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E-Service Intelligence

Methodologies, Technologies and Applications

With 190 Figures and 68 Tables

Springer

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Rough Sets and Conflict Analysis

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Commemorating the life and work of Zdzisław Pawlak[1].

Summary. E-service intelligence requires tools for approximate reasoning about vague concepts. The rough set based methods make it possible to deal with imperfect knowledge. In particular, approximate reasoning about vague concepts can be based on the rough set approach. This approach seems to be of fundamental importance to AI and cognitive sciences, especially in the areas of machine learning and pattern recognition, knowledge acquisition, decision analysis, data mining and knowledge discovery from databases, expert systems, and conflict analysis. We discuss the basic concepts of rough sets and we outline some current research directions on rough sets. Conflict analysis is one of the basic issues in e-service intelligence. The contribution of this article is an extension of the existing approach based on rough sets to conflict analysis.

1 Introduction

E-service intelligence has been identified as a new e-service direction with many potential applications in different areas such as governmental management, medicine, business, learning, banking, science (e.g., habitat monitoring with wireless sensor networks) or logistics. To realize intelligent presentation of web content, intelligent online services, personalized support or direct customer participation in organizational decision-making processes, it is neces-

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[1] Professor Zdzisław Pawlak passed away on 7 April 2006.

sary to develop methods that will make it possible to understand vague concepts and reasoning schemes expressed in natural language by humans who will cooperate with e-services. Hence, methods for approximation of vague concepts as well as methods for approximate reasoning along with reasoning performed in natural language are needed. In this article, we discuss some basic concepts of the rough set approach created for dealing with vague concepts and for approximate reasoning about vague concepts. Among the most important issues of e-service intelligence are also conflict analysis and negotiations. We also outline an approach for conflict analysis based on the rough set approach.

2 Preliminaries of Rough Sets

This section briefly delineates basic concepts in rough set theory. Basic ideas of rough set theory and its extensions, as well as many interesting applications can be found in books (see, e.g., [14, 18, 21, 22, 30, 35, 39, 40, 50, 52, 57, 67, 70, 71, 74, 84, 87, 90, 91, 100, 129, 151]), issues of the Transactions on Rough Sets [79, 80, 81, 82], special issues of other journals (see, e.g., [13, 48, 69, 78, 110, 130, 154, 155]), proceedings of international conferences (see, e.g., [1, 34, 51, 89, 109, 120, 137, 138, 142, 153, 156, 126, 127, 139]), tutorials (see, e.g., [38]), and on the internet such as www.roughsets.org, logic.mimuw.edu.pl, rsds.wsiz.rzeszow.pl.

2.1 Rough Sets: An Introduction

Rough set theory, proposed by Pawlak in 1982 [74, 73] can be seen as a new mathematical approach to vagueness.

The rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information (data, knowledge). For example, if objects are patients suffering from a certain disease, symptoms of the disease form information about patients. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. The *indiscernibility relation* generated in this way is the mathematical basis of rough set theory. This understanding of indiscernibility is related to the idea of Gottfried Wilhelm Leibniz that objects are indiscernible if and only if all available functionals take on them identical values (Leibniz's Law of Indiscernibility: The Identity of Indiscernibles) [2, 44]. However, in the rough set approach indiscernibility is defined relative to a given set of functionals (attributes).

Any set of all indiscernible (similar) objects is called an elementary set, and forms a basic granule (atom) of knowledge about the universe. Any union of some elementary sets is referred to as *crisp* (precise) set. If a set is not crisp then it is called *rough* (imprecise, vague).

Consequently, each rough set has *borderline cases* (*boundary–line*), i.e., objects which cannot be classified with certainty as members of either the set or its complement. Obviously crisp sets have no borderline elements at all. This means that borderline cases cannot be properly classified by employing available knowledge.

Thus, the assumption that objects can be "seen" only through the information available about them leads to the view that knowledge has granular structure. Due to the granularity of knowledge, some objects of interest cannot be discerned and appear as the same (or similar). As a consequence, vague concepts in contrast to precise concepts, cannot be characterized in terms of information about their elements. Therefore, in the proposed approach, we assume that any vague concept is replaced by a pair of precise concepts – called the lower and the upper approximation of the vague concept. The lower approximation consists of all objects which definitely belong to the concept and the upper approximation contains all objects which possibly belong to the concept. The difference between the upper and the lower approximation constitutes the boundary region of the vague concept. Approximations are two basic operations in rough set theory.

Hence, rough set theory expresses vagueness not by means of membership, but by employing a boundary region of a set. If the boundary region of a set is empty it means that the set is crisp, otherwise the set is rough (inexact). A nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely.

Rough set theory it is not an alternative to classical set theory but it is embedded in it. Rough set theory can be viewed as a specific implementation of Frege's idea of vagueness, i.e., imprecision in this approach is expressed by a boundary region of a set.

