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SOME PROBLEMS OF STRUCTURAL ANALYSIS OF PLATES WITH MIXED BOUNDARY CONDITIONS

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Introduction

During the last few years, there has been apparent a growing interest in problems of structural analysis of plates with mixed (discontinuous) boundary conditions. These problems can be solved in many ways: by reducing the problem to the solution of a system of Fredholm integral equations of the first kind, [1], [2], by methods of analytic functions, [3], [4], or directly by integrating the differential equation of structural analysis of plates, [5]. In the present paper, the problem of structural analysis of plates is stated in a more general way, considering elastic clamping and elastic support along the boundary and inside the region of the plate. The solution of the problem is reduced to a system of Fredholm equations of the second kind which in certain particular cases become Fredholm equations of the first kind. In Secs. 1 and 2 certain concrete solutions for rectangular plates are reviewed.

1. General Statement of the Problem

Consider a plate elastically clamped and elastically supported along n segments s_r, $r = 1, 2, \ldots, n$, of the boundary, and simply supported or free along the remaining segments. Let the external load p be distributed in an arbitrary way over the plate.

Denote by $M_r(S_r)$, $r = 1, 2, \ldots, n$, the moments of elastic clamping of the plate along the segments s_r of the boundary, and by $R_r(S_r)$, $r = 1, 2, \ldots, n$, the elastic reaction forces along these segments.

If we put $p = 0$, $M_r = 0$ and $R_r = 0$, $r = 1, 2, \ldots, n$, we are concerned with an unloaded plate with the segments s_r free. This plate will be called the basic system, for which kinematic invariability is required.

Let the basic system be subjected to the external load only. Denote by $w_0(P)$ the deflection at the point P of the basic system, due to this load.

Let the concentrated unit force $R_r(S_r) = 1$ now act at the point S_r of the edge s_r in the basic system. The deflection at the point P of the basic system due to this force will be denoted by $K_r(P, S_r)$. Similarly, the deflection at the point P of the basic system due to the action of the concentrated unit moment $M_r(S_r) = 1$ at the point S_r of the edge s_r will be denoted by $G_r(P, S_r)$. The vector of the moment is directed tangentially to the edge. We assume that the function $w_0(P)$, $K_r(P, S_r)$ and $G_r(P, S_r)$ can be determined in the basic system.

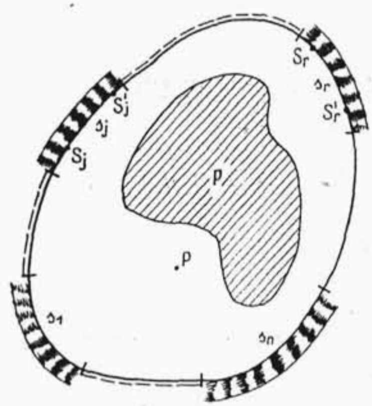

Fig. 1

Summing up the influences of the external load, the moments of elastic clamping M_r and the elastic support R_r, we obtain the deflection at the point P of the plate with mixed boundary conditions in the form of the integral expression

$$w(P) = w_0(P) + \sum_{r=1}^{r=n} \int_{s_r} [R_r(S_r)\, K_r(P, S_r) + M_r(S_r)\, G_r(P, S_r)]\, ds_r. \tag{1.1}$$

The mixed functions $R_r(S_r)$ and $M_r(S_r)$ appearing in the relation (1.1) are to be obtained by considering the boundary conditions for the segments s_j of the boundary.

These boundary conditions are

$$w(S_j') = -\frac{1}{k_j} R_j(S_j'), \tag{1.2}$$

$$\frac{\partial w(S_j')}{\partial n_j} = -\frac{1}{r_j} M_j(S_j'), \qquad (j = 1, 2, \ldots, n). \tag{1.3}$$

The first condition expresses the proportionate relation of the deflection of the point S_j' of the segment s_j to the elastic reaction fore $R_j(S_j')$ at that point. The proportionate relation coefficient is W i n k l e r's constant k_j.

The second condition expresses the relation between the moment $M_j(S_j')$ and the derivative of the deflection function in the direction of the normal n_j at the point S_j' of the segment s_j. The proportionate relation coefficient is the constant of elastic clamping r_j.

Performing on the expression (1.1) the operations indicated in the Eqs. (1.2) and (1.3), and moving the point P inside the region of the plate to coincide with the point S_j' of the segment s_j of the boundary, we obtain the system of Fredholm equations of the second kind

$$(1.4)\quad \begin{cases} -\dfrac{R_j(S_j')}{k_j} = w_0(S_j') + \displaystyle\sum_{r=1}^{r=n} \int_{s_r} [R_r(S_r)\,K_r(S_j', S_r) + M_r(S_r)\,G_r(S_j', S_r)]\,ds_r, \\ -\dfrac{M_j(S_j')}{r_j} = \dfrac{\partial w_0(S_j')}{\partial n_j} + \displaystyle\sum_{r=1}^{r=n} \int_{s_r} \left[R_r(S_r)\,\dfrac{\partial K_r(S_j', S_r)}{\partial n_j} + \right. \\ \qquad\qquad \left. + M_r(S_r)\,\dfrac{\partial G_r(S_j', S_r)}{\partial n_j}\right] ds_r \end{cases}$$

$$(j = 1, 2, \dots, n).$$

Solving this system, we obtain the functions $R_r(S_r)$ and $M_r(S_r)$, which, substituted in the relation (1.1), enable us to determine the deflection $w(P)$.

In the particular case where $k_j = \infty$ and $r_j = \infty$, $j = 1, 2, \dots, n$, we obtain a plate rigidly clamped along the segments s_j of the boundary. It is evident that in this case the system (1.4) becomes a system of Fredholm equations of the first kind.

If $r_j = \infty$, $j = 1, 2, \dots, n$, we obtain the case in which the segments s_j are elastically supported and the functions $\partial w/\partial n_j$ are equal to zero.

The system (1.4) becomes a system of mixed integral equations of the first and the second kind.

In the case of $k_j = \infty$, $j = 1, 2, \dots, n$, we are concerned with a plate elastically clamped and having rigid supports along the segments s_j.

This case can be reduced to a more simple system of integral equations if we take as basic another system in which the plate is simply supported along the segments s_r, $r = 1, 2, \dots, n$. If we denote the deflection at the point P of the basic system due to the action of external load by $\overline{w}_0(P)$ and the deflection in the basic system due to the action of the concentrated unit moment $M_r(S_r) = 1$ at the point S_r of the edge s_r — by $\overline{G}_r(P, S_r)$, the deflection at the point P of a plate with mixed boundary conditions will be expressed as

$$(1.5)\qquad w(P) = \overline{w}_0(P) + \sum_{r=1}^{r=n} \int_{s_r} M_r\,\overline{G}_r(P, S_r)\,ds_r.$$

From the boundary conditions (1.3) for the segments s_j, we obtain the system of Fredholm equations of the second kind

$$(1.6)\qquad -\frac{M_j(S_j')}{r_j} = \frac{\partial \overline{w}_0(S_j')}{\partial n_j} + \sum_{r=1}^{r=n} \int_{s_r} M_r(S_r)\,\frac{\partial \overline{G}_r(S_j', S_r)}{\partial n_j}\,ds_r$$

$$(j = 1, 2, \dots, n).$$

For $r_j = \infty$, $j = 1, 2, \ldots, n$, i.e. in the case of rigid clamping along the segments s_j of the boundary, this system becomes a system of Fredholm equations of the first kind.

Let us return, now, to the system of equations (1.4) and consider two limit cases:

(a) Let us disregard the elastic clamping along the segments s_j. Then, with $M_j = 0$, $j = 1, 2, \ldots, n$, the unknown functions $R_r(S_r)$ will be determined from the system of equations,

$$(1.7) \qquad -\frac{R_j(S_j')}{k_j} = w_0(S_j') + \sum_{r=1}^{r=n} \int_{s_r} R_r(S_r) K_r(S_j', S_r) \, ds_r \qquad (j = 1, 2, \ldots, n).$$

(b) Let us disregard the elastic support along the segments s_j. Then with $R_j = 0$, $j = 1, 2, \ldots, n$, we have

$$(1.8) \qquad -\frac{M_j(S_j')}{r_j} = \frac{\partial w_0(S_j')}{\partial n_j} + \sum_{r=1}^{r=n} \int_{s_r} M_r \frac{\partial G_r(S_j', S_r)}{\partial n_j} \, ds_r \qquad (j = 1, 2, \ldots, n).$$

Our considerations can easily be generalized to the case of a plate elastically clamped and supported along the segments s_j, $j = 1, 2, \ldots, n$, elastically supported along the segments l_i, $i = 1, 2, \ldots, p$, and elastically clamped along the segments m_r, $r = 1, 2, \ldots, t$, of the boundary.

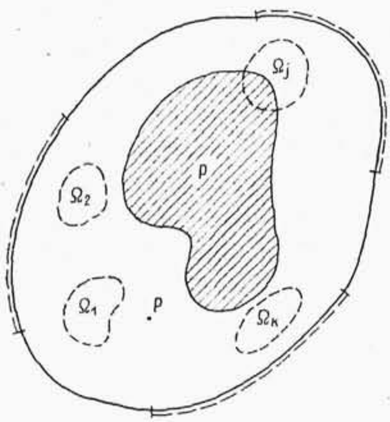

Fig. 2

The system of equations (1.4) will also be valid in the case in which clamped and supported edges s_j are not parts of the boundary but are located inside the region of the plate. Finally, our considerations can be generalized to plates elastically clamped and supported along segments constituting parts of the boundary and along segments located inside the region of the plate.

Another problem of practical importance can also be reduced to integral equations.

Let a plate, supported in any way on the boundary, have additional supports over the regions $\Omega_1, \Omega_2, \ldots, \Omega_p$ constituting parts of the region Ω of the plate.

The deflection of the plate will be determined by

$$w(P) = w_0(P) + \sum_{r=1}^{r=p} \iint_{\Omega_r} R_r(S_r) K_r(P, S_r)\, d\Omega_r. \tag{1.9}$$

In this relation, $w_0(P)$ denotes the deflection of the basic system (for which the plate supported on the boundary only is assumed), due to the action of the external load p, and $K_r(P, S_r)$ — the deflection at the point P of the basic system due to the action of the concentrated force $R_r(S_r) = 1$ at the point S_r of the region Ω_r. We assume that the quantities $w_o(P)$ and $K_r(P, S_r)$ can be determined in the basic system.

The unknown functions R_r will be determined using the boundary conditions for the regions Ω_j, $j = 1, 2, ..., p$:

$$w(S'_j) = -\frac{R_j(S'_j)}{k_j} \qquad (j = 1, 2, ..., p), \tag{1.10}$$

where an elastic foundation of the W i n k l e r type has been assumed. Moving the point P to coincide with the point S'_j of the region Ω_j, we obtain from the relation (1.9) and bearing in mind (1.10), the system of integral equations:

$$-\frac{R_j(S'_j)}{k_j} = w_0(S'_j) + \sum_{r=1}^{r=p} \iint_{\Omega_r} R_r(S_r) K_r(S'_j, S_r)\, ds_r. \tag{1.11}$$

This is a system of F r e d h o l m equations of the second kind. Let us observe that for $k_j = \infty$, $j = 1, 2, ..., p$, the system of equations (1.11) has no solution, the functions $R_j(S_j)$ and $w(S_j)$ being in this case equal to zero over the regions Ω_j. The solution must be sought in another way. The plate should be treated as a multiply-connected region rigidly clamped along the boundary of the region Ω_j, $j = 1, 2, ..., p$. Unknown functions representing the moments $M_j(S_j)$ will appear along these boundaries. These moments will be found from the system of integral equations (1.6), in which we should put $r_j = \infty$, $n = k$, and in which the basic system is represented by the same plate supported on the external boundary and simply supported on the boundaries of the regions Ω_j.

