POLSKA AKADEMIA NAUK
INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI

ARCHIWUM MECHANIKI STOSOWANEJ

ARCHIVES DE MÉCANIQUE APPLIQUÉE

1
IX

TOM IX WARSZAWA 1957 ZESZYT 1
PAŃSTWOWE WYDAWNICTWO NAUKOWE

THE STRESSES IN A THIN PLATE DUE TO A NUCLEUS OF THERMOELASTIC STRAIN

WITOLD NOWACKI (WARSZAWA)

Consider a thin plate (treated as two-dimensional problem) with free edges. Let the temperature of the surface element $d\Omega$ be T. The temperature of the remaining region is assumed to be zero. This discontinuous, concentrated thermal action will result in a state of stress and strain. To determine the stresses due to such a nucleus of thermoelastic deformation $(\alpha_t T d\Omega)$ is the object of this paper. From the solution of the problem thus stated, we shall obtain the stresses as functions of the considered point (x, y) and the point (ξ, η) where the nucleus of thermoelastic strain acts. In this way, we obtain the influence surfaces for the stresses, in other words Green's functions of our problem.

The knowledge of Green's functions will enable us to solve thermoelastic problems of two types. First, the stresses in the plate in which a finite region Ω is heated to the temperature T, the remaining region being kept under zero temperature, in other words in a discontinuous temperature field. Next, the knowledge of Green's functions will enable us to determine the stresses in a plate with an inclusion constituting a finite region Ω and made of a material of the same elastic properties and different thermal properties. This is because, on the basis of J. N. Goodier's considerations, [1], the problem of stresses due to the heating of the plate, with the inclusion, to the temperature T can be reduced to the problem of stress caused by a discontinuous temperature field in a homogeneous plate.

It is known that in thermoelastic problems the equations of displacement of the theory of elasticity for a plane state of stress can be reduced to a single differential equation, [2],

$$\nabla^2 \Phi = (1+\nu)\frac{\alpha_t T}{h}, \tag{1.1}$$

where Φ is the so-called thermal potential of displacement, T temperature, ν Poisson's ratio, α_t coefficient of thermal dilatation and h plate thickness.

In the case of a nucleus of thermoelastic strain, the Eq. (1.1) takes the form

$$\nabla^2 \Phi = \frac{(1+\nu)\,\alpha\, d\Omega\, T}{h}\, \delta(x-\xi)\,\delta(y-\eta), \tag{1.2}$$

where the right-hand member of the equation is equal to zero outside $d\Omega$ (δ being the Dirac function).

Knowing a particular integral of the Eq. (1.2) we can determine the stresses from the relations, [2],

$$\bar{\sigma}_x = -2G\frac{\partial^2 \Phi}{\partial y^2}, \qquad \bar{\sigma}_y = -2G\frac{\partial^2 \Phi}{\partial x^2}, \qquad \bar{\tau}_{xy} = 2G\frac{\partial^2 \Phi}{\partial x\,\partial y}, \tag{1.3}$$

where G is the modulus of elasticity in shear. The stresses (1.3) do not, in general, satisfy all edge conditions. Thus, another state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$ should be superposed over the state $(\bar{\sigma}, \bar{\tau})$. This new state of stress satisfies Airy's equation

$$\nabla^2 \nabla^2 F = 0, \tag{1.4}$$

where

$$\bar{\bar{\sigma}}_x = \frac{\partial^2 F}{\partial y^2}, \qquad \bar{\bar{\sigma}}_y = \frac{\partial^2 F}{\partial x^2}, \qquad \bar{\bar{\tau}}_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}. \tag{1.5}$$

The state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$ should be so chosen that the sum of the stresses $\bar{\sigma}+\bar{\bar{\sigma}}$, $\bar{\tau}+\bar{\bar{\tau}}$ at the edge may be equal to zero.

Two problems have, to the author's knowledge, been solved so far: the problem of the state of stress in an infinite plate heated to the temperature T, for a rectangular and an elliptic inclusion, [1], and that of the state of stress due to a nucleus of thermoelastic strain located on the symmetry axis of an infinite plate strip, [3].

The procedure was in both cases as follows: from the singular solution of the Eq. (1.1) for an infinite plate the stresses $(\bar{\sigma}, \bar{\tau})$ are determined and the edge conditions are corrected by means of the Airy function, thus determining the corresponding state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$.

A different method of solution will be adopted in this paper, using the analogy between the Eq. (1.2) and the equation of deflection of a membrane subjected to a concentrated force,

$$\nabla^2 w = -\frac{Q}{S}\,\delta(x-\xi)\,\delta(y-\eta), \tag{1.6}$$

where w denotes the deflection of the membrane, $Q = p\,d\Omega$ is the concentrated force, and S the stretching force. The right-hand member of the Eq. (1.6) is equal to zero except for the region $d\Omega$.

Since the Eq. (1.6) can be solved for various regions with the end condition $w = 0$, the Eq. (1.2) will be solved under the condition of $\Phi = 0$, using the known solution of the Eq. (1.6). Knowing the function Φ, we shall determine the state of stress $(\bar{\sigma}, \bar{\tau})$ from the Eqs. (1.3).

Since it was assumed that $\Phi = 0$ at the edge of the plate, the normal stress $\bar{\sigma}$ vanishes at the edge. The additional state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$, determined by means of the Airy function, will be so chosen that the normal stress $\bar{\bar{\sigma}}$ and the sum $\bar{\tau} + \bar{\bar{\tau}}$ may also be equal to zero at the edge of the plate.

The procedure outlined here will be explained by means of a few simple examples. We shall confine ourselves to the three most simple cases: an infinite strip, a semi-infinite strip and a semi-infinite plane.

2. The Case of an Infinite Strip

Consider an infinite strip of width a in which a nucleus of thermoelastic strain acts at the point $(\xi, 0)$, Fig. 1. It is seen that the solution of the differential equation (1.2), with the boundary condition $\Phi = 0$, can be represented in the form

$$\Phi = -\frac{2K}{a\pi h} \sum_{n=1}^{\infty} \sin \alpha_n \xi \times \sin \alpha_n x \int_0^{\infty} \frac{\cos \beta y \, d\beta}{\alpha_n^2 + \beta^2}, \tag{2.1}$$

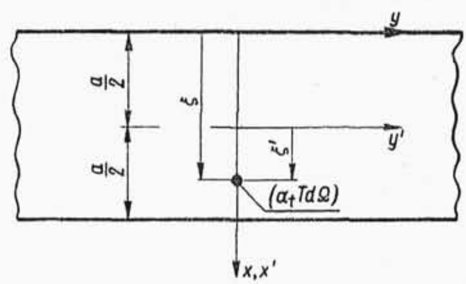

Fig. 1

where $K = (1+\nu)\alpha T \, d\Omega$ and $\alpha_n = n\pi/a$.

