\ \sigma_{\varphi\varphi} = \dfrac{2G}{1-2\nu} \displaystyle\int_0^\infty D(a)\, a^3 e^{-az} \left[2\nu J_0(ar) - (2\nu - 2 + az) \dfrac{J_1(ar)}{ar}\right] d a, \\ \bar{\bar{\sigma}}_{zz} = \dfrac{2G}{1-2\nu}\, z \displaystyle\int_0^\infty D(a)\, a^4 e^{-az} J_0(ar)\, d a, \\ \bar{\bar{\sigma}}_{rz} = -\dfrac{2G}{1-2\nu} \displaystyle\int_0^\infty D(a)\, a^3 e^{-az} (1-az) J_1(ar)\, d a. \end{cases}$$

From the second boundary condition (17), we have

$$D = \frac{1-2\nu}{2} \frac{A}{a^4} \sin a\, a.$$

Thus

$$(21)\quad \begin{cases} \sigma_{rr} = K \displaystyle\int_0^\infty \dfrac{e^{-az} \sin a\, a}{a} \left[(2-az) J_0(ar) + (2\nu - 2 + az) \dfrac{J_1(ar)}{ar}\right] d a, \\ \bar{\bar{\sigma}}_{\varphi\varphi} = K \displaystyle\int_0^\infty \dfrac{e^{-az} \sin a\, a}{a} \left[2\nu J_0(ar) - (2\nu - 2 + az) \dfrac{J_1(ar)}{ar}\right] d a, \\ \bar{\bar{\sigma}}_{zz} = Kz \displaystyle\int_0^\infty e^{-az} \sin a\, a\, J_0(ar)\, d a, \\ \sigma_{rz} = -K \displaystyle\int_0^\infty \dfrac{e^{-az}}{a} \sin a\, a\, (1-az) J_1(ar)\, d a. \end{cases}$$

The stress components (σ_{ij}) will be obtained by superposition:

$$\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij}. \tag{22}$$

Observe that $\sigma_{zz} = 0$, $\sigma_{rz} = 0$ at any point of the elastic half-space.

The stresses

$$\begin{cases} \sigma_{rr} = -(1+\nu)\,a\,\dfrac{4T_0G}{\pi r}\displaystyle\int_0^\infty \dfrac{e^{-\alpha z}}{\alpha^2}\sin\alpha a\,J_1(\alpha r)\,d\alpha, \\ \sigma_{\varphi\varphi} = -(1+\nu)\,a\,\dfrac{4T_0G}{\pi}\displaystyle\int_0^\infty \dfrac{e^{-\alpha z}}{\alpha}\sin\alpha a\left[J_0(\alpha r) - \dfrac{J_1(\alpha r)}{\alpha r}\right]d\alpha, \end{cases} \tag{23}$$

or

$$\begin{cases} \sigma_{rr} = -(1+\nu)\,a_t\,\dfrac{4T_0G}{\pi r^2}\displaystyle\int_0^r \eta^{1/2}\arcsin\left(\dfrac{2a}{\sqrt{z^2+(a+\eta)^2}+\sqrt{z^2+(a-\eta)^2}}\right)d\eta, \\ \sigma_{\varphi\varphi} = -(1+\nu)\,a_t\,\dfrac{4T_0G}{\pi}\left\{\arcsin\left[\dfrac{2a}{\sqrt{z^2+(a+r)^2}+\sqrt{z^2+(a-r)^2}}\right] - \right. \\ \qquad \left. -\dfrac{1}{r^2}\displaystyle\int_0^r \eta^{1/2}\arcsin\left[\dfrac{2a}{\sqrt{z^2+(a+\eta)^2}+\sqrt{z^2+(a-\eta)^2}}\right]d\eta\right\}, \end{cases} \tag{24}$$

are different from zero.

References

[1] E. Melan and H. Parcus, *Wärmespannungen stationärer Temperaturfelder*, Vienna 1953.

[2] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, London 1927.

Streszczenie

O PEWNYM PRZESTRZENNYM ZAGADNIENIU TERMOSPRĘŻYSTOŚCI O NIECIĄGŁYCH WARUNKACH BRZEGOWYCH

W pracy rozpatrzono stan naprężenia w półprzestrzeni sprężystej, wywołany ustalonym polem temperatury. Pole to spełnia równanie (1) wraz z nieciągłymi warunkami brzegowymi (2) w płaszczyźnie $z = 0$ ograniczającej półprzestrzeń sprężystą. Uzyskane rozwiązanie jest charakterystyczne pod tym względem, że w półprzestrzeni sprężystej znikają naprężenia σ_{zz} oraz σ_{rz}.

Резюме

О НЕКОТОРОЙ ПРОСТРАНСТВЕННОЙ ЗАДАЧЕ ТЕРМОУПРУГОСТИ С РАЗРЫВНЫМИ КРАЕВЫМИ УСЛОВИЯМИ

В работе рассматривается напряженное состояние в упругой полуплоскости, вызванное станционарным температурным полем. Это поле удовлетворяет уравнению (1), с разрывными краевыми условиями (2) в плоскости $z = 0$, ограничивающей упругую полуплоскость. Полученное решение характерно тем, что в упругой полуплоскости исчезают напряжения σ_{zz} и σ_{rz}.

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IBTP POLISH ACADEMY OF SCIENCES

Received March 12, 1957.