Some concrete problems of structural analysis of rectangular plates will be based on these general statements. Our considerations will be confined to two groups of problems, the first concerning plates with rigid supports and discontinuous boundary conditions, the second — plates elastically clamped and supported along certain segments and elastically supported over certain regions.

2. Rectangular Plates with Rigid Supports

We confine our considerations to rectangular plates with mixed boundary conditions at one edge only. The F r e d h o l m equations of the first kind will be derived in a manner differing from that in the first section of this paper. Thus, the advantages of a proper choice of basic systems will be seen more clearly.

Consider a rectangular plate loaded in any way. Let there be two continuous boundary conditions at each of the edges $x = 0$, $x = a$ and $y = b$, and let the boundary conditions at the edge $y = 0$ be homogeneous and expressed by

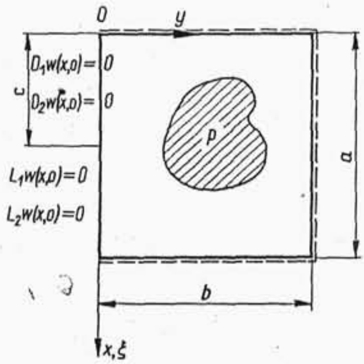

Fig. 3

$$(2.1)\qquad D_1 w(x, 0) = 0, \qquad D_2 w(x, 0) = 0,$$

for the interval $(0, c)$, and

$$(2.2)\qquad L_1 w(x, 0) = 0, \qquad L_2 w(x, 0) = 0,$$

for the interval (c, a), where D_1, D_2, L_1 and L_2 denote continuous linear differential operators.

$D_1 w(x, 0)$ and $D_2 w(x, 0)$ are unknown functions in the interval (c, a) and $L_1 w(x, 0)$ and $L_2 w(x, 0)$ — unknown functions in the interval $(0, c)$.

Let us take as basic the system in which the boundary conditions for the edge $y = 0$, both in the interval $(0, c)$ and (c, a), are $L_1 w(x, 0) = 0$ and $L_2 w(x, 0) = 0$.

Let the deflection surface in such a basic system due to the external load be denoted by $w_0(x, y)$. Let now the boundary condition for $L_1 w(x, 0)$ in the basic system, be such that $L_1 w(x, 0)$ is equal to zero over the interval $(0, a)$, except at the point $(\xi, 0)$, $0 < \xi < c$, where $L_1 w(\xi, 0) = 1$.

This state will provoke a deflection of the plate, denoted by $G_1(x, y; \xi, 0)$.

Similarly for $L_2 w(\xi, 0) = 1$ at the point $(\xi, 0)$ and $L_2 w(x, 0) = 0$ for $x \gtreqless \mathfrak{f}$ we obtain the deflection surface of the plate denoted by $G_2(x, y; \xi, 0)$.

The deflection of a plate with a discontinuous boundary condition for the interval $(0, a)$ will be described by the integral relation

$$(2.3)\qquad w(x, y) = w_0(x, y) + \int_0^c L_1 w(\xi, 0)\, G_1(x, y; \xi, 0)\, d\xi + \int_0^c L_2 w(\xi, 0)\, G_2(x, y; \xi, 0)\, d\xi.$$

The integrals appearing in this relation express the influences on plate deflection of the unknown functions $L_1 w$ and $L_2 w$ in the interval $(0, c)$.

Since the Eqs. (2.1) express the boundary conditions for the interval $(0, c)$, we obtain, performing the operations D_1 and D_2 on the relation (2.3) and moving the point (x, y) to the edge of the plate, the system of integral equations

$$
(2.4) \quad \begin{cases} D_1 w(x,0) = D_1 w_0(x,0) + \int\limits_0^c L_1 w(\xi,0) D_1 G_1(x,0;\xi,0)\, d\xi + \\ \qquad\qquad + \int\limits_0^c L_2 w(\xi,0) D_1 G_2(x,0;\xi,0)\, d\xi = 0, \\ D_2 w(x,0) = D_2 w_0(x,0) + \int\limits_0^c L_1 w(\xi,0) D_2 G_1(x,0;\xi,0)\, d\xi + \\ \qquad\qquad + \int\limits_0^c L_2 w(\xi,0) D_2 G_2(x,0;\xi,0)\, d\xi, \end{cases}
$$

where $0 \leqq \xi \leqq c,\ 0 \leqq x < c$.

We have thus obtained a system of two Fredholm equations of the first kind. In these equations we have, according to the theorem of E. Betti,

$$D_1 G_2 w(x,0;\xi,0) = D_2 G_1 w(x,0;\xi,0).$$

The basic system can also be chosen in another way, with $D_1 w(x, 0) = 0$ and $D_2 w(x, 0) = 0$ as boundary conditions for the edge $y = 0$.

If, in this system, $\overline{w}_0(x, y)$ denotes the deflection due to the external load, $N_1(x, y; \xi, 0)$ — the deflection due to the non-homogeneous boundary condition $L_1 w(x, 0) = 1$ for $\xi = x$ and $L_1 w(x, 0) = 0$ for $x \neq \xi$ and $N_2(x, y; \xi, 0)$ — the deflection due to the non-homogeneous boundary condition, $L_2 w(x, 0) = 1$ for $x = \xi$ and $L_2 w(x, 0) = 0$ for $\xi \neq x$, then the deflection of the plate with discontinuous boundary conditions for the edge $y = 0$ will be expressed by

$$
(2.5) \quad w(x,y) = \overline{w}_0(x,y) + \int\limits_c^a D_1 w(\xi,0) N_1(x,y;\xi,0)\, d\xi + \\ + \int\limits_c^a D_2 w(\xi,0) N_2(x,y;\xi,0)\, d\xi.
$$

Performing the operations L_1 and L_2 on the relation (2,5), and moving the point (x, y) to the edge $y = 0$ of the plate, we obtain the system of integral equations:

$$
(2.6) \quad \begin{cases} L_1 w(x,0) = L_1 \overline{w}_0(x,0) + \int\limits_c^a D_1 w(\xi,0) L_1 N_1(x,0;\xi,0)\, d\xi + \\ \qquad\qquad + \int\limits_c^a D_2 w(\xi,0) L_1 N_2(x,0;\xi,0)\, d\xi = 0, \\ L_2 w(x,0) = L_2 \overline{w}_0(x,0) + \int\limits_c^a D_1 w(\xi,0) L_2 N_1(x,0;\xi,0)\, d\xi + \\ \qquad\qquad + \int\limits_c^a D_2 w(\xi,0) L_2 N_2(x,0;\xi,0)\, d\xi, \end{cases}
$$

$$
c < \xi \leqq a, \qquad 0 \leqq x \leqq a,
$$

where

$$
L_1 N_2(x,0;\xi,0) = L_2 N_1(x,0;\xi,0).
$$

Solving the system (2.4), we obtain the functions $L_1 w(\xi, 0)$ and $L_2 w(\xi, 0)$, which substituted in the relation (2.3) enable us to determine the deflection of the plate $w(x, y)$ having discontinuous boundary conditions at the edge $y = 0$. Similarly, solving the system of integral equations (2.6), we obtain the functions $D_1 w(\xi, 0)$ and $D_2 w(\xi, 0)$. The knowledge of these functions enables us to determine the deflection of the plate $w(x, y)$ from the Eq. (2.5).

In a case where the boundary conditions $D_2 w(x, 0)$ and $L_2 w(x, 0)$ are identical, the system of integral equations reduces to the equation

$$
(2.7) \quad D_1 w(x,0) = D_1 w_0(x,y) + \int\limits_0^c L_1 w(\xi,0) D_1 G_1(x,0;\xi,0)\, d\xi = 0.
$$

Similarly the system of equations (2.6) reduces to

$$
(2.8) \qquad L_1 w(x,0) = L_1 \overline{w}_0(x,0) + \int\limits_c^a D_1 w(\xi,0) L_1 N_1(x,0;\xi,0)\, d\xi.
$$

Our considerations were confined to the solution for a plate having different boundary conditions over two segments of an edge. There is no difficulty in generalizing the above considerations to any number of segments. This can be done for one or more edges.

Returning to the system (2.4), let us rewrite it in a slightly different form:

$$
(2.9) \qquad \begin{cases} \int\limits_0^c X_1(\xi) K_{11}(x,\xi)\, d\xi + \int\limits_0^c X_2(\xi) K_{12}(x,\xi)\, d\xi = f_1(x), \\ \int\limits_0^c X_1(\xi) K_{21}(x,\xi)\, d\xi + \int\limits_0^c X_2(\xi) K_{22}(x,\xi)\, d\xi = f_2(x). \end{cases}
$$

The solution of this system of integral equations can be reduced to that of an infinite number of equations in which the unknowns are the Fourier coefficients for the functions $X_1(\xi)$ and $X_2(\xi)$.

Let us assume any set of functions $\varphi_\alpha(x)$, $\alpha = 1, 2, \ldots, \infty$, complete and orthogonal over the interval $(0, c)$, and an arbitrary set of functions $\psi_\beta(\xi)$, $\beta = 1, 2, \ldots, \infty$, complete and orthogonal over the interval $(0, a)$. Let us assume that the functions $X_1(\xi)$ and $X_2(\xi)$ can be expanded in a series with respect to the functions ψ_β, [6]:

$$X_1(\xi) = \sum_{\beta=1}^{\infty} X_{1,\beta}\, \psi_\beta(\xi), \qquad X_2(\xi) = \sum_{\beta=1}^{\infty} X_{2,\beta}\, \varphi_\beta(\xi),$$

where

$$X_{1,\beta} = \int_0^c X_1(\xi)\, \psi_\beta(\xi)\, d\xi, \qquad X_{2,\beta} = \int_0^c X_2(\xi)\, \psi_\beta(\xi)\, d\xi.$$

Let us multiply the system (2.1) by $\varphi_\alpha(x)$ and integrate it with respect to x over the interval $(0, a)$. We thus obtain an infinite system of equations:

$$\left\{\begin{aligned} &\sum_{\beta=1}^{\infty} \left(X_{1,\beta}\, a_{\alpha\beta}^{(1,1)} + X_{2,\beta}\, a_{\alpha\beta}^{(1,2)}\right) = f_{1,\alpha}, \\ &\sum_{\beta=1}^{\infty} \left(X_{1,\beta}\, a_{\alpha\beta}^{(2,1)} + X_{2,\beta}\, a_{\alpha\beta}^{(2,2)}\right) = f_{2,\alpha}, \end{aligned}\right. \qquad (\alpha = 1, 2, \ldots, \infty), \tag{2.10}$$

where

$$\left\{\begin{aligned} &a_{\alpha\beta}^{(i,j)} = \int_0^a \varphi_\alpha(x) \left[\int_0^c K_{\alpha j}(x, \xi)\, \psi_\beta(\xi)\, d\xi\right] dx \qquad (i, j = 1, 2), \\ &f_{\alpha,i} = \int_0^a \varphi_\alpha(x)\, f_i(x)\, dx \qquad (i = 1, 2). \end{aligned}\right. \tag{2.11}$$

The reduction of the system of integral equations (2.9) to the infinite number of equations for the Fourier coefficients of the functions $X_1(\xi)$ and $X_2(\xi)$, can also be done in another way which in our case will prove to be more convenient. Let us take only one complete set of functions orthogonal over the interval $(0, c)$, and let us expand the functions X_1 and X_2 in a series with respect to these functions:

$$X_1(\xi) = \sum_{\beta=1}^{\infty} Y_{1,\beta}\, \varphi_\beta(\xi), \qquad X_2(\xi) = \sum_{\beta=1}^{\infty} Y_{2,\beta}\, \varphi_\beta(\xi),$$

where

$$Y_{1,\beta} = \int_0^c X_1(\xi)\, \varphi_\beta(\xi)\, d\xi, \qquad Y_{2,\beta} = \int_0^c X_2(\xi)\, \varphi_\beta(\xi)\, d\xi.$$

Let us multiply the system of equations (2.9) by $\varphi_\alpha(x)$, and integrate with respect to x in the interval $(0, a)$. We thus obtain the infinite system of equations:

$$\left\{\begin{aligned}&\sum_{\beta=1}^{\infty}(Y_{1,\beta}b_{\alpha\beta}^{(1,1)}+Y_{2,\beta}b_{\alpha\beta}^{(1,2)})=f_{1,\alpha},\\&\sum_{\beta=1}^{\infty}(Y_{1,\beta}b_{\alpha\beta}^{(2,1)}+Y_{2,\beta}b_{\alpha\beta}^{(2,2)})=f_{2,\alpha},\end{aligned}\right.\qquad(\alpha=1,2,\dots,\infty),\tag{2.12}$$

where

$$\left\{\begin{aligned}&b_{\alpha\beta}^{(ij)}=\int_0^a\varphi_\alpha(x)\left[\int_0^c K_{\alpha j}(x,\xi)\,\varphi_\beta(\xi)\,d\xi\right]dx\qquad(i,j=1,2),\\&f_{i,\alpha}=\int_0^a\varphi_\alpha(x)\,f_i(x)\,dx\qquad(i=1,2).\end{aligned}\right.\tag{2.13}$$

This method of solution of the system of equations (2.9) will be applied to certain simple cases of bending of a rectangular plate with discontinuous boundary conditions at one edge of the plate.