We have used here the analogy between the Eq. (1.2) and the Eq. (1.6), since for a membrane strip of width a, supported on the edges and acted on by a concentrated force Q at the point $(\xi, 0)$, the deflection of the mebrane is expressed by the equation

$$w = \frac{2Q}{a\pi S} \sum_{n=1}^{\infty} \sin \alpha_n \xi \sin \alpha_n x \int_0^{\infty} \frac{\cos \beta y \, d\beta}{\alpha_n^2 + \beta^2}. \tag{2.2}$$

Since

$$\int_0^{\infty} \frac{\cos \beta y \, d\beta}{\alpha_n^2 + \beta^2} = \frac{1}{2} \frac{\pi}{\alpha_n} e^{-\alpha_n y},$$

the function Φ can be represented by the simple trigonometric series,

$$\Phi = -\frac{K}{h\pi} \sum_{n=1}^{\infty} \frac{e^{-\frac{n\pi y}{a}}}{n} \sin \frac{n\pi\xi}{a} \sin \frac{n\pi x}{a} \tag{2.3}$$

or by the closed formula,

$$\Phi = \frac{K}{h}\varphi(x, y; \xi, \eta), \tag{2.4}$$

where

$$\varphi(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{\operatorname{ch}\dfrac{\pi y}{a} - \cos\dfrac{\pi}{a}(x-\xi)}{\operatorname{ch}\dfrac{\pi y}{a} - \cos\dfrac{\pi}{a}(x+\xi)}.$$

Using the Eqs. (1.3), we obtain

$$\bar{\sigma}_x = -\frac{2GK}{h}\frac{\partial^2\varphi}{\partial y^2}, \qquad \bar{\sigma}_y = -\frac{2GK}{h}\frac{\partial^2\varphi}{\partial x^2}, \qquad \bar{\tau}_{xy} = \frac{2GK}{h}\frac{\partial^2\varphi}{\partial x\,\partial y}, \tag{2.5}$$

where

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{\partial^2\varphi}{\partial y^2} = \frac{\pi}{4a^2}\left\{\frac{1-\cos\dfrac{\pi}{a}(x+\xi)\operatorname{ch}\dfrac{\pi y}{a}}{\left[\cos\dfrac{\pi}{a}(x+\xi) - \operatorname{ch}\dfrac{\pi y}{a}\right]^2} - \frac{1-\cos\dfrac{\pi}{a}(x-\xi)\operatorname{ch}\dfrac{\pi y}{a}}{\left[\cos\dfrac{\pi}{a}(x-\xi) - \operatorname{ch}\dfrac{\pi y}{a}\right]^2}\right\},$$

$$\frac{\partial^2\varphi}{\partial x\,\partial y} = \frac{\pi}{4a^2}\operatorname{sh}\frac{\pi y}{a}\left\{\frac{\sin\dfrac{\pi}{a}(x+\xi)}{\left[\cos\dfrac{\pi}{a}(x+\xi) - \operatorname{ch}\dfrac{\pi y}{a}\right]^2} - \frac{\sin\dfrac{\pi}{a}(x-\xi)}{\left[\cos\dfrac{\pi}{a}(x-\xi) - \operatorname{ch}\dfrac{\pi y}{a}\right]^2}\right\}.$$

It is seen that, at the edges $x = 0$ and $x = a$, the normal stresses vanish. For $y \to \infty$, the shear stresses also vanish. In the neighbourhood of the nucleus of thermoelastic strain, the stresses increase infinitely.

It will be convenient, for subsequent considerations, to represent the shear stresses $\bar{\tau}_{xy}$ by the formula obtained directly from the Eq. (2.1):

$$\bar{\tau}_{xy} = 2G\frac{\partial^2\Phi}{\partial x\,\partial y} = \frac{4KG}{a\pi h}\int_0^\infty \beta\sin\beta y\left(\sum_{n=1}^\infty \frac{\alpha_n\sin\alpha_n\xi\cos\alpha_n x}{\alpha_n^2+\beta^2}\right)d\beta. \tag{2.6}$$

Bearing in mind that

$$\left\{\begin{aligned} &\sum_{n=1}^\infty \frac{\alpha_n\sin\alpha_n\xi}{\alpha_n^2+\beta^2} = \frac{a}{2}\vartheta_1(\xi,\beta),\\ &\sum_{n=1}^\infty \frac{\alpha_n(-1)^{n-1}\sin\alpha_n\xi}{\alpha_n^2+\beta^2} = \frac{a}{2}\vartheta_2(\xi,\beta), \end{aligned}\right. \tag{2.7}$$

where

$$\vartheta_1(\xi,\beta) = \frac{\operatorname{sh}\beta(a-\xi)}{\operatorname{sh}\lambda}, \qquad \vartheta_2(\xi,\beta) = \frac{\operatorname{sh}\beta\xi}{\operatorname{sh}\lambda} \qquad (\lambda = \beta a),$$

we can represent the shear stresses appearing at the edges by the expressions:

$$(2.8)\qquad \begin{cases} [\bar{\tau}_{xy}]_{x=0} = \dfrac{2\,KG}{\pi h} \displaystyle\int_0^\infty \vartheta_1(\xi, \beta)\,\beta \sin \beta y \, d\beta, \\ [\bar{\tau}_{xy}]_{x=a} = -\dfrac{2\,KG}{\pi h} \displaystyle\int_0^\infty \vartheta_2(\xi, \beta)\,\beta \sin \beta y \, d\beta. \end{cases}$$

The action of the nucleus of thermoelastic strain at the point $(\xi, 0)$ can be replaced by the action of sources located in a symmetric or antisymmetric manner in relation to the y'-axis (Fig. $2a$ and 2b).

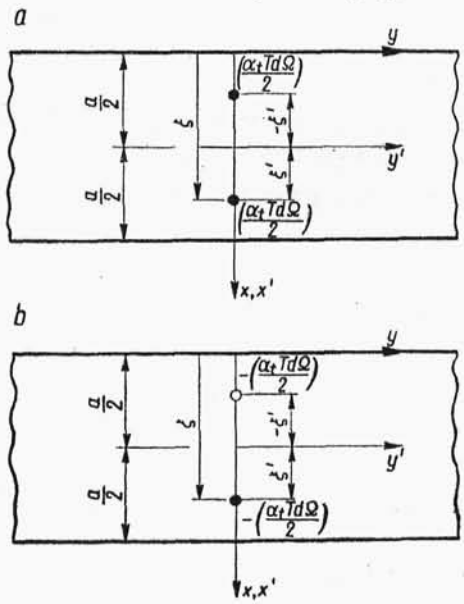

Fig. 2

For nuclei of intensity $\alpha_t T\, d\Omega/2$, symmetric in relation to y'-axis, we obtain in the x', y' — coordinate system

$$(2.9)\quad [\bar{\tau}^{(s)}_{x'y'}]_{x'=a/2} = -\frac{2\,KG}{\pi h}\,\frac{1}{2}\int_0^\infty \beta \sin \beta y' \times$$

$$\times \left[\eta_2\left(\beta, \frac{a}{2} + \xi'\right) + \eta_2\left(\beta, \frac{a}{2} - \xi'\right)\right] d\beta = -\frac{KG}{\pi h}\int_0^\infty \beta \sin \beta y'\, \varrho^{(s)}(\mu, \xi')\, d\beta,$$

where

$$\varrho^{(s)}(\mu, \xi') = \frac{\operatorname{ch} \beta \xi'}{\operatorname{ch} \mu}, \qquad \mu = \frac{\beta a}{2}.$$

For two nuclei of intensity $\alpha_t T\, d\Omega/2$, antisymmetric in relation to the y'-axis, we have

$$(2.10) \quad [\overline{\tau}^{(a)}_{x'y'}]_{x'=a/2} = -\frac{2\,KG}{\pi h}\,\frac{1}{2}\int_0^\infty \beta \sin\beta y' \times$$

$$\times\left[\eta_2\left(\beta, \frac{a}{2}+\xi'\right) - \eta_2\left(\beta, \frac{a}{2}-\xi'\right)\right] d\beta = -\frac{KG}{\pi h}\int_0^\infty \beta \sin\beta y'\, \varrho^{(a)}(\mu,\xi')\, d\beta,$$

where

$$\varrho^{(a)}(\mu,\xi') = \frac{\operatorname{sh}\beta\xi'}{\operatorname{sh}\mu}.$$

In order to suppress the shear stresses at the edges we must superpose such a state of stress $(\overline{\overline{\sigma}}_x, \overline{\overline{\sigma}}_y, \overline{\overline{\tau}}_{xy})$, that the sums of stresses $\overline{\sigma}_x + \overline{\overline{\sigma}}_x$, etc., satisfy all edge conditions.

