ROUGH CONTROLLERS

THEORETICAL FOUNDATIONS

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1. Introduction

Fuzzy controllers are recently very extensively investigated and developed. In this note we give another idea of controllers based on the rough set theory – called rough controllers.

Before we enter more specific considerations concerning the rough controllers we recall basic ideas underlying the rough set concept and next we give some auxiliary definitions necessary to formulate the control problem in the rough set setting.

2. Basic of the Rough Set Concept

Basic ideas of the rough set theory can be found in Pawlak (1991). In this section we will give only those notions which are necessary to formulate the rough control problem considered in this paper, i.e. approximations and the rough membership function.

Let U be a finite, nonempty set called the ***universe***, and let I be an equivalence relation on U, called an ***indiscernibility relation***. By $I(x)$ we mean the set of all y such that xIy, i.e. $I(x) = [x]_I$, i.e.– is an equivalence class of the relation I containing element x. The indiscernibility relation is meant to capture the fact that often we have limited information about elements of the universe and consequently are unable to discern them in view of the available information. Thus I represents our lack of knowledge about U.

We will define now two basic operations on sets in the rough set theory, called the *I-lower* and the *I-upper approximation*, and defined respectively as follows:

$$I_*(X) = \{x \in U\colon\ I(x) \subseteq X\},$$

$$I^*(X) = \{x \in U\colon\ I(x) \cap X \neq \emptyset\}.$$

The difference between the upper and the lower approximation will be called the *I- boundary* of X and will be denoted by $BN_I(X)$, i.e.

$$BN_I(X) = I^*(X) - I_*(X).$$

If $I^*(X) = I_*(X)$ we say the the set is ***I-exact*** otherwise the set X is ***I-rough***. Thus rough sets are sets with unsharp boundaries.

Usually in order to define a set we use the membership function. The membership function for rough sets is defined employing the equivalence relation I as follows:

$$\mu_X^I(x) = \frac{card\ (X \cap I(x))}{card\ \ I(x)}.$$

Obviously

$$\mu_X^I(x) \in [0,1].$$

The value of the membership function expresses the degree to which the element x belong to the set X in view of the indiscernibility relation I.

The above assumed membership function, can be used to define the two previously defined approximations of sets, as shown below:

$$I_*(X) = \{x \in U\colon \mu_X^I(x) = 1\},$$

$$I^*(X) = \{x \in U\colon \mu_X^I(x) > 0\}.$$

3. Rough Sets on the Real Line

In this section we reformulate the concepts of approximations and the rough membership function referring to the set of reals which will be needed in to formulate the control problem in the rough set setting.

Let $\mathbf{R}^+$ be the set of nonnegative reals and let $S \subseteq \mathbf{R}^+$ be the following sequence of reals $x_1, x_2, \ldots, x_i, \ldots$ such that $x_1 < x_2 < \ldots < x_i$. S will be called a ***categorization of*** $\mathbf{R}^+$ and the ordered pair $A = (\mathbf{R}^+, S)$ will be referred to as an ***approximation space***. Every categorization S of $\mathbf{R}^+$ induces partition $\pi(S)$ on $\mathbf{R}^+$ defined as $\pi(S) = \{0,\ (0,x_1),\ x_1,\ (x_1,x_2),$

x_2, (x_2,x_3), x_3, ... , x_i, (x_i,x_{i+1}), x_{i+1}... }, where (x_i,x_{i+1}) denotes an open interval. By $S(x)$ we will denote the interval containing x. In particular, if $x \in S$ then $S(x) = x$. Hence $S(x)$ denotes the equivalence class of the relation $\pi(S)$ containing x.

By $\bar{S}(x) = \langle x_i,x_{i+1}\rangle$ we denote the closed interval, called the *closure* of $S(x)$.

In what follows we will be interested in approximating closed intervals of the form $\langle 0,x\rangle = Q(x)$ for any $x \in \mathbf{R}^+$.

Suppose we are given an approximation space $A = (\mathbf{R}^+,S)$. (Let us remark that the categorization S can be viewed as an indiscernibility relation defined on $\mathbf{R}^+$).