A. Consider a rectangular plate simply supported on the edges $x = 0$, $x = a$ and $y = b$, rigidly fixed along the segment $(0, c)$ of the edge $y = 0$ and simply supported along the segment (c, a) of that edge. Let the plate be subjected to a uniform load p. The boundary conditions are: for the edges $x = 0$, $x = a$ and $y = 0$, we have $w = 0$ and $\nabla^2 w = 0$; for the edge $y=0$, over the interval $(0, c)$, we have

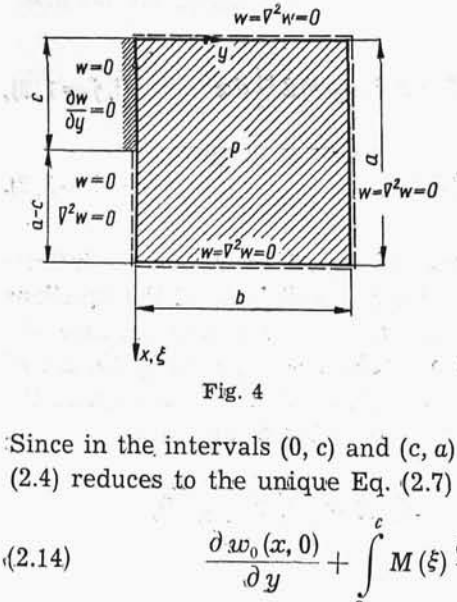

Fig. 4

$$D_1\,w(x,0)=\frac{\partial w(x,0)}{\partial y}=0$$

and

$$D_2\,w(x,0)=w(x,0)=0.$$

For the edge $y = 0$, over the interval (c, a), we have

$$L_1\,w(x,0)=-N\frac{\partial^2 w(x,0)}{\partial y^2}=0$$

and

$$L_2\,w(x,0)=w(x,0)=0.$$

Since in the intervals $(0, c)$ and (c, a) $w(x,0)=0$, the system of equations (2.4) reduces to the unique Eq. (2.7) which we rewrite as

$$\frac{\partial w_0(x,0)}{\partial y}+\int_0^c M(\xi)\,\frac{\partial G_1(x,0;\xi,0)}{\partial y}\,d\xi=0,\tag{2.14}$$

$$0<\xi<c,\qquad 0<x<a,$$

where

$$D_1 = \frac{\partial}{\partial y}, \qquad L_1 w(\xi, 0) = -N \frac{\partial^2 w(\xi, 0)}{\partial y^2} = M(\xi)$$

is the clamping moment of the plate along the segment $(0, c)$; $G_1(x, y; \xi, 0)$ — the deflection due to the action of the concentrated moment at the point $(\xi, 0)$ of the segment $(0, c)$ in the basic system (a plate simply supported on all edges); $w_0(x, y)$ — the deflection in the basic system, due to the load p.

Using the known solutions for w_0 and G_1, [7]:

$$(2.15) \qquad \begin{cases} w_0(x, y) = \dfrac{4p}{Na} \displaystyle\sum_{n=1,3,\ldots}^{\infty} \dfrac{1}{\alpha_n^5} \Big[1 - \cosh \alpha_n y + \Big(1 - \dfrac{\lambda_n}{2 \sinh \lambda_n} \Big) \times \\ \quad \times \operatorname{tgh} \dfrac{\lambda_n}{2} \sinh \alpha_n y + \dfrac{1}{2} \alpha_n y \Big(\sinh \alpha_n y - \operatorname{tgh} \dfrac{\lambda_n}{2} \cosh \alpha_n y \Big) \Big] \sin \alpha_n x, \\ G_1(x, y; \xi, 0) = \dfrac{1}{Na} \displaystyle\sum_{n=1,2,\ldots}^{\infty} \dfrac{\sin \alpha_n \xi}{\alpha_n^2 \sinh \lambda_n} \Big[\alpha_n y \cosh \alpha_n (b - y) - \\ \qquad - \lambda_n \dfrac{\sinh \alpha_n y}{\sinh \lambda_n} \Big] \sin \alpha_n x, \end{cases}$$

$$\alpha_n = \frac{n\pi}{a}, \qquad \lambda_n = \alpha_n b,$$

we can rewrite the Eq. (2.14) in the form

$$(2.16) \qquad \int_0^c M(\xi) K(x, \xi) \, d\xi = f(x),$$

where

$$K(x, \xi) = \sum_{n=1}^{\infty} \frac{Q_n}{\alpha_n} \sin \alpha_n \xi \sin \alpha_n x, \qquad f(x) = -2p \sum_{n=1,3,\ldots}^{\infty} \frac{F_n}{\alpha_n^4} \sin \alpha_n x,$$

$$Q_n = \operatorname{ctgh} \lambda_n - \frac{\lambda_n}{\sinh^2 \lambda_n}, \qquad F_n = \operatorname{tgh} \frac{\lambda_n}{2} - \frac{\frac{\lambda_n}{2}}{\cosh \frac{\lambda_n}{2}}.$$

For the set of orthogonal functions, we take the trigonometric functions

$$\varphi_\alpha(x) = \sqrt{\frac{2}{\pi}} \sin \frac{\alpha \pi x}{c} \qquad (\alpha = 1, 2, \ldots, \infty, \quad 0 < x < c).$$

Multiplying the Eq. (2.16) by $\varphi_\alpha(x)$, integrating from 0 to c, and assuming that $M(\xi)$ can be represented by the trigonometric series

$$M(\xi) = \sum_{\beta=1}^{\infty} M_\beta \sin \frac{\beta \pi x}{c} \qquad (0 < \xi < c),$$

we obtain, after the operations indicated, the infinite system of equations

$$\sum_{\beta=1}^{\infty} M_\beta b_{\alpha\beta} = f_\alpha \qquad (\alpha = 1, 2, \ldots, \infty), \tag{2.17}$$

where

$$b_{\alpha\beta} = \alpha\beta \cos \alpha\pi \cos \beta\pi \sum_{n=1,3,\ldots}^{\infty} \frac{\sin^2 \frac{n\pi}{\varrho} Q_n}{n(n^2 - \varrho^2\alpha^2)(n^2 - \varrho^2\beta^2)},$$

$$f_\alpha = -\frac{2pa^2}{\varrho\pi^2} \alpha \cos \alpha\pi \sum_{n=1,3,\ldots}^{\infty} \frac{\sin \frac{n\pi}{\varrho} F_n}{n^4(n^2 - \varrho^2\alpha^2)}, \qquad \left(\varrho = \frac{a}{c}\right).$$

Observe that in the case of $c = a$, i. e. for continuous boundary conditions along the edge $y = 0$ of the plate, the system of equations (2.17) will take the simplified form

$$\left\{\begin{aligned} & M_\alpha b_{\alpha\alpha} = f_\alpha, \\ & b_{\alpha\alpha} = \frac{Q_\alpha \alpha^3}{4\alpha\pi}, \\ & f_\alpha = -p \frac{F_\alpha a^5}{\pi^4 \alpha^4}; \quad (\alpha = 1, 3, \ldots, \infty). \end{aligned}\right. \tag{2.18}$$

Then

$$M(x) = -\frac{4pa^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{F_n}{Q_n n^3} \sin \frac{n\pi x}{a}$$

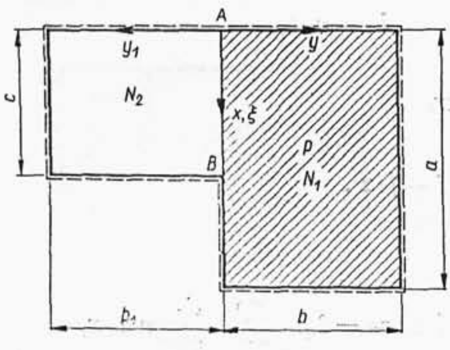

Fig. 5

and for $b \to \infty$, i. e. for a semi-infinite strip rigidly fixed at the edge $y = 0$, we have

$$M(x) = -\frac{4pa^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin \alpha_n x}{n^3} = -\frac{p}{2}(ax - x^2) \tag{2.19}$$

according to the known result, [8].

Consider a more general case — a system of two rectangular plates (Fig. 5) joined over the rigid support AB. Let these plates be simply supported on their boundaries and let the plate I be loaded by a load $p = \text{const}$. The boundary conditions at the edge $y = 0$ of the plate I are discontinuous. In the interval (0, c) we have $w_{\text{I}}(x, 0) = 0, w_{\text{II}}(x, 0) = 0$ and $\partial w_{\text{I}}(x, 0)/\partial y + \partial w_{\text{II}}(x, 0)/\partial y_1 = 0$, and in the interval (c, a) $w_{\text{I}}(x, 0) = 0$, $m_y(x, 0) = -N\partial^2 w_{\text{I}}(x, 0)/\partial y^2 = 0$. The reaction moment $-N\partial^2 w_{\text{I}}(x, 0)/\partial y^2 = -N\partial^2 w_{\text{II}}(x, 0)/\partial y_1^2 = M(x)$ in the interval (0, c) is taken as the unknown function of the problem.