Consider first the case of symmetric nuclei of intensity $\alpha_t T d\Omega/2$. The following differential equation should be solved:

$$(2.11) \qquad \nabla^2\nabla^2 F^{(s)} = 0,$$

with the edge conditions

$$(2.12) \qquad \overline{\overline{\sigma}}_{x'} = \frac{\partial^2 F^{(s)}}{\partial y'^2} = 0, \qquad \overline{\overline{\tau}}^{(s)}_{x'y'} = -\frac{\partial^2 F^{(s)}}{\partial x'\,\partial y'} = -\overline{\tau}^{(s)}_{x'y'},$$

at the edge $x' = a/2$.

The function $F^{(s)}$ will be assumed in the form of the Fourier integral:

$$(2.13) \qquad F = \frac{1}{h}\int_0^\infty \frac{1}{\beta^2}\,[A \operatorname{ch}\beta x' + B\beta x' \operatorname{sh}\beta x']\cos\beta y'\, d\beta.$$

The boundary conditions (2.12) lead to the relations

$$(2.14) \qquad A = -\frac{KG}{\pi}\,\frac{\beta\ \mu \operatorname{sh}\mu\, \varrho^{(s)}(\mu,\xi')}{\operatorname{sh}\mu \operatorname{ch}\mu + \mu}, \qquad B = -A\,\frac{\operatorname{ch}\mu}{\mu \operatorname{sh}\mu}.$$

Using the Eqs. (1.5) we obtain the stresses $\overline{\overline{\sigma}}^{(s)}_{x'}$, etc.:

$$(2.15) \quad \begin{cases} \overline{\overline{\sigma}}^{(s)}_{x'} = \dfrac{KG}{\pi h}\displaystyle\int_0^\infty \frac{\beta \operatorname{ch}\beta\xi'}{\mu + \operatorname{sh}\mu \operatorname{ch}\mu}\,(\mu \operatorname{th}\mu \operatorname{ch}\beta x' - \beta x' \operatorname{sh}\beta x')\cos\beta y'\, d\beta, \\[2ex] \overline{\overline{\sigma}}^{(s)}_{y'} = -\dfrac{KG}{\pi h}\displaystyle\int_0^\infty \frac{\beta \operatorname{ch}\beta\xi'}{\mu + \operatorname{sh}\mu \operatorname{ch}\mu}\,[(\mu \operatorname{th}\mu - 2)\operatorname{ch}\beta x' - \\ \qquad\qquad - \beta x' \operatorname{sh}\beta x']\cos\beta y'\, d\beta, \\[2ex] \overline{\overline{\tau}}^{(s)}_{x'y'} = -\dfrac{KG}{\pi h}\displaystyle\int_0^\infty \frac{\beta \operatorname{ch}\beta\xi'}{\mu + \operatorname{sh}\mu \operatorname{ch}\mu}\,[(\mu \operatorname{th}\mu - 1)\operatorname{sh}\beta x' - \\ \qquad\qquad - \beta x' \operatorname{ch}\beta x']\sin\beta y'\, d\beta. \end{cases}$$

It can be seen from these equations that for $x' = a/2$ the stress $\bar{\sigma}^{(s)}_{x'}$ vanishes and $\bar{\bar{\tau}}^{(s)}_{x'y'} = -\bar{\tau}^{(s)}_{x'y'}$. For $x' = 0$ and $y' = 0$, the stress $\bar{\bar{\tau}}^{(s)}_{x'y'}$ vanishes.

Consider the case of two nuclei of intensity $\alpha_t T\, d\Omega/2$, antisymmetric in relation to the y'-axis. In order to suppress the shear stresses $\bar{\tau}_{x'y'}$ appearing at the edges $x' = \pm a/2$, we choose such a state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$ that the differential equation

$$\nabla^2 \nabla^2 F^{(a)} = 0 \tag{2.16}$$

is satisfied, together with the boundary conditions for $x' = a/2$:

$$\bar{\bar{\sigma}}^{(a)}_{x'} = \frac{\partial^2 F^{(a)}}{\partial y'^2} = 0; \qquad \bar{\bar{\tau}}^{(a)}_{x'y'} = -\frac{\partial^2 F^{(a)}}{\partial x' \partial y'} = -\bar{\tau}^{(a)}_{x'y'}. \tag{2.17}$$

The function $F^{(a)}$ will be assumed in the form of the Fourier integral:

$$F^{(a)} = \frac{1}{h} \int_0^\infty \frac{1}{\beta^2} [A \operatorname{sh} \beta x' + B \beta x' \operatorname{ch} \beta x'] \cos \beta y' \, d\beta. \tag{2.18}$$

From the conditions (2.17), we obtain the following values of A and B in function of the parameter β:

$$A = -\frac{KG}{\pi} \frac{\beta \mu \operatorname{ch} \mu\, \varrho^{(a)}(\mu, \xi')}{\operatorname{sh} \mu \operatorname{ch} \mu - \mu}, \qquad B = -A \frac{\operatorname{th} \mu}{\mu}. \tag{2.19}$$

The knowledge of the function A and B enables the determination of the stresses:

$$\begin{cases} \bar{\bar{\sigma}}^{(a)}_{x'} = \dfrac{KG}{\pi h} \displaystyle\int_0^\infty \frac{\beta \operatorname{sh} \beta \xi'}{\operatorname{sh} \mu \operatorname{ch} \mu - \mu} (\mu \operatorname{cth} \mu \operatorname{sh} \beta x' - \beta x' \operatorname{ch} \beta x') \cos \beta y' \, d\beta, \\ \bar{\bar{\sigma}}^{(a)}_{y'} = -\dfrac{KG}{\pi h} \displaystyle\int_0^\infty \frac{\beta \operatorname{sh} \beta \xi'}{\operatorname{sh} \mu \operatorname{ch} \mu - \mu} [(\mu \operatorname{cth} \mu - 2) \operatorname{sh} \beta x' - \\ \qquad\qquad - \beta x' \operatorname{ch} \beta x'] \cos \beta y' \, d\beta, \\ \bar{\bar{\tau}}^{(a)}_{x'y'} = -\dfrac{KG}{\pi h} \displaystyle\int_0^\infty \frac{\beta \operatorname{sh} \beta \xi'}{\operatorname{sh} \mu \operatorname{ch} \mu - \mu} [(\mu \operatorname{cth} \mu - 1) \operatorname{ch} \beta x' - \\ \qquad\qquad - \beta x' \operatorname{sh} \beta x'] \sin \beta y' \, d\beta. \end{cases} \tag{2.20}$$

Observe that for $x' = a/2$ the normal stresses $\bar{\bar{\sigma}}^{(a)}_{x'}$ vanish and $\bar{\bar{\tau}}^{(a)}_{x'y'} = -\bar{\tau}^{(a)}_{x'y'}$. For $x' = 0$ the normal stresses vanish and for $y' = 0$ the shear stresses vanish. Finally, for $\xi' = 0$, the normal and shear stresses vanish for x' and y' different from zero.