By the the *S-lower* and the *S-upper* approximation of $Q(x)$, denoted by $S_*(Q(x))$ and $S^*(Q(x))$ respectively, we mean sets defined below:

$$S_*(Q(x)) = \{y \in \mathbf{R}^+ : S(y) \subseteq Q(x)\}$$

$$S^*(Q(x)) = \{y \in \mathbf{R}^+ : S(y) \cap Q(x) \neq \emptyset\}.$$

The above definitions of approximations of interval $\langle 0,x)$ can be understood as approximations of the real number x which are simple the ends of the interval $S(x)$, therefore we will use the following abbreviations: $S_*(Q(x)) = S_*(x)$ and $S^*(Q(x)) = S^*(x)$. Thus $S(x) = (S_*(x), S^*(x))$. If general if x and y are ends of a closed or an open interval X, then $y - x$ will be called the *length* of X and will be denoted $\Delta(X)$. In particular $\Delta(S(x))$ will be denoted by $\Delta_S(x)$.

In other words given any real number x and a set of reals S, by the S-lower and the S-upper approximation of x we mean the numbers $S_*(x)$ and $S^*(x)$, which can be defined as

$$S_*(x) = Max\{y \in S : y \leq x\}$$

$$S^*(x) = Min\{y \in S : y \geq x\}.$$

We will say that the number x is *exact* in $A = (\mathbf{R}^+,S)$ iff

$S_*(x) = S^*(x)$, otherwise the number x is *inexact (rough)* in $A = (\mathbf{R}^+, S)$. Of course x is exact iff $x \in S$. Thus ever inexact number x can be presented as pair of exact numbers $S_*(x)$ and $S^*(x)$ or as the interval $S(x)$. For example if $\mathbf{N}$ is the set of all non negative integers then every real number x such that non $x \in \mathbf{N}$ is inexact in the approximation space $A = (\mathbf{R}^+, \mathbf{N})$. In general if $A = (\mathbf{R}^+, S)$ is an approximation space then the categorization S can be interpreted as a scale by means of which reals from $\mathbf{R}^+$ are measured with some approximation due to the scale S.

The introduced ideas of the rough set on the real line correspond exactly to those defined for arbitrary sets and can be seen as a special case of the general definition.

Now we give the definition of the next basic notion in the rough set approach - the rough membership function - referring to the real line.

The rough membership function for the set of reals will have the form

$$\mu_{Q(x)}(y) = \frac{\Delta(Q(x) \cap S(y))}{\Delta_S(y)}.$$

The membership function $\mu_{Q(x)}(y)$ says to what degree any element y belongs to the interval $Q(x)$. We will however need a specific case of the rough membership function for which $y = x$. In this case the rough membership function can be presented as

$$\mu(x) = \frac{x - S_*(x)}{\Delta_S(x)}$$

4. Rough Functions

In this section we will define basic concepts concerning real functions which are necessary to define rough controllers.

Suppose we are given real a function $f : X \to Y$, where both X and Y are sets of non negative reals and let $A = (X, S)$ and $B = (Y, P)$ be two approximation spaces.

By the (S,P)-*lower approximation* of f we understand the function $f_*: X \to Y$ such that

$$f_*(x) = P_*(f(x)) \text{ for every } x \in X.$$

Similarly the (S,P)-*upper approximation* of f is defined as

$$f^*(x) = P^*(f(x)) \text{ for every } x \in X.$$

We say that a function f is ***exact*** in x iff $f_*(x) = f^*(x)$; otherwise the function f is *inexact (rough)* in x. The number $f^*(x) - f_*(x)$ is the error of approximation of f in x.

A function f is *(S,P)-continuous (roughly continuous)* in x iff

$$f(\overline{S}(x)) \subseteq \overline{P}(f(x)).$$

If f is roughly continuous in x for every $x \in X$ we say that f is *(S,P)-continuous*.

The function $f_S: S \to Y$ such that $f_S(x) = f(x)$ for any $x \in S$ will be called a *S-discrete representation* of f or in short *S-discretization* of f..

Our main task is to give interpolation algoritmhs for discrete representation f_S giving the best approximation of f.

Let us first consider the *linear interpolation* formula. The linear interpolation of f will be dentoted by f_α and is defined as follows:

$$f_\alpha(x) = f(S_*(x)) \pm \mu^S(x)\ \Delta f(S(x)),$$

where $\Delta f(S(x)) = f(S^*(x)) - f(S_*(x))$.
The number

$$\frac{|f_\alpha(x) - f(x)|}{f(x)}$$

will be called the realative error of the interpolation of f in x. The maximal errror of interpolation will be called the error of interpolation of f. We will be also intrested in the following problem. Given a function $f: X \rightarrow Y$ and a number $0 \leq \varepsilon \leq 1$. Find categorizations S and P such the the error of interpolation of f is less than ε.

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Ciąg dalszy nastąpi.

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