The integral equation of the problem will take the form

$$\int_0^c M(\xi)\left[\frac{\partial G_{1,\text{I}}(x, 0; \xi, 0)}{\partial y} + \frac{\partial G_{1,\text{II}}(x, 0; \xi, 0)}{\partial y_1}\right] d\xi + \frac{\partial w_{0,\text{I}}(x, 0)}{\partial y} = 0, \tag{2.20}$$

where

$$\left\{\begin{aligned} &\frac{\partial G_{1,\text{I}}(x, 0; \xi, 0)}{\partial y} = \frac{1}{N_1 a}\sum_{n=1}^{\infty}\frac{Q_{n,\text{I}}}{\alpha_n}\sin\alpha_n\xi\sin\alpha_n x,\\ &\qquad Q_{n,\text{I}} = \operatorname{ctgh}\lambda_n - \frac{\lambda_n}{\sinh^2\lambda_n}, \qquad \lambda_n = \alpha_n b,\\ &\frac{\partial G_{1,\text{II}}(x, 0; \xi, 0)}{\partial y_1} = \frac{1}{N_2 c}\sum_{m=1}^{\infty}\frac{Q_{m,\text{II}}}{\beta_m}\sin\beta_m\xi\sin\beta_m x,\\ &\qquad Q_{m,\text{II}} = \operatorname{ctgh}\mu_m - \frac{\mu_m}{\sinh^2\mu_m}, \quad \beta_m = \frac{m\pi}{c}, \quad \mu_m = \beta_m b_1,\\ &\frac{\partial w_{0,\text{I}}(x, 0)}{\partial y} = \frac{2p}{N_1 a}\sum_{n=1,3,\dots}^{\infty}\frac{F_{n,\text{I}}}{\alpha_n^4}\sin\alpha_n x, \quad F_{n,\text{I}} = \operatorname{tgh}\frac{\lambda_n}{2} - \frac{\frac{\lambda_n}{2}}{\cosh\frac{\lambda_n}{2}}. \end{aligned}\right. \tag{2.20.1}$$

We take the following set of orthogonal functions:

$$\varphi_\alpha(x) = \sqrt{\frac{2}{\pi}}\sin\frac{\alpha\pi x}{c} \qquad (\alpha = 1, 2, \dots, \infty).$$

Then the Eq. (2.20) is multiplied and integrated from 0 to c. The function $M(\xi)$ is expanded in the series

$$M(\xi) = \sum_{\beta=1}^{\infty} M_\beta \sin\frac{\beta\pi\xi}{c}.$$

As a result, we obtain the infinite system of equations

$$(2.21)\quad \alpha\cos\alpha\pi\sum_{\beta=1}^{\infty} M_\beta\beta\cos\beta\pi\sum_{n=1,3,\dots}^{\infty}\frac{Q_{n,\mathrm{I}}\sin^2\frac{n\pi}{\varrho}}{n(n^2-\varrho^2\alpha^2)(n^2-\varrho^2\beta^2)}+$$

$$+\frac{N_1}{N_2}\frac{\pi^2}{4\varrho^2}\frac{M_\alpha Q_{\alpha,\mathrm{II}}}{\alpha}=-\frac{2pa^2}{\varrho\pi}\alpha\cos\alpha\pi\sum_{n=1,3,\dots}^{\infty}\frac{F_{n,\mathrm{I}}\sin\frac{n\pi}{\varrho}}{n^4(n^2-\varrho^2\alpha^2)}$$

$$\left(\alpha=1,2,\dots,\infty,\qquad \varrho=\frac{a}{c}\right).$$

In the particular case of $N_2\to\infty$, we obtain the rigid clamping of the plate I over the interval (0, c). The system of equations (1.21) reduces to the system (1.17).

Finally, in the particular case $c=a$, the system of equations (1.21) leads to the simple relations

$$(2.22)\qquad M_\alpha=-\frac{4p}{a}\frac{F_\alpha}{\alpha_\alpha^3\left(Q_{\alpha,\mathrm{I}}+Q_{\alpha,\mathrm{II}}\frac{N_1}{N_2}\right)}\qquad(\alpha=1,3,\dots,\infty),$$

from which it follows

$$M(x)=-\frac{4p}{a}\sum_{n=1,3,\dots}^{\infty}\frac{F_n\sin\alpha_n x}{\alpha_n^3\left(Q_{n,\mathrm{I}}+Q_{n,\mathrm{II}}\frac{N_1}{N_2}\right)}.$$

For $b\to\infty$ and $b_1\to\infty$, we obtain the simple expression for $M(x)$

$$(2.23)\quad M(x)=-\frac{4p}{a\left(1+\frac{N_1}{N_2}\right)}\sum_{n=1,3,\dots}^{\infty}\frac{\sin\alpha_n x}{\alpha_n^3}=-\frac{p}{2\left(1+\frac{N_1}{N_2}\right)}(ax-x^2).$$

We propose to describe another way leading to the system of equations (2.21).

Let us cut the system of two plates (Fig. 5) along the line AB, to obtain two plates I and II simply supported on all edges. The plate I is subjected to the load p and the moment

$$M(\xi)=\sum_{m=1}^{\infty}A_n\sin(n\pi\xi/a)$$

in the interval (0, c).

The plate II is subjected to the moment

$$M(\xi) = \sum_{m=1}^{\infty} M_m \sin \frac{m \pi \xi}{c}.$$

The derivative $\partial w_{\mathrm{I}}(x, 0)/\partial y$ of the plate I is determined by

$$\frac{\partial w_{\mathrm{I}}(x, 0)}{\partial y} = \frac{2p}{N_1 a} \sum_{n=1,3,\dots}^{\infty} \frac{F_{n,\mathrm{I}}}{\alpha_n^4} \sin \alpha_n x + \frac{1}{2N_1} \sum_{n=1,2,\dots}^{\infty} \frac{A_n Q_{n,\mathrm{I}}}{\alpha_n} \sin \alpha_n x.$$

For the plate II we have similarly

$$\frac{\partial w_{\mathrm{II}}(x, 0)}{\partial y} = \frac{1}{2N_2} \sum_{m=1,2,\dots}^{\infty} \frac{M_m Q_{m,\mathrm{II}}}{\beta_m} \sin \beta_m x.$$

In the interval $(0, c)$, the continuity condition $\partial w_{\mathrm{I}}(x, 0)/\partial y + \partial w_{\mathrm{II}}(x, 0)/\partial y_1 = 0$ should be satisfied. This condition will be satisfied if the coefficients $\sin \alpha_n x$ appearing in the expression for $\partial w_{\mathrm{I}}(x, 0)/\partial y$ are developed in series with respect to $\sin \beta_m x$:

$$\sin \alpha_n x = \sum_{m=1}^{\infty} a_{n,m} \sin \beta_m x \qquad \text{for} \qquad 0 < x < c,$$

where

$$a_{n,m} = \frac{2}{c} \sin \alpha_n c \, \frac{\beta_m \cos \beta_m c}{\alpha_n^2 - \beta_m^2}.$$

The continuity condition leads, therefore, to the relation

$$\sum_{n=1,2,\dots}^{\infty} \frac{A_n Q_{n,\mathrm{I}}}{\alpha_n} a_{n,m} + \frac{N_1}{N_2} \frac{M_m Q_{m,\mathrm{II}}}{\beta_m} = -\frac{4p}{a} \sum_{n=1,3,\dots}^{\infty} \frac{a_{n,m}}{\alpha_n^4} \tag{2.24}$$

$$(m = 1, 2, \dots, \infty).$$

Observe that

$$M(x) = \sum_{n=1}^{\infty} A_n \sin \alpha_n x = \sum_{m=1}^{\infty} M_m \sin \beta_m x. \tag{2.25}$$

Expanding the moment $M(x)$ in the Fourier series, it should be remembered that $M = 0$ in the interval (c, a).

This will be taken into consideration by expanding the function

$$\Phi(x) = \begin{cases} \sin \beta_m x & \text{for} \quad 0 < x < c, \\ 0 & \text{for} \quad c < x < a, \end{cases}$$

with respect to $\sin \alpha_n x$.

We have

$$\Phi(x) = \sum_{n=1}^{\infty} b_{n,m} \sin \alpha_n x,$$

where

$$\mathfrak{b}_{n,m}=\frac{2}{c}\beta_m\sin\beta_m c\,\frac{\sin\alpha_n c}{\alpha_n^2-\beta_m^2}\,.$$

Substituting $\Phi(x)$ in the Eq. (2.25), we obtain

$$A_n=\sum_{\beta=1}^{\infty}\mathfrak{b}_{\beta,n}M_\beta. \tag{2.26}$$

Introducing A_n in the Eq. (2.24), and changing the subscript m into a, we obtain, after some simple transformations, the system of equations (2.21). The method described avoids the use of the integral equation shifting the solution to the domain of differential equations; it is an extension of W. B a s i l e w i t s c h's method, [9], for problems of torsion of bars, to problems connected with the biharmonic equation.

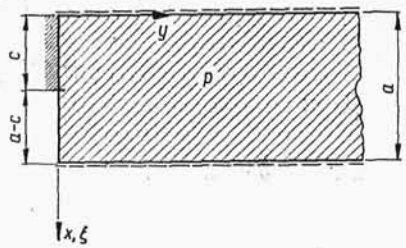

Fig. 6

B. Consider a semi-infinite strip simply supported on the edges $x=0$ and $x=a$, rigidly fixed along the segment $(0,c)$ of the edge $y=0$, and free along the segment (c,a) of the same edge. Let the plate be subjected to a uniform load p. The boundary conditions then are

at the edges $x=0$ and $x=a$: $w=0,\ \nabla^2 w=0$,

in the interval $(0,c)$: $D_1 w(x,0)=w(x,0)=0,\ D_2 w(x,0)=\dfrac{\partial w(x,0)}{\partial y}=0$,

in the interval (c,a): $L_1 w(x,0)=-N\left[\dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2}\right]_{y=0}=0$,

$$L_2 w(x,0)=-N\left[\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial y\,\partial x^2}\right]_{y=0}=0.$$

If $R(\xi)$ denotes the unknown reaction force and $M(\xi)$ the reaction moment in the interval $(0,c)$, the system of equation (2.4) takes the form

$$\left\{\begin{aligned} &w_0(x,0)+\int_0^c R(\xi)\,G_1(x,0;\xi,0)\,d\xi+\int_0^c M(\xi)\,G_2(x,0;\xi,0)\,d\xi=0,\\ &\frac{\partial w_0(x,0)}{\partial y}+\int_0^c R(\xi)\frac{\partial G_1(x,0;\xi,0)}{\partial y}\,d\xi+\int_0^c M(\xi)\frac{\partial G_2(x,0;\xi,0)}{\partial y}\,d\xi=0,\end{aligned}\right. \tag{2.27}$$

where $G_1(x, y; \xi, 0)$ is the deflection surface of the basic system (the plate simply supported along the edges $x = 0$ and $x = a$ the edge $y = 0$ remaining free) due to the concentrated force $R = 1$ acting at the point $(\xi, 0)$ of the interval $(0, c)$. $G_2(x, y; \xi, 0)$ denotes the deflection of the basic system due to the concentrated bending moment $M = 1$ at the point $(\xi, 0)$ of the edge. Finally $w_0(x, y)$ denotes the deflection of the basic system due to the load p.