For a nucleus of intensity $\alpha_t T\, d\Omega$, at the point $(\xi, 0)$, in other words for an asymmetrically located nucleus, the stress components will be expressed by the equations:

$$\sigma_x = \bar{\sigma}_x + \bar{\bar{\sigma}}^{(s)}_x + \bar{\bar{\sigma}}^{(a)}_x, \qquad \sigma_y = \sigma_y + \bar{\bar{\sigma}}^{(s)}_y + \bar{\bar{\sigma}}^{(a)}_y, \qquad \tau_{xy} = \tau_{xy} + \bar{\bar{\tau}}^{(s)}_{xy} + \bar{\bar{\tau}}^{(a)}_{xy}. \tag{2.21}$$

For a nucleus of thermoelastic strain acting at the point (ξ, η) of the plate we shall use the above formulae, substituting $(y-\eta)$ for y and $y'-\eta'$ for y'.

If, therefore, $\sigma(x, y; \xi, \eta) = \bar{\sigma}(x, y; \xi, \eta) + \sigma(x, y; \xi, \eta)$ denotes the stresses at the point (x, y) due to the action of a unit nucleus of thermoelastic strain, the stress $\sigma^*(x, y)$, due to the temperature field $T(\xi, \eta)$ distributed over the region Ω of the plate, will be expressed by the integral,

$$\sigma^*(x, y) = \iint_{\Omega} T(\xi, \eta)\, \sigma(x, y; \xi\, \eta)\, d\xi\, d\eta. \tag{2.22}$$

3. A Semi-infinite Strip

A semi-infinite strip can be treated as an infinite strip in which a positive nucleus acts at the point (ξ, η) and a negative nucleus at the point $(\xi, -\eta)$. In consequence of this assumption, we obtain $\bar{\sigma}_y = 0$ on the axis $y = 0$. Using the Eq. (2.2), the function Φ is expressed by the relation

$$\Phi = -\frac{2K}{a\pi h} \sum_{n=1}^{\infty} \sin a_n \xi \sin a_n x \int_0^{\infty} \frac{\cos\beta(y-\eta) - \cos\beta(y+\eta)}{a_n^2 + \beta^2}\, d\beta, \tag{3.1}$$

or

$$\Phi = -\frac{4K}{a\pi h} \sum_{n=1}^{\infty} \sin a_n \xi \sin a_n x \int_0^{\infty} \frac{\sin\beta y \sin\beta\eta}{a_n^2 + \beta^2}\, d\beta. \tag{3.2}$$

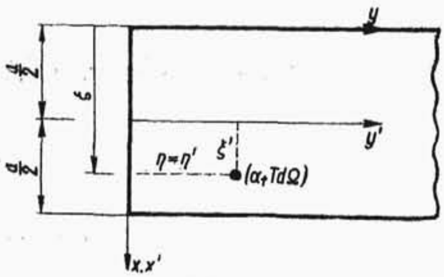

Fig. 3

In order to determine the stress $(\bar{\sigma}, \bar{\tau})$, it is necessary to represent the function Φ by a closed formula. Using the Eqs. (2.2), (2.3) we obtain.

$$\Phi = \frac{K}{h}\left[\varphi_1(x, y; \xi, \eta) - \varphi_2(x, y; \xi, \eta)\right], \tag{3.3}$$

where

$$\begin{cases} \varphi_1 = \dfrac{1}{4\pi} \ln \dfrac{\operatorname{ch}\dfrac{\pi}{a}(y-\eta) - \cos\dfrac{\pi}{a}(x-\xi)}{\operatorname{ch}\dfrac{\pi}{a}(y-\eta) - \cos\dfrac{\pi}{a}(x+\xi)}, \\[2ex] \varphi_2 = \dfrac{1}{4\pi} \ln \dfrac{\operatorname{ch}\dfrac{\pi}{a}(y+\eta) - \cos\dfrac{\pi}{a}(x-\xi)}{\operatorname{ch}\dfrac{\pi}{a}(y+\eta) - \cos\dfrac{\pi}{a}(x+\xi)}. \end{cases} \tag{3.4}$$

Thus

$$(3.5)\quad \begin{cases} \sigma_x = -\dfrac{2KG}{h}\dfrac{\partial^2(\varphi_1-\varphi_2)}{\partial y^2}, \qquad \sigma_y = -\dfrac{2GK}{h}\dfrac{\partial^2(\varphi_1-\varphi_2)}{\partial x^2}, \\ \tau_{xy} = \dfrac{2GK}{h}\dfrac{\partial^2(\varphi_1-\varphi_2)}{\partial x\,\partial y}. \end{cases}$$

It can easily be verified that for $x=0$ and $x=a$ the stresses $\bar\sigma_x$ vanish, as also the stresses $\bar\sigma_y$ for $y=0$, and that in the neighbourhood of the nucleus of thermoelastic strain the stresses increase infinitely.

The shear stress $\bar\tau_{xy}$ can be represented by the equation

$$(3.6)\quad \bar\tau_{xy} = 2G\frac{\partial^2\Phi}{\partial x\,\partial y} = -\frac{8KG}{a\pi h}\int\limits_0^\infty \beta\sin\beta\eta\cos\beta y\left(\sum_{n=1}^{\infty}\frac{\alpha_n\sin\alpha_n\xi\cos\alpha_n x}{\alpha_n^2+\beta^2}\right)d\beta.$$

Bearing in mind

$$(3.7)\quad \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{\alpha_n\sin\alpha_n\xi}{\alpha_n^2+\beta^2} = \frac{a}{2}\frac{\operatorname{sh}\beta(a-\xi)}{\operatorname{sh}\lambda} = \frac{a}{2}\eta_1(\beta,\xi), \\ \displaystyle\sum_{n=1}^{\infty}\frac{\alpha_n(-1)^{n-1}\sin\alpha_n\xi}{\alpha_n^2+\beta^2} = \frac{a}{2}\eta_2(\beta,\xi), \quad \int\limits_0^\infty\frac{\beta\sin\beta\eta\,d\beta}{\alpha_n^2+\beta^2} = \frac{\pi}{2}e^{-\alpha_n\eta}, \end{cases}$$

we find

$$(3.8)\quad \begin{cases} [\bar\tau_{xy}]_{x=a} = \dfrac{4KG}{\pi h}\displaystyle\int\limits_0^\infty \beta\eta_2(\beta,\xi)\sin\beta\eta\cos\beta y\,d\beta, \\ [\bar\tau_{xy}]_{y=0} = -\dfrac{4KG}{\pi h}\displaystyle\sum_{n=1}^{\infty}\alpha_n e^{-\alpha_n\eta}\sin\alpha_n\xi\cos\alpha_n x. \end{cases}$$

Proceeding as in the case of a strip with one nucleus of thermoelastic strain, we shall consider first the case of two nuclei of intensity $(\alpha_t T\,d\Omega/2)$ symmetric in relation to the y'-axis, and then the case of the same nuclei antisymmetric in relation to that axis (Fig. 4a and 4b).

