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Abstract . Содержание . Streszczenie

In this note we define rough functions, i.e. functions arguments and values of which are not known exactly. The definition of rough continuity of functions is given and some elementary topological properties of rough continuity are stated.

The presented approach may be considered as an alternative to "fuzzy" philosophy in mathematics.

Приближенные функции

В работе дано определение понятия приближенной функции как результат приближенного множества и отношения. Дано определение приближенной непрерывности функции а также несколько основных особенностей приближенной непрерывности.

Представленный подход обсуждается как альтернатива размытых функций.

Funkcje przybliżone

W pracy zdefiniowano pojęcie funkcji przybliżonej, jako konsekwencje przybliżonego zbioru i relacji. Podano następnie definicje przybliżonej ciągłości funkcji i podano kilka podstawowych własności przybliżonej ciągłości.

Przedstawione podejście może być traktowane jako alternatywa funkcji "rozmytych".

1. INTRODUCTION

This note is a continuation of [1] and [2], where the notions of a rough set and a rough relation were introduced; here we introduce the notion of a rough function.

Intuitively speaking, rough function is a function arguments and values of which are not known exactly, but only with some accuracy determined by a "indiscernibility relation", which express the accuracy of our observations, measurements or descriptions.

Rough functions may be applied in measurement theory pattern recognitions, some fields of artificial intelligence, and other branches, however we shall not discuss the applications in this paper.

The proposed approach can be considered as an alternative to "fuzzy" philosophy.

2. ROUGH SETS

Before we define rough functions we recall the notion of a rough set and a rough relation after [1] and [2].

The pair $A = \langle X,R \rangle$, where X is some set, and R an equivalence relation on X (called indiscernibility relation) will be called an <u>approximation space</u>.

Equivalence classes of R are called <u>elementary sets</u> in A, and every union of elementary sets in A, is called a composed set of A.

If $Y \subset X$, then the least composed set in A containing Y will be called the <u>best upper approximation</u> of Y in A, and will be denoted by $\overline{A}Y$. The greatest composed set in A contained in Y will be called the <u>best lower approximation</u> of Y in A, and will be denoted by $\underline{A}Y$.

The following properties of approximations are valid for every approximation space $A = \langle X,R \rangle$, and sets $Y,Z \subset X$.

1) $\overline{A}Y \supset Y \supset \underline{A}Y$,
2) $\overline{A}1 = \underline{A}1 = 1$,
3) $\overline{A}\emptyset = \underline{A}\emptyset = \emptyset$,
4) $\overline{A}\,\overline{A}Y = \underline{A}\,\overline{A}Y = \overline{A}Y$,
5) $\underline{A}\,\underline{A}Y = \overline{A}\,\underline{A}Y = \underline{A}Y$,
6) $\overline{A}(Y \cup Z) = \overline{A}Y \cup \overline{A}Z$,
7) $\underline{A}(Y \cap Z) = \underline{A}Y \cap \underline{A}Z$,
8) $\overline{A}Y = -\underline{A}(-Y)$,
9) $\underline{A}Y = -\overline{A}(-Y)$,
10) $\overline{A}(Y \cap Z) \subset \overline{A}Y \cap \overline{A}Z$,
11) $\underline{A}(Y \cup Z) \supset \underline{A}Y \cup \underline{A}Y$,
12) $\overline{A}Y - \overline{A}Z \subset \overline{A}(Y-Z)$,
13) $\underline{A}Y - \underline{A}Z \supset \underline{A}(Y-Z)$.

We use 1 to denote the whole set X of the approximation space $A = \langle X,R \rangle$, and $-Y$ is to denote $1-Y$ or $X-Y$.

3. ROUGH RELATION

We shall define here binary rough relations only. N- arguments relations can be defined in a similar way, however we shall not consider these relations here.

Let $B = \langle Y,Q \rangle$, and $C = \langle Z,P \rangle$ be two approximation spaces. By a product of B and C we shall mean the space $A = \langle X,R \rangle$, where $X = Y \times Z$ and $R \subset (Y \times Z)^2$ is defined as follows: $R((y_1,z_1),(y_2,z_2))$ iff $Q(y_1,y_2)$ and $P(z_1,z_2)$, where $y_1,y_2 \in Y$, $z_1,z_2 \in Z$. Product of B and C is denoted by $B \times C$.

Any equivalence class of the relation R will be called an <u>elementary relation</u> in $A = \langle X,R \rangle$. Union of any number of elementary relations in A will be called a <u>composed relation</u> in A.

Let $A = B \times C$, where $B = \langle Y,Q \rangle$, and $C = \langle Z,P \rangle$, and let $S \subset Y \times Z$ be some binary relation on $Y \times Z$.

By the <u>best upper approximation</u> of S in A (denoted $\overline{A}S$) we shall mean the least composed relation in A containing S.

By the <u>best lower approximation</u> of S in A (denoted $\underline{A}S$) we shall mean the greatest composed relation in A contained in S.

Some elementary properties of rough relations are given in [2].

4. ROUGH FUNCTIONS

Let $B = \langle Y,Q \rangle$, $C = \langle Z,P \rangle$ be two approximation spaces and let $A = B \times C$ be a product space.

If $f : Y \to Z$, then from the definition of upper approximation of a relation (f is considered as a relation) we have:

$\overline{A}f = \{e \in E_A : e \cap f \neq \emptyset\}$, where E_A is the set of all equivalence classes in the approximation space $A = \langle X,R \rangle$.

The above property is depicted in Fig. 1.