These functions have the form, [2]:

$$
(2.28)\quad \begin{cases}
G_1(x, y; \xi, 0) = \\
\qquad = \dfrac{4}{aN(1-\nu)(3+\nu)} \displaystyle\sum_{n=1}^{\infty} \dfrac{e^{-\alpha_n y}}{\alpha_n^3} \left[1 + \dfrac{1-\nu}{2} \alpha_n y\right] \sin \alpha_n \xi \sin \alpha_n x, \\
G_2(x, y; \xi, 0) = \\
= \dfrac{2}{aN(1-\nu)(3+\nu)} \displaystyle\sum_{n=1}^{\infty} \dfrac{e^{-\alpha_n y}}{\alpha_n^2} [1 + \nu - (1-\nu)\alpha_n y] \sin \alpha_n \xi \sin \alpha_n x, \\
w_0(x, y) = \dfrac{4p}{aN} \displaystyle\sum_{n=1,3,\dots}^{\infty} \dfrac{1}{\alpha_n^5} \left[1 + \dfrac{\nu}{3+\nu} \left(\dfrac{1+\nu}{1-\nu} - \alpha_n y\right) e^{-\alpha_n y}\right] \sin \alpha_n x.
\end{cases}
$$

Introducing these functions in the Eq. (2.27) we have

$$
(2.29)\quad \begin{cases}
\displaystyle\int_0^c R(\xi) \left(\sum_{n=1,2,\dots}^{\infty} \dfrac{1}{n^3} \sin \alpha_n \xi \sin \alpha_n x \right) d\xi - \\
\qquad - \dfrac{(1+\nu)\pi}{2a} \displaystyle\int_0^c M(\xi) \left(\sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin \alpha_n \xi \sin \alpha_n x \right) d\xi = \\
\qquad\qquad = -\dfrac{pa^2}{\pi^2}(3-\nu) \displaystyle\sum_{n=1,3,\dots}^{\infty} \dfrac{1}{n^5} \sin \alpha_n x, \\
-\dfrac{(1+\nu)\pi}{2a} \displaystyle\int_0^c R(\xi) \left(\sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin \alpha_n \xi \sin \alpha_n x \right) d\xi + \\
+ \dfrac{\pi^2}{a^2} \displaystyle\int_0^c M(\xi) \left(\sum_{n=1}^{\infty} \dfrac{1}{n} \sin \alpha_n \xi \sin \alpha_n x \right) d\xi = \dfrac{2pa\nu}{\pi} \sum_{n=1,3,\dots}^{\infty} \dfrac{1}{n^4} \sin \alpha_n x.
\end{cases}
$$

These integral equations can be reduced to an infinite system of equations the unknowns in which are the Fourier coefficients R_β and M_β for the functions $R(\xi)$ and $M(\xi)$:

$$
R(\xi) = \sum_{\beta=1}^{\infty} R_\beta \sin \frac{\beta\pi\xi}{c}, \qquad M(\xi) = \sum_{\beta=1}^{\infty} M_\beta \sin \frac{\beta\pi\xi}{c}.
$$

We have:

$$
(2.30)\qquad \begin{cases} \sum\limits_{\beta=1}^{\infty} (R_\beta \mathfrak{b}_{\alpha\beta}^{(1,1)} + M_\beta \mathfrak{b}_{\alpha\beta}^{(1,2)}) = \mathfrak{f}_{1,\alpha}, \\ \sum\limits_{\beta=1}^{\infty} (R_\beta \mathfrak{b}_{\alpha\beta}^{(2,1)} + M_\beta \mathfrak{b}_{\alpha\beta}^{(2,2)}) = \mathfrak{f}_{2,\alpha}, \end{cases} \qquad (\alpha = 1, 2, \ldots, \infty),
$$

where

$$
(2.31)\qquad \begin{cases} \mathfrak{b}_{\alpha\beta}^{(1,1)} = \alpha\beta \cos\alpha\pi \cos\beta\pi \sum\limits_{n=1,3,\ldots}^{\infty} \dfrac{\sin^2 \dfrac{n\pi}{\varrho}}{n^3 (n^2 - \varrho^2 \alpha^2)(n^2 - \varrho^2 \beta^2)}, \\ \mathfrak{b}_{\alpha\beta}^{(1,2)} = \mathfrak{b}_{\alpha\beta}^{(2,1)} = \\ \qquad = -\dfrac{(1+\nu)\pi}{2a} \alpha\beta \cos\alpha\pi \cos\beta\pi \sum\limits_{n=1,3,\ldots}^{\infty} \dfrac{\sin^2 \dfrac{n\pi}{\varrho}}{n^2 (n^2 - \varrho^2 \alpha^2)(n^2 - \varrho^2 \beta^2)} \\ \mathfrak{b}_{\alpha\beta}^{(2,2)} = \dfrac{\pi^2}{a^2} \alpha\beta \cos\alpha\pi \cos\beta\pi \sum\limits_{n=1,3,\ldots}^{\infty} \dfrac{\sin^2 \dfrac{n\pi}{\varrho}}{n (n^2 - \varrho^2 \alpha^2)(n^2 - \varrho^2 \beta^2)}. \\ \mathfrak{f}_{1,\alpha} = -\dfrac{pa}{\pi\varrho} (3-\nu)\, \alpha \cos\alpha \sum\limits_{n=1,3,\ldots}^{\infty} \dfrac{\sin \dfrac{n\pi}{\varrho}}{n^5 (n^2 - \varrho^2 \alpha^2)}, \\ \mathfrak{f}_{2,\alpha} = \dfrac{2p\nu}{\varrho} \alpha \cos\alpha \sum\limits_{n=1,3,\ldots}^{\infty} \dfrac{\sin \dfrac{n\pi}{\varrho}}{n^4 (n^2 - \varrho^2 \alpha^2)}. \end{cases}
$$

In the particular case of simple support over the interval (0, c), the clamping moments vanish and, for the determination of R_β, we have

$$
(2.32)\qquad \sum_{\beta=1}^{\infty} R_\beta \mathfrak{b}_{\alpha\beta}^{(1,1)} = \mathfrak{f}_{1,\alpha} \qquad (\alpha = 1, 2, \ldots, \infty).
$$

If the plate is clamped in the interval (0, c) in such a way that the reaction force $R(\xi)$ vanishes, we obtain the system

$$
(2.33)\qquad \sum_{\beta=1}^{\infty} M_\beta \mathfrak{b}_{\alpha\beta}^{(2,2)} = \mathfrak{f}_{2,\alpha} \qquad (\alpha = 1, 2, \ldots, \infty).
$$

This case will be realized in the example of an infinite strip (Fig. 7) having a slot along the $y = 0$ axis and subjected to a uniform load p. The shearing forces on the axis of symmetry $y = 0$ vanish, and the moments $M(\xi)$ in the interval $(0, c)$ are still present.

In the case of $b \to \infty$ ($Q_n \to 1$, $F_n \to 1$), the system of equations (2.33) is identical (including the constant) with the system (2.17).

C. Consider a rectangular plate simply supported on the boundary and having an additional rigid support AB (Fig. 8). Let the plate be subjected to a uniform load p. The same plate simply supported on the boundary will be taken as the basic system. Denoting by $w_0(x, y)$ the deflection of the plate due to the load p in the basic system: by $G_1(x, y; \xi, 0)$ — the deflection in the basic system due to the action of the force $R = 1$ at the point $(\xi, 0)$; and by $R(\xi)$ the reaction force in the interval $(0, c)$ according to Fig. 8 — we have from the boundary condition $w(x, 0)$ in the interval $(0, c)$, the integral equation

$$(2.34) \qquad \int_0^c R(\xi)\, G_1(x, 0; \xi, 0)\, d\xi + w_0(x, 0) = 0.$$

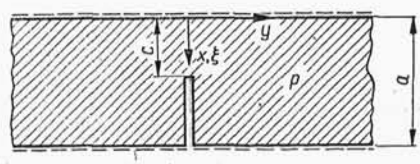

Fig. 7

Expanding $R(\xi)$ in the Fourier series,

$$R(\xi) = \sum_{\beta=1}^{\infty} R_\beta \sin \frac{\beta \pi \xi}{c},$$

multiplying (2.34) by $\sin(\alpha \pi x/c)$, and integrating over the interval $(0, c)$, we obtain the system of equations

$$(2.35) \qquad \sum_{\beta=1}^{\infty} R_\beta\, b_{\alpha\beta} = f_\alpha \qquad (\alpha = 1, 2, \ldots, \infty).$$

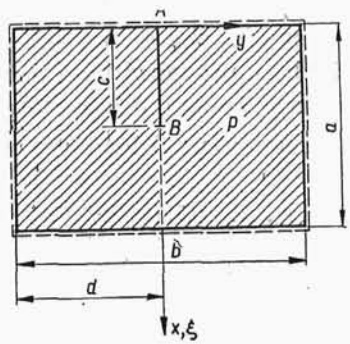

Fig. 8

In the particular case where an infinite strip is simply supported on the edges $x = 0$ and $x = a$, and has an additional support along the segment c, the whole of the plate area being subjected to the load p, we have

$$w_0(x, y) = w_0(x) = \frac{p}{24\,N}(x^4 - 2\,a x^3 + a^3 x) = \frac{4\,p}{N a} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin a_n x}{a_n^5},$$

$$G_1(x, y; \xi, 0) = \frac{1}{2\,N} \sum_{n=1,2,\ldots}^{\infty} \frac{1}{a_n^3}(1 + a_n y) e^{-a_n y} \sin a_n \xi \sin a_n x, \qquad \text{for } y > 0.$$

The system of equations (2.35) takes the form

$$(2.36)\qquad \alpha\cos\alpha\pi\sum_{\beta=1}^{\infty}R_\beta\,\beta\cos\beta\pi\sum_{n=1,3,\dots}^{\infty}\frac{\sin^2\frac{n\pi}{\varrho}}{n^3(n^2-\varrho^2\alpha^2)(n^2-\varrho^2\beta^2)}=$$

$$=-\frac{8\,pa}{\pi\varrho}\,\alpha\cos\alpha\pi\sum_{n=1,3,\dots}^{\infty}\frac{\sin\frac{n\pi}{\varrho}}{n^5(n^2-\varrho^2\alpha^2)}\qquad(\alpha=1,2,\dots,\infty).$$

This is identical (with the accuracy to the constant) with the system (2.32).

For a rigid support over the interval $(0,a)$ we have the system of equations:

$$(2.37)\qquad \begin{cases} R_\alpha=\dfrac{16\,pa}{\pi^2\alpha^2}, \\ R(x)=-\dfrac{16\,pa}{\pi^2}\displaystyle\sum_{\alpha=1,3,\dots}^{\infty}\frac{\sin\frac{\alpha\pi x}{a}}{\alpha^2}. \end{cases}$$

D. Consider the deformation of a semi-infinite strip simply supported on all edges, provoked by a deformation of the edge $y=0$. Let this edge be rotated through $\partial w(x,0)/\partial y=h(x)$, where $h(x)$ is a given function.

With no external load, we obtain from (2.7) the integral equation

$$(2.38)\qquad \int_0^a M(\xi)\frac{\partial G_1(x,0;\xi,0)}{\partial y}\,d\xi=\frac{\partial w(x,0)}{\partial y}=h(x),\qquad 0<x,\ \xi<a,$$

or

$$\int_0^a M(\xi)\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi\sin\alpha_n x}{\alpha_n}\,d\xi=aNh(x).$$

Solving this equation, we have, similarly to the foregoing case,

$$M_n=2\,N\,\alpha_n\,b_n,$$

where b_n are the Fourier coefficients for the function $h(x)$.

The fixing moment on the line $y=0$ will, therefore, take the form

$$(2.39)\qquad M(x)=2\,N\sum_{n=1,2,\dots}^{\infty}\alpha_n\,b_n\sin\alpha_n x.$$

If the rotation of the edge is such that

$$h(x) = \begin{cases} \varphi_0 x & \text{for} \quad 0 < x < \dfrac{a}{2}, \\ \varphi_0 (a - x) & \text{for} \quad \dfrac{a}{2} < x < a, \end{cases}$$

we have (bearing in mind that

$$h(x) = \frac{4a\varphi_0}{\pi^2} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} \sin \alpha_n x):$$

$$M(x) = \frac{8\varphi_0 N}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \sin \frac{n\pi x}{a}$$

or

$$M(x) = -\frac{2\varphi_0 N}{\pi} \ln \frac{1 - \sin \frac{\pi x}{a}}{1 + \sin \frac{\pi x}{a}}.$$

For $x = a/2$ we obtain a discontinuity of the logarithmic type of the function $M(x)$. Consider a semi-infinite strip having a free edge along the line $y = 0$. Let us deform the edge $y = 0$ so that $w(x, 0) = g(x)$ becomes a known function. The integral equation of the problem has the form

$$\text{(2.40)} \qquad \int_0^a R(\xi) \left(\sum_{n=1}^{\infty} \frac{1}{n^3} \sin \alpha_n \xi \sin \alpha_n x \right) d\xi = \frac{aN(3+\nu)(1-\nu)}{2} g(x)$$

$$0 < x, \ \xi < a.$$

If s_n denotes the Fourier coefficients for the function $g(x)$ in the interval $(0, a)$, the solution of the integral equation is reduced to

$$\text{(2.41)} \qquad R(x) = \sum_{n=1}^{\infty} R_n \sin \alpha_n,$$

where

$$R_n = \frac{N(1-\nu)(3+\nu)}{2\pi^3} a^3 \alpha_n^3 s_n.$$

It is seen that to secure the convergence of the series (2.41), the coefficients s_n should contain a factor equal to $1/\alpha_n^4$ at least.