Consider first the case of symmetric nuclei of intensity $(\alpha_t T\,d\Omega/2)$. In the x', y'-system we obtain:

$$(3.9)\quad \begin{cases} [\bar\tau^{(s)}_{x'y'}]_{x'=a/2} = \dfrac{2KG}{\pi h}\displaystyle\int\limits_0^\infty \beta\,\varrho^{(s)}(\mu,\xi')\sin\beta\eta'\cos\beta y'\,d\beta, \\ [\bar\tau^{(s)}_{x'y'}]_{y'=0} = \dfrac{4KG}{ah}\displaystyle\sum_{n=1,3,}^{\infty}\alpha_n e^{-\alpha_n\eta'}\cos\alpha_n\xi'\sin\alpha_n x'. \end{cases}$$

The corresponding additional stresses $\overline{\overline{\sigma}}_{x'}^{(s)}$, etc., will be obtained from the solution of A i r y' s equation

$$\nabla^2 \nabla^2 F^{(s)} = 0, \tag{3.10}$$

with the boundary conditions

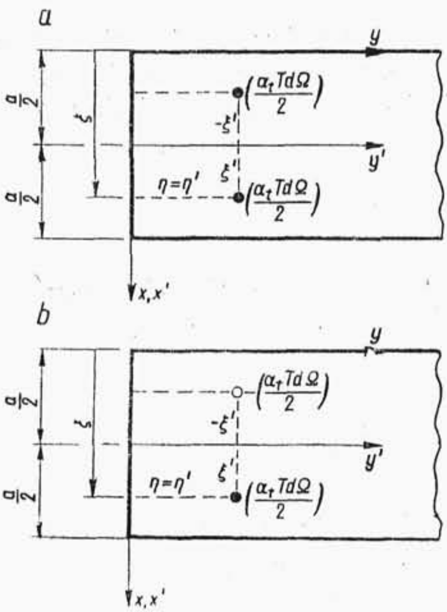

Fig. 4

$$\left\{\begin{aligned} &\overline{\overline{\sigma}}_{x'}^{(s)} = \frac{\partial^2 F^{(s)}}{\partial y'^2} = 0, \\ &\overline{\overline{\tau}}_{x'y'}^{(s)} = -\frac{\partial^2 F^{(s)}}{\partial x' \partial y'} = -\overline{\tau}_{x'y'}^{(s)} \\ &\qquad \text{for} \quad x' = \frac{a}{2}, \\ &\overline{\overline{\sigma}}_{y'}^{(s)} = \frac{\partial^2 F^{(s)}}{\partial x'^2} = 0, \\ &\overline{\overline{\tau}}_{x'y'}^{(s)} = -\frac{\partial^2 F^{(s)}}{\partial x' \partial y'} = -\overline{\tau}_{x'y'}^{(s)} \\ &\qquad \text{for} \quad y' = 0. \end{aligned}\right. \tag{3.11}$$

These conditions will be satisfied assuming the function in the form:

$$F^{(s)} = \frac{1}{h} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{\alpha_n^2} (A_n^{(s)} + B_n^{(s)} \alpha_n y') e^{-\alpha_n y'} \cos \alpha_n x' + \frac{1}{h} \int_0^\infty \frac{1}{\beta^2} (A^{(s)} \operatorname{ch} \beta x' + B^{(s)} \beta x' \operatorname{sh} \beta x') \sin \beta y' \, d\beta. \tag{3.12}$$

The boundary conditions (3.11) lead to the following relations:

$$\left\{\begin{aligned} &A_n^{(s)} = 0, \qquad A^{(s)} \operatorname{ch} \mu + B^{(s)} \mu \operatorname{sh} \mu = 0 \qquad (\mu = a\beta/2), \\ &\sum_{n=1,3,\ldots}^{\infty} B_n^{(s)} \sin \alpha_n x' - \int_0^\infty [(A^{(s)} + B^{(s)}) \operatorname{sh} \beta x' + B^{(s)} \beta x' \operatorname{ch} \beta x'] \, d\beta = \\ &\qquad = -\frac{4\,KG}{a} \sum_{n=1,3,\ldots}^{\infty} \alpha_n e^{-\alpha_n \eta'} \cos \alpha_n \xi' \sin \alpha_n x', \\ &\sum_{n=1,3,\ldots}^{\infty} B_n^{(s)} (1 - \alpha_n y') e^{-\alpha_n y'} \sin \frac{n\pi}{2} - \int_0^\infty [(A^{(s)} + B^{(s)}) \operatorname{sh} \mu + \\ &\qquad + B^{(s)} \mu \operatorname{ch} \mu] \cos \beta y' \, d\beta = -\frac{2\,KG}{\pi} \int_0^\infty \beta \varrho^{(s)}(\mu, \xi') \sin \beta \eta' \cos \beta y' \, d\beta. \end{aligned}\right. \tag{3.13}$$

If the following expansions are substituted in these equations:

$$(3.14)\quad \begin{cases} \operatorname{sh}\beta x' = \sum_{n=1}^{\infty} E_{n\beta}\sin\alpha_n x', \qquad \beta x'\operatorname{ch}\beta x' = \sum_{n=1}^{\infty} F_{n\beta}\sin\alpha_n x', \\ e^{-\alpha_n y'}(1-\alpha_n y') = \int_0^{\infty} C_{n,\beta}\cos\beta y'\,d\beta, \end{cases}$$

we obtain the system of two equations:

$$(3.15)\quad \begin{cases} B_n^{(s)} + \int_0^{\infty} A^{(s)}\left[r(\mu)E_{n\beta} - g(\mu)F_{n\beta}\right]d\beta = -\dfrac{4KG}{a}\alpha_n e^{-\alpha_n\eta'}\cos\alpha_n\xi', \\ \sum_{k=1}^{\infty} B_k^{(s)} C_{k\beta}\sin\dfrac{k\pi}{2} + A^{(s)}t(\mu) = -\dfrac{2KG}{\pi}\beta\varrho^{(s)}(\mu,\xi')\sin\beta\eta', \end{cases}$$

where

$$r(\mu) = \frac{\mu\operatorname{sh}\mu - \operatorname{ch}\mu}{\mu\operatorname{sh}\mu}, \qquad g(\mu) = \frac{\operatorname{cth}\mu}{\mu}, \qquad t(\mu) = \frac{\mu + \operatorname{sh}\mu\operatorname{ch}\mu}{\mu\operatorname{sh}\mu}.$$

Bearing in mind

$$(3.16)\quad \begin{cases} E_{n\beta} = \dfrac{4\beta}{a}\,\dfrac{\operatorname{ch}\mu\sin\frac{n\pi}{2}}{\alpha_n^2+\beta^2}, \\ F_{n\beta} = \dfrac{4\beta}{a}\,\dfrac{\sin\frac{n\pi}{2}}{\alpha_n^2+\beta^2}\left(\mu\operatorname{sh}\mu + \dfrac{\alpha_n^2-\beta^2}{\alpha_n^2+\beta^2}\operatorname{ch}\mu\right), \\ C_{n\beta} = \dfrac{4}{\pi}\,\dfrac{\alpha_n\beta^2}{(\alpha_n^2+\beta^2)^2}, \end{cases}$$

we can reduce the system (3.15) to the form

$$(3.17)\quad \begin{cases} B_n^{(s)} + \dfrac{32n^2}{a\pi^2}\sin\dfrac{n\pi}{2}\displaystyle\int_0^{\infty}\dfrac{A^{(s)}\operatorname{ch}^2\mu\,d\mu}{\left(n^2+\frac{4\mu^2}{\pi^2}\right)^2\operatorname{sh}\mu} = -\dfrac{4KG\pi}{a^2}n' e^{-\frac{n\pi}{a}\eta'}\cos\dfrac{n\pi}{a}\xi', \\ \dfrac{16\mu^2 a}{\pi^4}\displaystyle\sum_{k=1,3,\ldots}^{\infty} B_k^{(s)}\dfrac{k\sin\frac{k\pi}{2}}{\left(k^2+\frac{4\mu^2}{\pi^2}\right)^2} + A^{(s)}t(\mu) = -\dfrac{4KG}{\pi a}\mu\varrho^{(s)}(\mu,\xi')\sin\dfrac{2\mu\eta'}{a}. \end{cases}$$