Rough set theory has attracted attention of many researchers and practitioners all over the world, who have contributed essentially to its development and applications. Rough set theory overlaps with many other theories. Despite this overlap, rough set theory may be considered as an independent discipline in its own right. The rough set approach seems to be of fundamental importance in artificial intelligence and cognitive sciences, especially in research areas such as machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, knowledge discovery, decision analysis, and expert systems. The main advantage of rough set theory in data analysis is that it does not need any preliminary or additional information about data like probability distributions in statistics, basic probability assignments in Dempster–Shafer theory, a grade of membership or the value of possibility in fuzzy set theory. One can observe the following about the rough set approach:

- introduction of efficient algorithms for finding hidden patterns in data,
- determination of optimal sets of data (data reduction),
- evaluation of the significance of data,
- generation of sets of decision rules from data,

- easy-to-understand formulation,
- straightforward interpretation of obtained results,
- suitability of many of its algorithms for parallel processing.

2.2 Indiscernibility and Approximation

The starting point of rough set theory is the indiscernibility relation, which is generated by information about objects of interest (see Sect. 2.1). The indiscernibility relation expresses the fact that due to a lack of information (or knowledge) we are unable to discern some objects employing available information (or knowledge). This means that, in general, we are unable to deal with each particular object but we have to consider granules (clusters) of indiscernible objects as a fundamental basis for our theory.

From a practical point of view, it is better to define basic concepts of this theory in terms of data. Therefore we will start our considerations from a data set called an *information system*. An information system is a data table containing rows labeled by objects of interest, columns labeled by attributes and entries of the table are attribute values. For example, a data table can describe a set of patients in a hospital. The patients can be characterized by some attributes, like *age, sex, blood pressure, body temperature*, etc. With every attribute a set of its values is associated, e.g., values of the attribute *age* can be *young, middle*, and *old*. Attribute values can be also numerical. In data analysis the basic problem we are interested in is to find patterns in data, i.e., to find a relationship between some set of attributes, e.g., we might be interested whether *blood pressure* depends on *age and sex*.

Suppose we are given a pair $\mathcal{A} = (U, A)$ of non-empty, finite sets U and A, where U is the *universe* of *objects*, and A – a set consisting of *attributes*, i.e. functions $a : U \longrightarrow V_a$, where V_a is the set of values of attribute a, called the *domain* of a. The pair $\mathcal{A} = (U, A)$ is called an *information system* (see, e.g., [72]). Any information system can be represented by a data table with rows labeled by objects and columns labeled by attributes. Any pair (x, a), where $x \in U$ and $a \in A$ defines the table entry consisting of the value $a(x)$[2].

Any subset B of A determines a binary relation $I(B)$ on U, called an *indiscernibility relation*, defined by

$$x I(B) y \text{ if and only if } a(x) = a(y) \text{ for every } a \in B, \tag{1}$$

where $a(x)$ denotes the value of attribute a for object x.

Obviously, $I(B)$ is an equivalence relation. The family of all equivalence classes of $I(B)$, i.e., the partition determined by B, will be denoted by $U/I(B)$, or simply U/B; an equivalence class of $I(B)$, i.e., the block of the partition U/B, containing x will be denoted by $B(x)$ (other notation used: $[x]_B$ or

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[2] Note, that in statistics or machine learning such a data table is called a sample [25].

$[x]_{I(B)}$). Thus in view of the data we are unable, in general, to observe individual objects but we are forced to reason only about the accessible granules of knowledge (see, e.g., [70, 74, 94]).

If $(x, y) \in I(B)$ we will say that x and y are *B-indiscernible*. Equivalence classes of the relation $I(B)$ (or blocks of the partition U/B) are referred to as *B-elementary sets* or *B-elementary granules*. In the rough set approach the elementary sets are the basic building blocks (concepts) of our knowledge about reality. The unions of *B-elementary sets* are called *B-definable sets*[3].

For $B \subseteq A$ we denote by $Inf_B(x)$ the B-signature of $x \in U$, i.e., the set $\{(a, a(s)) : a \in A\}$. Let $INF(B) = \{Inf_B(s) : s \in U\}$. Then for any objects $x, y \in U$ the following equivalence holds: $xI(B)y$ if and only if $Inf_B(x) = Inf_B(y)$.

The indiscernibility relation will be further used to define basic concepts of rough set theory. Let us define now the following two operations on sets $X \subseteq U$

$$B_*(X) = \{x \in U : B(x) \subseteq X\}, \tag{2}$$

$$B^*(X) = \{x \in U : B(x) \cap X \neq \emptyset\}, \tag{3}$$

assigning to every subset X of the universe U two sets $B_*(X)$ and $B^*(X)$ called the *B-lower* and the *B-upper approximation* of X, respectively. The set

$$BN_B(X) = B^*(X) - B_*(X), \tag{4}$$

will be referred to as the *B-boundary region* of X.