3. Rectangular Plates with Discontinuous Elastic Supports and Resting on a Discontinuous Elastic Foundation

In this section we propose to confine ourselves to the most simple cases of discontinuous elastic supports along a single edge only.

Consider a rectangular plate simply supported at the edges $x = 0$, $x = a$ and $y = b$ and free in the interval (c, a) (Fig. 9). Let this plate be elastically clamped and supported in the interval $(0, c)$ of the edge $y = 0$.

The problem reduces to the solution of the system of two equations (1.4) which will be rewritten here in a somewhat different form, taking the opposite directions of R and M:

$$(3.1)\qquad \begin{cases} R(x) = k w_0(x) - k \int\limits_0^c R(\xi) K(x,\xi)\, d\xi - k \int\limits_0^c M(\xi) G(x,\xi)\, d\xi, \\[2ex] M(x) = r \varphi_0(x) - r \int\limits_0^c R(\xi) N(x,\xi)\, d\xi - r \int\limits_0^c M(\xi) H(x,\xi)\, d\xi, \end{cases}$$

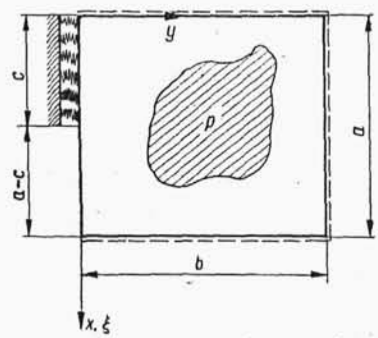

Fig. 9

where

$$N(x,\xi) = \left[\frac{\partial K(x,y;\xi,0)}{\partial y}\right]_{y=0},$$

$$H(x,\xi) = \left[\frac{\partial G(x,y;\xi,0)}{\partial y}\right]_{y=0},$$

$$\varphi_0 = \left[\frac{\partial w_0(x,y)}{\partial y}\right]_{y=0}.$$

This can be reduced to one Fredholm equation of the second kind:

$$(3.2)\qquad \Phi(x) = \lambda \int\limits_0^{2c} L(x,\xi)\Phi(\xi)\, d\xi + F(x),$$

where

$$(3.3.1)\qquad \Phi(x) = \begin{cases} R(x) & \text{for} \quad 0 \leqq x < c, \\ M(x-c) & \text{for} \quad c \leqq x < 2c, \end{cases}$$

$$F(x) = \begin{cases} k w_0(x) & \text{for} \quad 0 \leqq x < c, \\ r\varphi_0(x) & \text{for} \quad c \leqq x < 2c, \end{cases}$$

and

$$(3.3.2)\qquad \lambda L(x,\xi) = \begin{cases} -kK(x,\xi) & \text{for} \quad 0 \leqq x < c, \quad 0 \leqq \xi < c, \\ -kG(x,\xi) & \text{for} \quad 0 \leqq x < c, \quad c \leqq \xi < 2c; \\ -rN(x,\xi) & \text{for} \quad c \leqq x < 2c, \quad 0 \leqq \xi < c, \\ -rH(x,\xi) & \text{for} \quad c \leqq x < 2c, \quad 0 \leqq \xi < 2c, \end{cases}$$

λ being a constant. We can apply the iteration method to the Eq. (3.2) and, to find the function $\Phi(\xi)$ with the required degree of accuracy, estimate the errors of the successive approximations. The kernels K, G and N are bounded and the kernel H satisfies the integrability condition

$$\int_0^c H^2(x,\xi)\,d\xi < C. \tag{3.4}$$

Solving the system (3.1) or the equations (3.2) we shall assume the more weak condition (3.4) for the estimation of errors. The following condition should, therefore, be assumed for $|\lambda|$:

$$|\lambda| \leq \frac{1}{\sqrt{\int_0^{2c}\int_0^{2c} L^2(x,\xi)\,dx\,d\xi}}. \tag{3.5}$$

For λ satisfying the condition (3.5), we can construct a resolving kernel for the Eq. (3.2) and estimate the absolute error from, [10],

$$\delta = D\sqrt{C}\,\frac{|\lambda|^{n+1}B^n}{1-|\lambda|B}, \tag{3.6}$$

where

$$C > \int_0^{2c} L^2(x,\xi)\,d\xi, \qquad B^2 = \int_0^{2c}\int_0^{2c} L^2(x,\xi)\,dx\,d\xi, \qquad D^2 = \int_0^{2c} F^2(x)\,dx.$$

The solution of the integral equation (3.2) takes the form

$$\Phi(x) = F(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^{2c} L_m(x,\xi)\,F(\xi)\,d\xi, \tag{3.7}$$

where $L_m(x,\xi)$ denotes the m-th iterated kernel. $\Phi(x)$, being known, it is easy to find the functions $R(x)$ and $M(x)$ from the relations (3.3.1).

The solution takes a particularly simple form in cases where the plate is only elastically supported in the interval (0, c) of the edge $y = 0$. The system of equations (3.1) reduces then to the unique integral equation

$$R(x) = k\,w_0(x) - k\int_0^c R(\xi)\,K(x,\xi)\,d\xi,$$

or

$$R(x) = k w_0(x) + \lambda\int_0^c R(\xi)\,K^*(x,\xi)\,d\xi, \tag{3.8}$$

where

$$K^*(x,\xi) = -\frac{k}{\lambda}K(x,\xi).$$

The function $K^*(x, \xi)$ has, by rejecting the constants, been obtained from $K(x, \xi)$.

Denoting by $K_m^*(x, \xi)$ the m-th iterated kernel, the solution of the Eq. (3.8) can be written as

$$R(x) = k w_0(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^c K_m^*(x, \xi)\, d\, \xi. \tag{3.9}$$

The function $k w_0(x) < D$ being, in our examples, bounded and the kernel $K^*(x, \xi) < B$ continuous and bounded, the interval of convergence of the parameter λ is determined by

$$|\lambda| \leqq \frac{1}{Bc}, \tag{3.10}$$

and the maximum value of the error after the n-th iteration is

$$\delta = D \frac{|\lambda|^{n+1} (Bc)^{n+1}}{1 - |\lambda| Bc}. \tag{3.11}$$

If the plate is only elastically clamped in the interval $(0, c)$ of the edge $y = 0$, the system (3.1) reduces to the equation

$$M(x) = r \varphi_0(x) + \lambda \int_0^c M(\xi) H^*(x, \xi)\, d\, \xi, \tag{3.12}$$

where

$$H^*(x, \xi) = -\frac{r}{\lambda} H(x, \xi).$$

The integral

$$\int_0^c [k \varphi_0(x)]^2 < D$$

being limited and the kernel — in the examples which will be treated below — bounded,

$$\int_0^c H^{*2}(x, \xi)\, d\, \xi < C_1.$$

By denoting

$$\int_0^c \int_0^c H^{*2}(x, \xi)\, dx\, d\xi = B^2,$$

we can determine the interval of convergence of the parameter λ from the formula

$$|\lambda| \leqq \frac{1}{B}, \tag{3.13}$$

and the maximum value of the absolute error for the n-th approximation from the equation

$$\delta = D\sqrt{C_1}\frac{|\lambda|^{n+1}B^n}{1-|\lambda|B}. \tag{3.14}$$

The equation (3.12) will also be valid in cases where the edge $y = 0$ is simply supported along the segment $(0, a)$ and has an additional elastic clamping along the segment $(0, c)$.

The basic system in this case will be, however, the plate simply supported along the segment $(0, a)$ of the edge $y = 0$.

The procedure will be illustrated by three simple examples.

(a) Consider a plate strip simply supported on the edges and loaded along these edges by the moments M. Let this strip rest on a transverse discontinuous elastic support of length $c = a/2$ (Fig. 10).

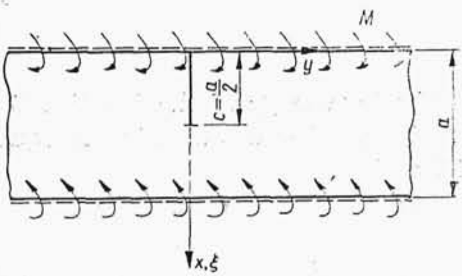

Fig. 10

The integral equation of the problem takes the form

$$R(x) = A(ax - x^2) + \lambda\int_0^{a/2} R(\xi)K^*(x,\xi)\,d\xi, \tag{3.15}$$

where

$$A = \frac{kM}{2N}, \qquad \lambda = -\frac{a^2 k}{2\pi^3 N},$$

$$K^*(x,\xi) = \sum_{n=1,2,\ldots}^{\infty}\frac{1}{n^3}\sin\alpha_n x\sin\alpha_n\xi, \qquad \alpha_n = \frac{n\pi}{a}.$$

The n-th approximation will take the form

$$R_n(x) = R_{n-1}(x) + \lambda^n A\sum_{j_1 j_2\ldots j_n}^{\infty}\gamma_{j_1}\left(\prod_{i=2}^{i=n}\frac{\varepsilon_{j_{i-1},\,j_i}}{j_i^3}\right)\sin\alpha_{j_n}x, \tag{3.16}$$

and the final solution of the integral equation (3.15) is

$$R(x) = A(ax - x^2) + A\left[\sum_{n=1}^{\infty}\lambda^n\sum_{j_1 j_2\ldots j_n}^{\infty}\gamma_{j_1}\left(\prod_{i=2}^{i=n}\frac{\varepsilon_{j_{i-1},\,j_i}}{j_i^3}\right)\sin\alpha_{j_n}x\right], \tag{3.17}$$

where

$$(3.18) \qquad \varepsilon j_{i-1}, j_i = \frac{\frac{j_i \pi}{a} \sin \frac{\pi j_{i-1}}{2} \cos \frac{\pi j_i}{2} - \frac{j_{i-1} \pi}{a} \cos \frac{j_{i-1} \pi}{2} \sin \frac{j_i \pi}{2}}{\frac{\pi^2}{a^2} (j_{i-1}^2 - j_i^2)},$$

$$\gamma_{j_1} = \frac{1 - \left(1 + j_1 \frac{\pi^2}{4}\right) \cos \frac{j_1 \pi}{2}}{\left(\frac{j_1 \pi}{a}\right)^3}.$$

The knowledge of the function $R(x)$ enables us to determine of the function $w(x) = R/k$ and, in consequence, the deflection surface $w(x, y)$ of the plate.

In our example, we assume

$$D = \frac{Aa^2}{4}, \qquad B = \sum_{n=1}^{\infty} \frac{1}{n^3} > \sum_{n=1}^{\infty} \frac{1}{n^3} \sin a_n x \sin a_n \xi; \qquad B \approx 1,2.$$

Then

$$(3.19) \qquad |\lambda| = \frac{a^2 k}{2 N \pi^3} \leqq \frac{1}{B \frac{a}{2}}, \qquad \mu = \frac{a^3 k}{N} \leqq 103,$$

$$(3.20) \qquad \delta \leqq \frac{M}{8a} \frac{(0{,}00968)^{n+1} \mu^{n+2}}{1 - \mu \cdot 0{,}00968}.$$

The table of relative errors $\psi = \delta / R_{max}\ 100\%$ in function of n and μ is given below. It should be observed that the values of the errors in Table 1

Table 1

n	2	3	4	5	
ψ	0,103%	0,01%	0,001%	0,0001%	for $\mu = 10$
ψ	24,6%	11,9%	5,75%	2,80%	for $\mu = 50$

are excessive in view of the very rough estimation of the error. Fig. 11 represents the diagram of the first and second approximation for the function $R(x)$ with $\mu = 10$ and $\mu = 50$. It is seen from that figure that the real error is, for $\mu = 50$ and $n = 2$, considerably lower than that calculated from the equation (3.19). It does not exceed 5%.