Substituting the function $A^{(s)}$ from the second of the Eqs. (3.17) in the first, we obtain an infinite system of equations involving unknown coefficients $B_n^{(s)}$. In this system, which has the form

$$B_n^{(s)} + \sum_{k=1}^{\infty} B_k^{(s)} \mathfrak{a}_{nk}(\mu) = \mathfrak{b}_n(\mu) \qquad (n = 1, 3, 5, \ldots), \tag{3.18}$$

complicated Fourier integrals appear in the expressions $\mathfrak{a}_{nk}(\mu)$ and $\mathfrak{b}_n(\mu)$. They should be calculated numerically. Confining ourselves to a few equations of the system (3.18), approximate values of $B_n^{(s)}$ can be obtained. From the second equation of the group (3.17) we obtain approximate values of $A^{(s)}$. The knowledge of the integration constants will enable us to determine the Airy function (3.12) and, in consequence, the additional stresses

$$\bar{\sigma}_{x'}^{(s)} = \frac{\partial^2 F^{(s)}}{\partial y'^2}, \qquad \bar{\sigma}_{y'}^{(s)} = \frac{\partial^2 F^{(s)}}{\partial x'^2}, \qquad \bar{\tau}_{x'y'}^{(s)} = -\frac{\partial^2 F^{(s)}}{\partial x' \partial y'}. \tag{3.19}$$

Consider now the action of two nuclei of intensity $\alpha_t T d\Omega/2$, located antisymmetrically in relation to the y'-axis. In the system of coordinates x', y' we have

$$\begin{cases} [\bar{\tau}_{x'y'}^{(a)}]_{x'=a/2} = \dfrac{2KG}{\pi h} \displaystyle\int_0^{\infty} \beta \varrho^{(a)}(\mu, \xi') \sin \beta \eta' \cos \beta y' \, d\beta, \\ [\bar{\bar{\tau}}_{x'y'}^{(a)}]_{x'=a/2} = -\dfrac{4KG}{ah} \displaystyle\sum_{n=2,4,\ldots}^{\infty} \alpha_n e^{-\alpha_n \eta'} \sin \alpha_n \xi' \cos \alpha_n x'. \end{cases} \tag{3.20}$$

The corresponding stresses $\bar{\bar{\sigma}}_{x'}^{(a)}$, $\bar{\bar{\sigma}}_{y'}^{(a)}$ and $\bar{\bar{\tau}}_{x'y'}^{(a)}$ can be found from the solution of the differential equation

$$\nabla^2 \nabla^2 F^{(a)} = 0, \tag{3.21}$$

with the boundary conditions

$$\begin{cases} \bar{\bar{\sigma}}_{x'}^{(a)} = \dfrac{\partial^2 F^{(a)}}{\partial y'^2} = 0, \qquad \bar{\bar{\tau}}_{x'y'}^{(a)} = -\dfrac{\partial^2 F^{(a)}}{\partial x' \partial y'} = -\bar{\tau}_{x'y'}^{(a)} \quad \text{for} \quad x' = \dfrac{a}{2}, \\ \bar{\bar{\sigma}}_{y'}^{(a)} = \dfrac{\partial^2 F^{(a)}}{\partial x'^2} = 0, \qquad \bar{\bar{\tau}}_{x'y'}^{(a)} = -\dfrac{\partial^2 F^{(a)}}{\partial x' \partial y'} = -\bar{\tau}_{x'y'}^{(a)} \quad \text{for} \quad y' = 0. \end{cases} \tag{3.22}$$

The function $F^{(a)}$ will be assumed in the form

$$F^{(a)} = \frac{1}{h} \sum_{n=2,4,\ldots}^{\infty} \frac{1}{\alpha_n^2} (A_n^{(a)} + B_n^{(a)} \alpha_n y') e^{-\alpha_n y'} \sin \alpha_n x + \frac{1}{h} \int_0^{\infty} \frac{1}{\beta^2} (A^{(a)} \operatorname{sh} \beta x' + B^{(a)} \beta x' \operatorname{ch} \beta x') \sin \beta y' \, d\beta. \tag{3.23}$$

The boundary conditions (3.22) lead to the system of equations:

$$(3.24)\quad \begin{cases} A_n^{(a)}=0, \qquad A_n^{(a)}\,\mathrm{sh}\,\mu+B^{(a)}\,\mu\,\mathrm{ch}\,\mu=0, \\ \sum\limits_{n=2,4,\dots}^{\infty} B_n^{(a)}\cos\alpha_n x'+\int\limits_0^{\infty}[(A^{(a)}+B^{(a)})\,\mathrm{ch}\,\beta x'+B^{(a)}\beta x'\,\mathrm{sh}\,\beta x']\,d\beta= \\ \qquad\qquad =-\dfrac{4KG}{a}\sum\limits_{n=2,4,\dots}^{\infty}\alpha_n e^{-\alpha_n\eta'}\sin\alpha_n\xi'\cos\alpha_n x', \\ \sum\limits_{n=2,4,\dots}^{\infty} B_n^{(a)}e^{-\alpha_n y'}(1-\alpha_n y')\cos\dfrac{n\pi}{2}+\int\limits_0^{\infty}[A^{(a)}+B^{(a)})\,\mathrm{ch}\,\mu+ \\ \qquad +B^{(a)}\mu\,\mathrm{sh}\,\mu]\cos\beta y'\,d\beta=\dfrac{2KG}{\pi}\int\limits_0^{\infty}\beta\,\varrho^{(a)}(\mu,\xi')\sin\beta\eta'\cos\beta y'\,d\beta. \end{cases}$$

Next, trigonometric series and a Fourier integral can be used to express the functions

$$\mathrm{ch}\,\beta x'=\sum_{n=1}^{\infty}G_{n\beta}\cos\alpha_n x', \qquad \beta x'\,\mathrm{sh}\,\beta x'=\sum_{n=1}^{\infty}H_{n\beta}\cos\alpha_n x',$$

$$(1-\alpha_n y')\,e^{-\alpha_n y'}=\int_0^{\infty}C_{n\beta}\cos\beta y'\,d\beta,$$

where

$$G_{n\beta}=\frac{4}{a}\,\mathrm{ch}\,\mu\,\frac{\alpha_n\sin\frac{n\pi}{2}}{\alpha_n^2+\beta^2},$$

$$H_{n\beta}=\frac{2\beta\,\alpha_n\sin\frac{n\pi}{2}}{\alpha_n^2+\beta^2}\left[\mathrm{sh}\,\mu-\frac{4\beta}{a}\,\frac{\mathrm{ch}\,\mu}{\alpha_n^2+\beta^2}\right], \qquad C_{n\beta}=\frac{4}{\pi}\,\frac{\alpha_n\beta^2}{(\alpha_n^2+\beta^2)^2}.$$