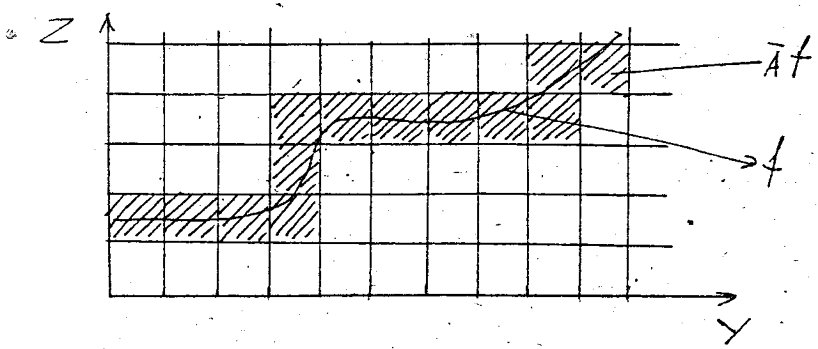

Fig. 1

Let $A = \langle X,R \rangle$ be a product approximation space of $B = \langle Y,Q \rangle$ and $C = \langle Z,P \rangle$, and let $f : Y \to Z$. The <u>r-representation</u> of f in A, denoted $\hat{A}f$, we shall define as

$$\hat{A}f = \bar{A}f/R,$$

where $\bar{A}f/R$ is a quotient relation (the relation $\bar{A}f$ divided by the equivalence relation R).

The <u>r-approximation</u> of f in A, denoted A^*f, is any function $g : Y/Q \to Z/C$, such that $g \subset \hat{A}f$ and $D_g = D_{\hat{A}f}$. ($D_g, D_{\hat{A}f}$ denote domains of the function g and the relation $\hat{A}f$ respectively).

We shall say that the function $f : Y \to Z$ is <u>roughly continuous</u> (<u>r-continuous</u>) in $A = \langle X,R \rangle$, where $A = B \times C$, and $B = \langle Y,Q \rangle$, $C = \langle Z,P \rangle$, in the point $y_0 \in Y$ if

$$y_0 \in \bar{B}Y' \Rightarrow f(y_0) \in \bar{C}f(Y'),$$

for every $Y' \subset Y$.

<u>Theorem</u>. The function $f : Y \to Z$ is r-continuous in $A = \langle X,R \rangle$, in the point $y_0 \in Y$ iff $y \in e_{y_0}$ implies $f(y) \in e_{f(y_0)}$) where e_{y_0} and $e_{f(y_0)}$ are elementary sets in B and C containing points y_0 and $f(y_0)$ respectively.

The function $f : Y \to Z$ is <u>r-continuous</u> in A iff f is <u>r-continuous</u> in A for every $y \in Y$.

<u>Theorem</u>. The function $f : Y \to Z$ is r-continuous in A iff the r-representation of f, $\hat{A}f$ is a function, or in other words if f have exactly one r-approximation A^*f in A.

<u>Theorem</u>. If $f : Y \to Z$ is r-continuous in A, then

$$f(\bar{B}Y') \subset \bar{C}f(Y'),$$

for every $Y' \subset Y$.

<u>Theorem</u>. If $f : Y \to Z$ is continuous then there exist an approximation space $A = \langle X,R \rangle$, such that f is r-continuous in A.

If in the approximation space $A = \langle X,R \rangle$ every equivalence class of the indiscernibility relation R contains exactly one element we shall call A a selective approximation space.

<u>Theorem</u>. Let $A = \langle X,R \rangle$ be a selective approximation space and let $A = B \times C$, where $B = \langle Y,Q \rangle$ and $C = \langle Z,P \rangle$. Every function $f : Y \to Z$ is r-continuous in A, and for every function $f : Y \to Z$, $A^*f = f$; moreover

$$f(\bar{B}Y') = \bar{C}f(Y'),$$

for every $Y' \subset Y$.

4. FINAL REMARKS

The intuitive meaning of r-continuity is the following. A function $f : Y \to Z$ is r-continuous in a given approximation space A, if f does not change its values "to fast" in comparison to observation possibilities, defined be an indiscernibility relation of A.

For example the function shown in Fig. 1 is not r-continuous in the approximation system shown on the picture, however the function shown in Fig. 2 is r-continuous in the corresponding approximation space.

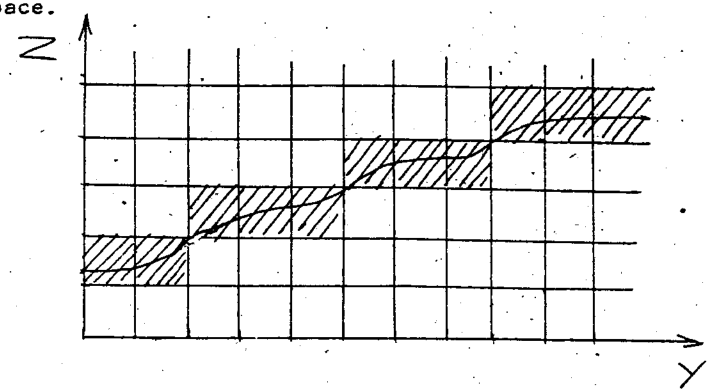

Fig. 2

Of course the same function f can be r-continuous in one approximation space, and be not r-continuous in another one. For example if we increase the accuracy of observation, i.e., take "finer" approximation space, the function shown in Fig. 1 can be r-continuous. One can also give an example of a function which is r-continuous in one approximation space but not r-continuous in a "finer" approximation space.

In other words in our approach the notion of continuity is related to the exactness of observation.

It seems to be not very difficult task to express basic notions of standard analysis in terms of "rough" approach, obtaining thus a tool to deal with some problems of computer sciences and may be physics too.

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[1] Z. Pawlak, Rough Sets, ICS PAS, Reports, 431, 1981.

[2] Z. Pawlak, Rough relations, ICS PAS, Reports, 435, 1981.