In the limit case of an elastic support along the segment $c = a$ of the line $y = 0$, we can obtain a solution of the equation (3.15) valid for all values of λ:

$$R(x) = \frac{4kMa^2}{N\pi^3} \sum_{m=1,3,\dots}^{\infty} \frac{\sin \dfrac{m\pi x}{a}}{m^3 + \dfrac{a^3 k}{4\pi^3 N}}.$$

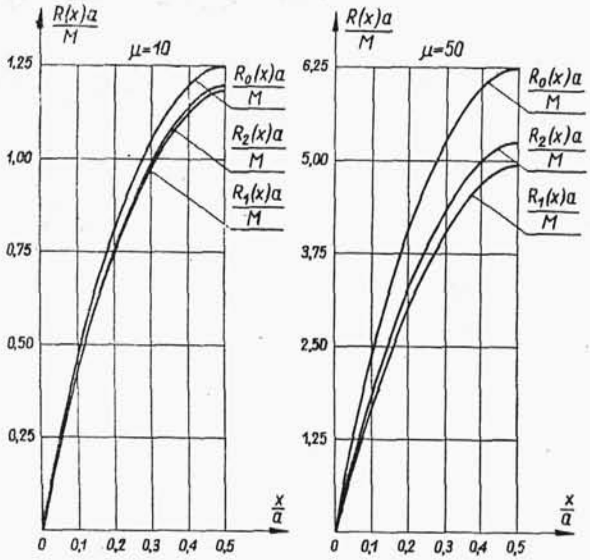

Fig. 11

(b) Consider a plate strip as in the preceding example, the only difference consisting in the load, which will be here uniformly distributed over the whole region and equal to p (Fig. 12).

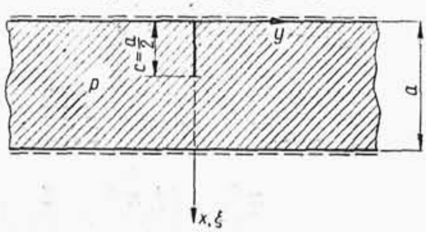

Fig. 12

The integral equation of the problem will take the form

(3.21) $R(x) =$

$$= \frac{kqa^4}{24N}\left(\frac{x^4}{a^4} - \frac{2x^3}{a^3} + \frac{x}{a}\right) - \frac{ak^2}{2N\pi^3}\int_0^{a/2} R(\xi)\left(\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\alpha_n x \sin\alpha_n \xi\right) d\xi.$$

The solution of this problem can be written analogously to (3.17), in the form

$$(3.22)\qquad R(x)=A_0\left(\frac{x^4}{a^4}-\frac{2x^3}{a^3}+\frac{x}{a}\right)+$$
$$+A_0\left[\sum_{n=1}^{\infty}\lambda^n\sum_{j_1 j_2 \dots j_n}^{\infty}\gamma_{j_1}\left(\prod_{i=2}^{i=n}\frac{\varepsilon j_{i-1}, j_i}{j_i^3}\right)\sin a_{j_n} x\right],$$

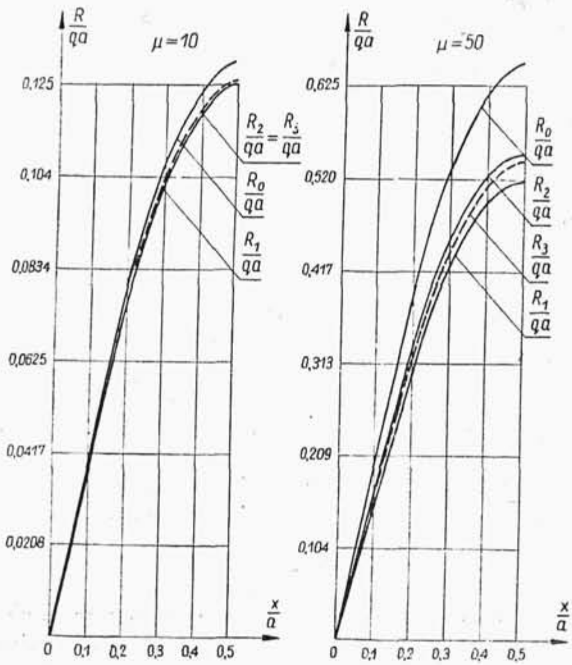

Fig. 13

where

$$A_0=\frac{kqa^4}{24N},\qquad \lambda=-\frac{2ak}{2N\pi^3},$$

$$\gamma_{j_1}=\frac{24a}{(j_1\pi)^5}\left\{1-\cos\frac{j_1\pi}{2}\left[1+\left(\frac{j_1\pi}{8}\right)^2+\frac{5}{384}(j_1\pi)^4\right]\right\}\frac{1}{j_1^3},$$

and $\varepsilon j_{i-1}, j_i$ is expressed by the Eq. (3.18).

Denoting

$$D=A_0\frac{5}{16},\qquad B=\sum_{n=1}^{\infty}\frac{1}{n^3}>\sum_{n=1}^{\infty}\frac{1}{n^3}\sin a_n\xi\sin a_n x;\qquad B\approx 1,2,$$

we have

$$|\lambda| \leqq \frac{1}{B\frac{a}{2}}, \qquad \mu = \frac{a^3 k}{N} \leqq 103, \qquad \delta = \frac{5}{384} qa \frac{(0{,}00968)^{n+1} \mu^{n+2}}{1 - 0{,}00968\,\mu}.$$

The relative error is represented as a function of μ and n in Table 2. Fig. 13 shows the second and third approximation for the function $R(x)$ for $\mu = 10$ and $\mu = 50$.

Table 2

n	1	2	3	4	
ψ	1,07%	0,1%	0,01%	0,001%	for $\mu = 10$
ψ	63%	30%	14,5%	7%	for $\mu = 50$

In the particular case $c = a$ we have

$$R(x) = \frac{kqa^4}{N\pi^5} \sum_{m=1,3,\dots}^{\infty} \frac{\sin\frac{m\pi x}{a}}{m^2\left(m^3 + \frac{a^3 k}{4N\pi^3}\right)}.$$

(c) Consider a semi-infinite strip simply supported at the edges $x = 0$, $x = a$ and $y = 0$, and elastically clamped along the segment $(0, c)$ of the edge $y = 0$.

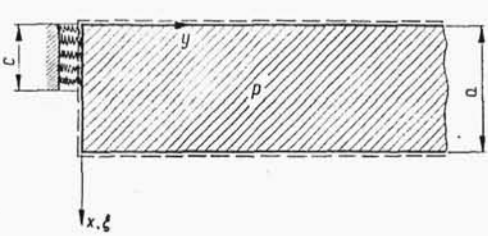

Fig. 14

The integral equation takes the form

$$M(x) = \frac{2qa^3 r}{N\pi^4} \sum_{n=1,3,\dots}^{\infty} \frac{\sin \alpha_n x}{n^4} + \frac{r}{4\pi N} \int_0^{a/2} M(\xi) \ln \frac{1 - \cos\frac{\pi}{a}(x-\xi)}{1 - \cos\frac{\pi}{a}(x+\xi)}\, d\xi. \tag{3.23}$$

Expanding the kernel of the integral equation in the Fourier series, we have

$$M(x) = A \sum_{m=1,3,\dots}^{\infty} \frac{\sin \alpha_n x}{n^4} + \lambda \int_0^{a/2} M(\xi) \sum_{n=1}^{\infty} \frac{\sin \alpha_n \xi \sin \alpha_n x}{n}\, d\xi, \tag{3.24}$$

where

$$\lambda = -\frac{r}{\pi N}, \qquad A = \frac{2\,q a^3 r}{N \pi^4}.$$

The solution of the equation (3.24) takes the form

$$M(x) = A \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^4} \sin \alpha_n x + A \sum_{n=1}^{\infty} \lambda^n \sum_{j_1 j_2 \dots j_n}^{\infty} \frac{1}{j_1^4} \left(\prod_{i=1}^{i=n} \frac{\varepsilon j_i, j_{i+1}}{j_{i+1}} \right) \sin \alpha j_{n+1} x, \tag{3.25}$$

where j_1 assumes the values 1, 3, 5, ..., the other j_i ($i = 1, 2, 3, \dots$) — the values 1, 2, 3, ..., and

$$\varepsilon j_i, j_{i+1} = \frac{\dfrac{j_{i+1}\pi}{a} \sin\dfrac{j_i \pi}{2} \cos\dfrac{j_{i+1}\pi}{2} - \dfrac{j_i \pi}{a} \cos\dfrac{j_i \pi}{2} \sin\dfrac{j_{i+1}\pi}{2}}{\dfrac{\pi^2}{a^2}(j_i^2 - j_{i+2}^2)}.$$

It can be shown that the integral of the square of the kernel of Eq. (3.24) exists. It can be estimated that $C_1 \approx 0{,}645a$ and $B^2 \approx 0{,}2915a^2$. Thus, from the Eqs. (3.13) and (3.14) it follows that

$$|\lambda| \leqq \frac{1}{B} = \frac{1{,}85}{a}, \qquad \mu = \left|\frac{ra}{N}\right| \leqq 5{,}82, \qquad \delta = q a^2 \frac{\dfrac{1}{\pi}(0{,}172)^n \mu^{n+2}\, 0{,}0126}{1 - 0{,}172\,\mu}.$$

Table 3

n	1	2	3	4	
ψ	4,35%	0,75%	0,13%	0,022%	for $\mu = 1$
ψ	80%	36%	18,5%	9,5%	for $\mu = 3$

The relative error ψ as a function of μ and n is shown in Table 3. Fig. 15 represents the diagrams of $M(x)$ for the zero, the first and the second approximation. It is seen from these diagrams that the estimation of the series is very rough. The real error will be several times smaller than its value as calculated.

Thus, for $\mu = 3$ the error of the second approximation will be not greater than 15%.