We substitute these functions in the Eqs. (3.24), and reduce (3.24) to the system of two equations:

$$(3.25)\quad \begin{cases} B_n^{(a)}+\int\limits_0^{\infty}A^{(a)}[\mathrm{c}(\mu)\,G_{n\beta}-\mathrm{d}(\mu)\,H_{n\beta}]\,d\beta=-\dfrac{4KG}{a}\,\alpha_n\,e^{-\alpha_n\eta'}\sin\alpha_n\xi', \\ \sum\limits_{n=2,4,\dots}^{\infty}B_n^{(a)}\,C_{n\beta}\cos\dfrac{n\pi}{2}+A^{(a)}\,\mathrm{f}(\mu)=\dfrac{2KG}{\pi}\,\beta\,\varrho^{(a)}(\mu,\xi')\sin\beta\eta', \end{cases}$$

where

$$\mathrm{c}(\mu)=\frac{\mu\,\mathrm{ch}\,\mu-\mathrm{sh}\,\mu}{\mu\,\mathrm{ch}\,\mu}, \qquad \mathrm{d}(\mu)=\frac{\mathrm{sh}\,\mu}{\mu\,\mathrm{ch}\,\mu}, \qquad \mathrm{f}(\mu)=\frac{\mathrm{sh}\,\mu\,\mathrm{ch}\,\mu-\mu}{\mu\,\mathrm{ch}\,\mu}.$$

The system of equations (3.25) is valid for $n=2, 4, 6, \dots$. On the other hand, $G_{n\beta}$ and $H_{n\beta}$ vanish for even n; it follows that the integral appearing in the first equation vanishes. For even n, the sum appearing in the second equation of the group (3.25) also vanishes. It is evident that the coefficients $B_n^{(a)}$, $A^{(a)}$ and $B^{(a)}$ can be directly obtained from the relations:

$$B_n^{(a)} = -\frac{4KG}{a} a_n e^{-a_n \eta} \sin a_n \xi' \qquad (n = 2, 4, 6, \ldots),$$

$$A^{(a)} = \frac{2KG}{\pi} \frac{\beta \varrho^{(a)}(\mu, \xi') \sin \beta \eta'}{f(\mu)}, \qquad B^{(a)} = -A^{(a)} \frac{th \mu}{\mu}.$$

Thus is determined the function $F^{(a)}$, enabling us to determine the stresses $\overline{\overline{\sigma}}_{x'}^{(a)}$, $\overline{\overline{\sigma}}_{y'}^{(a)}$ and $\overline{\overline{\tau}}_{x',y}^{(a)}$ from the Eqs. (1.5). The stresses due to the action of a nucleus of thermoelastic strain of intensity $a_t T d\Omega$ located at the point (ξ, η) will be obtained by means of superposition, e. g.: $\sigma_x = \overline{\sigma}_x + \overline{\overline{\sigma}}_x^{(a)} + \overline{\overline{\sigma}}_x^{(s)}$, etc.

4. An Elastic Semi-infinite Plate

Consider an elastic semi-infinite plate with a nucleus of thermoelastic strain at the point $(\xi, 0)$, Fig. 5. This case is equivalent to the case of two nuclei, one positive, at the point $(\xi, 0)$, the other negative, at the point $(-\xi, 0)$. In this case, we have $\Phi = 0$ on the straight line $x = 0$.

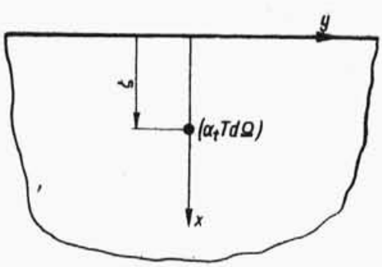

Fig. 5

Determining the right-hand side of the Eq. (1.2) by means of the Fourier integral

$$(4.1) \quad \frac{(1+\nu) a T d\Omega}{h} \delta(x-\xi) \delta(y) = \frac{(1+\nu) a T d\Omega}{h \pi^2} \int_0^\infty \int_0^\infty \cos \beta y [\cos a(x-\xi) - \cos a(x+\xi)] \, da \, d\beta,$$

and the function Φ by means of the integral

$$(4.2) \qquad \Phi = \int_0^\infty \int_0^\infty G(a, \beta) \cos \beta y \, [\cos a(x-\xi) - \cos a(x+\xi)] \, da \, d\beta,$$

we obtain the solution of the Eq. (1.2) in the form

$$(4.3) \quad \Phi = -\frac{2K}{h\pi^2} \int_0^\infty \int_0^\infty \frac{\cos \beta y \sin a\xi \sin a x}{a^2 + \beta^2} \, da \, d\beta, \qquad K = (1+\nu) a T d\Omega$$

or, after integrating,

$$(4.4) \qquad \Phi = -\frac{P}{4\pi h} \ln \frac{y^2 + (x+\xi)^2}{y^2 + (x-\xi)^2}.$$

This function has a singularity at the point $(\xi, 0)$. For $x \to \infty$, $y \to \infty$ the function Φ tends to zero.

From the Eqs. (1.3) we determine the stresses

$$
(4.5)\quad \begin{cases} \bar{\sigma}_x = -2\,G\dfrac{\partial^2 \Phi}{\partial y^2} = \dfrac{KG}{\pi h}\left\{\dfrac{(x+\xi)^2 - y^2}{[(x+\xi)^2 + y^2]^2} - \dfrac{(x-\xi)^2 - y^2}{[(x-\xi)^2 + y^2]^2}\right\} = -\bar{\sigma}_y, \\[2ex] \bar{\tau}_{xy} = 2\,G\dfrac{\partial^2 \Phi}{\partial x\,\partial y} = \dfrac{2\,KGy}{\pi h}\left\{\dfrac{x+\xi}{[(x+\xi)^2 + y^2]^2} - \dfrac{x-\xi}{[(x+\xi)^2 + y^2]^2}\right\}. \end{cases}
$$

Let us observe that these stresses increase infinitely when the point considered approaches $(\xi, 0)$. The stresses $\bar{\sigma}_x$ and $\bar{\sigma}_y$ are equal to zero for $x = 0$ and for $x \to \infty$, $y \to \infty$. For $x = 0$, the shear stress $\bar{\tau}_{xy}$ takes the value

$$
(4.6)\qquad [\bar{\tau}_{xy}]_{x=0} = \frac{4\,KGy\,\xi}{\pi h\,(\xi^2 + y^2)^2}.
$$

Over this state of stress should be superposed another state of stress $(\bar{\bar{\sigma}}, \bar{\bar{\tau}})$, such that

$$
(4.7)\qquad [\bar{\bar{\tau}}_{xy}]_{x=0} = -\frac{4\,KG}{\pi h}\,\frac{y\,\xi}{(\xi^2 + y^2)^2} = -[\bar{\tau}_{xy}]_{x=0} \qquad \text{and} \qquad [\bar{\bar{\sigma}}_x]_{x=0} = 0
$$

at the edge $x = 0$.