In the particular case of $c = a$, that is with an elastic clamping along the whole edge $y = 0$, the accurate solution of the Eq. (3.23) will be obtained in the form of the series

$$M(x) = \frac{2\,q a^3 r}{N \pi^4} \sum_{m=1,3,\dots}^{\infty} \frac{\sin\dfrac{m\pi x}{a}}{m^3 \left(m - \dfrac{ar}{2\pi N}\right)}.$$

The problem of a plate resting on a discontinuous elastic foundation will be illustrated by the example of a square plate simply supported on the edges and having an additional support in the form of a rectangular elastic region $\Omega_1 = (a/2)\,(a/2)$, (Fig. 16). The load is $p = \text{const.}$

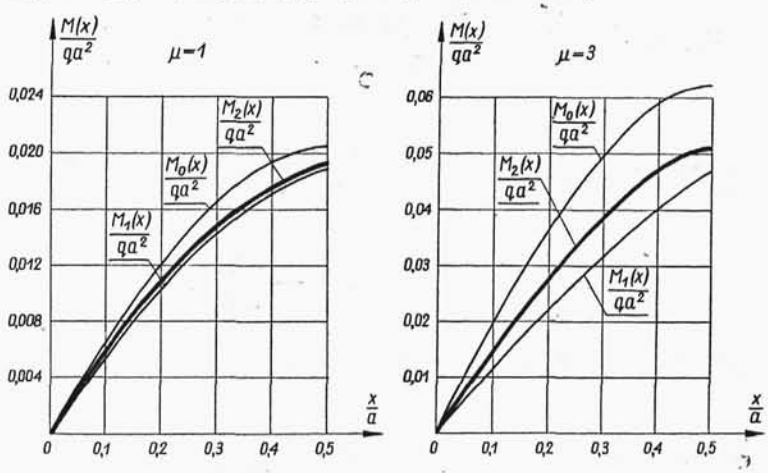

Fig. 15

The integral equation takes the form

$$R(x, y) = k w_0(x, y) - k \iint_{\Omega_1} R(\xi, \eta)\, K(x, y; \xi, \eta)\, d\,\Omega_1,$$

or

$$R(x, y) = k w_0(x, y) + \lambda \iint_{\Omega_1} R(\xi, \eta)\, K^*(x, y; \xi, \eta)\, d\,\Omega_1, \tag{3.26}$$

where

$$K^*(x, y; \xi, \eta) = -\frac{k}{\lambda}\, K(x, y; \xi, \eta).$$

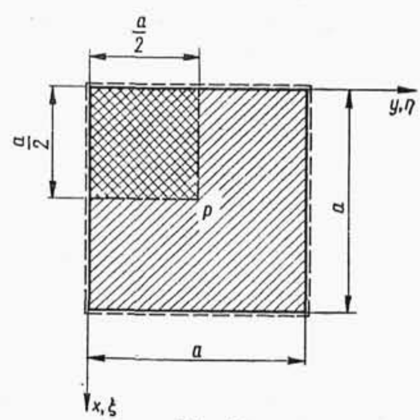

Fig. 16

The kernel of the integral equation being in our case bounded:

$$B \geqq |K^*(x, y; \xi, \eta)|$$

and

$$D \geqq |k w_0(x, y)|,$$

we obtain the condition for λ

$$|\lambda| \leqq \frac{1}{B\,\Omega_1}, \qquad \Omega_1 = \frac{a^2}{4},$$

the maximum error being given by

$$\delta = D\,\frac{|\lambda|^{n+1}(B\,\Omega_1)^{n+1}}{1 - |\lambda|\,B\,\Omega_1}.$$

Expressing $w_0(x,y)$ and $K^*(x,y;\xi,\eta)$ by the double trigonometric series known from structural analysis of plates, the integral equation (3.26) will be represented in the form

$$(3.27) \quad R(x,y) = A \sum_{m_1=1,3,\ldots}^{\infty} \sum_{j_1=1,3,\ldots}^{\infty} \frac{\sin \alpha_{m_1} x \sin \alpha_{j_1} y}{m_1 j_1 (m_1^2 + j_1^2)^2} +$$

$$+ \lambda \int_0^{a/2} \int_0^{a/2} R(\xi,\eta) \left[\sum_{m_2=1,2,\ldots}^{\infty} \sum_{j_2=1,2,\ldots}^{\infty} \frac{\sin \alpha_{m_2} x \sin \alpha_{j_2} y}{(m_2^2 + j_2^2)^2} \sin \alpha_{m_2} \xi \sin \alpha_{j_2} \eta \right] d\xi \, d\eta,$$

where

$$A = \frac{16 k a^4 p}{N \pi^6}, \qquad \lambda = -\frac{4 k a^2}{N \pi^4}.$$

The solution of the integral equation leads to the formula

$$(3.28) \quad R(x,y) = A \sum_{m_1=1,3,\ldots}^{\infty} \sum_{j_1=1,3,\ldots}^{\infty} \frac{\sin \alpha_{m_1} x \sin \alpha_{j_1} y}{m_1 j_1 (m_1^2 + j_1^2)^2} +$$

$$+ A \sum_{n=1}^{\infty} \lambda^n \sum_{m_1 m_2 \ldots m_n}^{\infty} \sum_{j_1, j_2 \ldots j_n}^{\infty} \left[\prod_{i=1}^{i=n+1} \frac{1}{m_1 j_1} \frac{\varepsilon_{m_i m_{i+1}} \varepsilon_{j_i j_{i+1}}}{(m_i^2 + j_i^2)^2} \right] \sin \alpha_{m_{n+1}} x \sin \alpha j_{n+1} y,$$

where

$$\varepsilon_{m_i, m_{i+1}} = \frac{\frac{m_i \pi}{a} \sin \frac{m_{i+1} \pi}{2} \cos \frac{m_i \pi}{2}}{\frac{\pi^2}{a^2} (m_{i+1}^2 - m_i^2)}, \qquad \varepsilon_{j_i, j_{i+1}} = \frac{\frac{j_i \pi}{a} \sin \frac{j_{i+1} \pi}{2} \cos \frac{j_i \pi}{2}}{\frac{\pi^2}{a^2} (j_{i+1}^2 - j_i^2)}.$$

It should be observed that the subscripts m_i and j_i will take, for $i = 2, 3, \ldots$, the successive values $1, 2, 3, \ldots$.

In our case, it is easy to show that

$$B = 0{,}273, \qquad D = \frac{A}{4}.$$

Thus,

$$|\lambda| \leqq \frac{1}{B \frac{a^2}{4}} = \frac{14{,}7}{a^2}, \qquad \mu = \frac{k a^4}{N} \leqq 356, \qquad \delta = \frac{16}{\pi^6} q \frac{\left(\frac{1}{356}\right)^{n+1} \mu^{n+2}}{1 - \frac{1}{356}\mu}.$$

The relative error ψ for $n = 1, \ldots, 4$ and $\mu = 50$, $\mu = 100$ is shown in Table 4.

The solutions obtained above by the iteration method are valid for a certain range of λ. Bearing in mind that the parameter λ involves the coefficient k (or r) of the elastic foundation, the plate rigidity N and the geometric dimensions of the plate, it is seen that for many practical cases the range of λ may be insufficient.

Table 4

n	1	2	3	4	
ψ	2,28%	0,32%	0,045%	0,006%	for $\mu = 50$
ψ	11%	3,08%	0,86%	0,24%	for $\mu = 100$

The solution can be obtained, however, for any value of λ, by splitting the kernel of the integral equation into a separable part and the «remainder», the number of terms of the separable kernel depending on the desired range of λ. The problem reduces, therefore, to the determination by the iteration method of the resolving kernel for the integral equation and the solution of a system of n equations with n unknowns. The number of equations will be in direct proportion to the range of λ.

In our examples involving elastic supports, we were concerned with integral equations of the type

$$\Phi(x) = \lambda \int_0^c \Phi(\xi) K(x, \xi)\, d\xi + F(x). \tag{3.29}$$

Here the kernel is either bounded or the integral of the square of it exists.

This kernel can be split into two parts:

$$K(x, \xi) = P(x, \xi) + K'(x, \xi).$$

In our cases the kernel was of the type,

$$K(x, \xi) = \sum_{j=1}^{\infty} \gamma_j \sin \frac{j\pi x}{a} \sin \frac{j\pi \xi}{a}.$$

Thus,

$$P(x, \xi) = \sum_{j=1}^{j=m} \gamma_j \sin \frac{j\pi x}{a} \sin \frac{j\pi \xi}{a}, \qquad K'(x, \xi) = \sum_{j=m+1}^{\infty} \gamma_j \sin \frac{j\pi x}{a} \sin \frac{j\pi \xi}{a}.$$

The Eq. (3.29) can be written in the form

$$\Phi(x) - \lambda \int_0^c K'(x, \xi) \Phi(\xi)\, d\xi = F(x) + \lambda \int_0^c \Phi(\xi) P(x, \xi)\, d\xi. \tag{3.30}$$

Let the left-hand member be treated as a known function. The partial solution of the Eq. (3.30) will be

$$(3.31)\qquad \Phi(x) = F(x) + \lambda \int_0^c \Phi(\xi) P(x,\xi)\, d\xi +$$

$$+ \lambda \int_0^c \left\{ \sum_{n=1}^{\infty} \lambda^{n-1} K'_n(x,t) \left[F(t) + \lambda \int_0^c P(t,\xi) \Phi(\xi)\, d\xi \right] \right\} dt,$$

where $K'_n(x,\xi)$ is the n-th iterated kernel for the kernel $K'(x,\xi)$ and $(1/\lambda)\sum_{n=1}^{\infty} \lambda^n K'_n(x,\xi) = \Gamma'(x,\xi,\lambda)$ — the resolving kernel for the Eq. (3.31). The parameter λ should satisfy the conditions:

$$(3.32)\qquad \begin{cases} |\lambda| \leqq \dfrac{1}{B'c} & \text{if the kernel is bounded } [K'(x,\xi) \leqq B'], \\[2ex] |\lambda| \leqq \dfrac{1}{B'} & \text{if integral of the square of the kernel exists} \end{cases}$$

$$\left[B'^2 = \int_0^c \int_0^c K'^2(x,\xi)\, dx\, d\xi, \qquad \int_0^c K^2(x,\xi)\, d\xi < C_1 \right].$$

The number of terms of the separable kernel $P(x,\xi)$ should be in direct proportion to λ. For a given λ, we should determine B' and $K'(x,\xi)$ from the Eqs. (3.32). Hence, the number of terms of the kernel $P(x,\xi)$. Thus, in the example of a strip with a transverse elastic support of $c = a/2$ in length we had $B \approx 1, 2$. If we assume $m = 4$, we obtain $B' = 0{,}0124$ and the permissible value of λ will increase 100 times.

The final solution of the integral equation (3.30) has the form

$$(3.33)\qquad \Phi(x) = f(x) + \lambda \sum_{i=1}^{i=m} c_i a_i(x),$$

where

$$a_i(x) = \sin \alpha_i x + \lambda \int_0^c \Gamma'(x,t,\lambda) \sin \alpha_i t\, dt,$$

$$f(x) = F(x) + \lambda \int_0^c \Gamma'(x,t,\lambda) F(t)\, dt.$$

The coefficients

$$c_i = \int_0^c b_i(\xi) \Phi(\xi)\, d\xi, \qquad b_i = \gamma_i \sin \alpha_i \xi,$$

appearing in the Eq. (3.33) will be obtained from the solution of a system of m equations with m unknowns:

$$c_i - \lambda \sum_{p=1}^{p=m} a_{ip} c_p = f_i, \tag{3.34}$$

where

$$a_{ip} = \int_0^b b_i(\xi) a_i(\xi) d\xi.$$

In the case of a rigid support or clamping along the segment c, the foundation constants k and r tend to infinity. Hence $\lambda \to \infty$.

This is possible if $B' \to 0$, $K'(x, \xi) \to 0$ and the subscript m in the sum $P(x, \xi)$ tends to infinity. At the same time, the resolving kernel tends to zero.

Let us introduce into the Eqs. (3.34) the new quantities $\bar{\lambda} k = \lambda$, $\bar{f}_i k = f_i$, and rewrite the system in the new form

$$\frac{c_i}{k} - \bar{\lambda} \sum_{p=1}^{p=m} a_{ip} c_p = \bar{f}_i \qquad (i = 1, 2, \ldots, m). \tag{3.35}$$

For discontinuous perfectly rigid supports ($k \to \infty$ or $r \to \infty$) we obtain, when $m \to \infty$, the system of equations

$$\bar{f}_i + \bar{\lambda} \sum_{p=1}^{\infty} \bar{a}_{ip} c_p = 0 \qquad (i = 1, 2, \ldots, \infty), \tag{3.36}$$

where $\bar{a}_{ip}$ should be obtained from a_{ip} assuming that $\Gamma' = 0$.

The system of equations (3.36) corresponds to the infinite system of algebraic equations, which was obtained in the first part of this paper as the general solution of the Fredholm equations of the first kind appearing in that problem.

The considerations concerning plates with mixed boundary conditions presented in this paper can easily be transposed to problems of forced vibrations of plates subjected to constant compressive or tensile forces acting in the middle plane.

In the particular case of free vibrations or buckling of a plate, we shall obtain homogeneous integral equations in which the frequency of free vibrations or the buckling force appears as a parameter.

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