The A i r y function will be assumed in the form

$$
(4.8)\qquad F = \frac{1}{h}\int_0^\infty \frac{1}{\beta^2}(A + B\,\beta\,x)\,e^{-\beta x}\cos\beta y\,d\beta.
$$

It will be convenient to represent the stress $\bar{\tau}_{xy}$ in the form

$$
(4.9)\qquad \bar{\tau}_{xy} = \frac{4\,KG}{\pi^2 h}\int_0^\infty\int_0^\infty \frac{\alpha\beta\sin\beta y\sin\alpha\xi\cos\alpha x}{\alpha^2 + \beta^2}\,d\alpha\,d\beta,
$$

following directly from the integral expression (4.3).

For $x = 0$ we find

$$
(4.10)\qquad [\bar{\tau}_{xy}]_{x=0} = \frac{2\,KG}{\pi h}\int_0^\infty e^{-\beta\xi}\,\beta\sin\beta y\,d\beta.
$$

From the conditions (4.7) we find

$$
A = 0, \qquad B = -\frac{2\,KG}{\pi}\,\beta\,e^{-\beta\xi}.
$$

Thus

$$
(4.11)\qquad F = -\frac{2\,KG}{\pi h}\,x\int_0^\infty e^{-\beta(x+\xi)}\cos\beta y\,d\beta.
$$

The additional stresses will be expressed by the equations:

$$(4.12)\quad \begin{cases} \bar{\bar{\sigma}}_x = \dfrac{\partial^2 F}{\partial y^2} = \dfrac{2\,KG}{\pi h}\, x \displaystyle\int_0^\infty e^{-\beta(x+\xi)}\, \beta^2 \cos\beta y\, d\beta, \\ \bar{\bar{\sigma}}_y = \dfrac{\partial^2 F}{\partial x^2} = \dfrac{2\,KG}{\pi h} \displaystyle\int_0^\infty \beta\,(2-\beta x)\, e^{-\beta(x+\xi)} \cos\beta y\, d\beta, \\ \bar{\bar{\tau}}_{xy} = \dfrac{\partial^2 F}{\partial x\, \partial y} = -\dfrac{2\,KG}{\pi h} \displaystyle\int_0^\infty \beta\,(1-\beta x)\, e^{-\beta(x+\xi)} \sin\beta y\, d\beta, \end{cases}$$

or

$$(4.13)\quad \begin{cases} \bar{\bar{\sigma}}_x = \dfrac{4\,KG}{\pi h}\, x\,(x+\xi)\, \dfrac{(x+\xi)^2 - 3\,y^2}{[(x+\xi)^2 + y^2]^2}, \\ \bar{\bar{\sigma}}_y = \dfrac{4\,KG}{\pi h}\, \dfrac{1}{[(x+\xi)^2+y^2]^2} \left[(x+\xi)^2 - y^2 - x\,(x+\xi)\, \dfrac{(x+\xi)^2 - 3\,y^2}{(x+\xi)^2 + y^2} \right], \\ \bar{\bar{\tau}}_{xy} = -\dfrac{4\,KG}{\pi h}\, \dfrac{1}{(x+\xi)^2 + y^2} \left[(x+\xi) - x\, \dfrac{3\,(x+\xi)^2 - y^2}{(x+\xi)^2 + y^2} \right]. \end{cases}$$

Let us observe that for $x=0$ we have $\bar{\bar{\sigma}}_x = 0$ and $\bar{\bar{\tau}}_{xy} = -\bar{\bar{\tau}}_{xy}$. For $x=\xi$ the stresses show no singularities. Finally, for $x \to \infty$, $y \to \infty$ the stresses tend to zero.

Shifting the nucleus of thermoelastic strain to the point (ξ, η), y should be replaced in the Eqs. (4.1) to (4.13) by $y-\eta$.

For a temperature distribution such that T is a continuous function in the region Ω and $T=0$ outside this region, the stress $\sigma^*(x,y)$ will be determined by integration:

$$\sigma^*(x,y) = \iint_{(\Omega)} \sigma(x,y;\xi,\eta)\, T(\xi,\eta)\, d\xi\, d\eta.$$

References

[1] J. N. Goodier, *On the Integration of the Thermo-Elastic Equations*, Phil. Mag., Vol. 23, 1937, p. 1017.

[2] E. Melan a. H. Parcus, *Wärmespannungen stationärer Temperaturfelder*, Vienna 1953.

[3] G. Paria, *Stresses in an Infinite Strip Due to a Nucleus of Thermo-Elastic Strain in it*, Bull. Calcutta Mathem. Soc., Vol. 45, 3 (1953).

Streszczenie

O NAPRĘŻENIACH WYWOŁANYCH W TARCZY DZIAŁANIEM JĄDRA TERMOSPRĘŻYSTEGO ODKSZTAŁCENIA

Celem pracy jest wyznaczenie stanu naprężenia wywołanego w tarczy działaniem jądra termosprężystego odkształcenia. Rozwiązanie tego podstawowego zagadnienia — wyznaczenia naprężeń w postaci funkcji Greena — pozwala drogą całkowania na wyznaczenie stanu naprężenia wywołanego działaniem nieciągłego pola temperatury. Znajomość funkcji Greena pozwala też na wyznaczenie stanu naprężenia w tarczy podgrzanej do temperatury T, posiadającej inkluzję o innych własnościach termicznych niż reszta tarczy.

W pracy wykorzystano formalną analogię, jaka istnieje pomiędzy równaniem różniczkowym potencjału termosprężystego i równaniem różniczkowym ugięcia błony. Dzięki tej analogii spełniono część warunków brzegowych; mianowicie na brzegu tarczy uzyskano zerowe wartości naprężeń normalnych, ale różne od zera wartości naprężeń ścinających.

Dla zniweczenia naprężeń ścinających dodajemy do uzyskanego stanu naprężenia drugi stan naprężeń, uzyskany z rozwiązania zagadnienia brzegowego dla płaskiego stanu naprężenia za pomocą funkcji Airy.

W pracy szczegółowo rozpatrzono stan naprężenia wywołany działaniem jądra termosprężystego odkształcenia: (1) w paśmie tarczowym nieskończenie długim, (2) w półpaśmie tarczowym, (3) w półpłaszczyźnie tarczowej.

Резюме

О НАПРЯЖЕНИЯХ, ВЫЗЫВАЕМЫХ В ДИСКЕ ДЕЙСТВИЕМ ЯДРА ТЕРМОУПРУГОЙ ДЕФОРМАЦИИ

Работа задается целью определить напряжения, вызванные в диске действием ядра термоупругой деформации. Решение этой основной проблемы, определения напряжений в виде функций Грина, даст возможность определить, при помощи интегрирования, напряжения, вызванные действием прерывного поля температуры. Знание функции Грина позволит вместе с тем, определить напряжения в диске, подогретом до температуры T, с включением, обладающим другими термическими свойствами, чем остальная часть диска.

В работе используется формальная аналогия, существующая между дифференциальным уравнением термоупругого потенциала и дифференциальным уравнением прогиба мембраны. Благодаря этой аналогии, удовлетворяются некоторые граничные условия, а именно на

контуре диска получены нулевые значения нормальных напряжений, но различные от нуля значения касательных напряжений.

Для нейтрализации касательных напряжений, прибавляют к полученному напряженному состоянию второе напряженное состояние, получаемое из решения граничной задачи для плоского напряженного состояния при помощи функции Эри.

Работа подробно рассматривает напряженное состояние, вызванное действием ядра термоопругой деформации: (1) в бесконечно длинной полосе, (2) в полуполосе, (3) в полуплоскости.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA
IBTP POLISH ACADEMY OF SCIENCES

Received August 3, 1956