From the definition we obtain the following interpretation:

- The *lower approximation* of a set X with respect to B is the set of all objects, which can be for *certain* classified as X using B (are *certainly X* in view of B).
- The *upper approximation* of a set X with respect to B is the set of all objects which can be *possibly* classified as X using B (are *possibly X* in view of B).
- The *boundary region* of a set X with respect to B is the set of all objects, which can be classified neither as X nor as not-X using B.

In other words, due to the granularity of knowledge, rough sets cannot be characterized by using available knowledge. Therefore with every rough set we associate two *crisp* sets, called *lower* and *upper approximation*. Intuitively, the lower approximation of a set consists of all elements that *definitely* belong to the set, whereas the upper approximation of the set constitutes of all elements that *possibly* belong to the set, and the *boundary region* of the set consists of all elements that cannot be classified uniquely to the set or its complement, by employing available knowledge. The approximation definition is clearly depicted in Figure 1.

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[3] One can compare data tables corresponding to information systems with relations in relational databases [26].
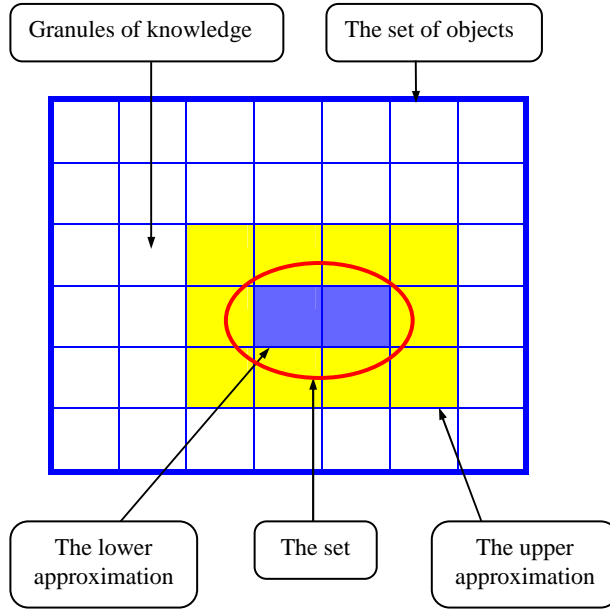
Fig. 1. A rough set

The approximations have the following properties:

$$B_*(X) \subseteq X \subseteq B^*(X), \tag{5}$$
$$B_*(\emptyset) = B^*(\emptyset) = \emptyset, B_*(U) = B^*(U) = U,$$
$$B^*(X \cup Y) = B^*(X) \cup B^*(Y),$$
$$B_*(X \cap Y) = B_*(X) \cap B_*(Y),$$
$$X \subseteq Y \text{ implies } B_*(X) \subseteq B_*(Y) \text{ and } B^*(X) \subseteq B^*(Y),$$
$$B_*(X \cup Y) \supseteq B_*(X) \cup B_*(Y),$$
$$B^*(X \cap Y) \subseteq B^*(X) \cap B^*(Y),$$
$$B_*(-X) = -B^*(X),$$
$$B^*(-X) = -B_*(X),$$
$$B_*(B_*(X)) = B^*(B_*(X)) = B_*(X),$$
$$B^*(B^*(X)) = B_*(B^*(X)) = B^*(X).$$

Let us note that the inclusions in (5) cannot be in general substituted by the equalities. This has some important algorithmic and logical consequences.

Now we are ready to give the definition of rough sets.

If the boundary region of X is the empty set, i.e., $BN_B(X) = \emptyset$, then the set X is *crisp* (*exact*) with respect to B; in the opposite case, i.e., if $BN_B(X) \neq \emptyset$, the set X is referred to as *rough* (*inexact*) with respect to B.

Thus any rough set, in contrast to a crisp set, has a non-empty boundary region.

One can define the following four basic classes of rough sets, i.e., four categories of vagueness:

$$B_*(X) \neq \emptyset \text{ and } B^*(X) \neq U, \text{ iff } X \text{ is } \textit{roughly B-definable}, \qquad (6)$$
$$B_*(X) = \emptyset \text{ and } B^*(X) \neq U, \text{ iff } X \text{ is } \textit{internally B-indefinable},$$
$$B_*(X) \neq \emptyset \text{ and } B^*(X) = U, \text{ iff } X \text{ is } \textit{externally B-indefinable},$$
$$B_*(X) = \emptyset \text{ and } B^*(X) = U, \text{ iff } X \text{ is } \textit{totally B-indefinable}.$$

The intuitive meaning of this classification is the following.

If X is roughly B-*definable*, this means that we are able to decide for some elements of U that they belong to X and for some elements of U we are able to decide that they belong to $-X$, using B.

If X is internally B-indefinable, this means that we are able to decide about some elements of U that they belong to $-X$, but we are unable to decide for any element of U that it belongs to X, using B.

If X is externally B-indefinable, this means that we are able to decide for some elements of U that they belong to X, but we are unable to decide, for any element of U that it belongs to $-X$, using B.

If X is totally B-indefinable, we are unable to decide for any element of U whether it belongs to X or $-X$, using B.

Thus a set is *rough* (imprecise) if it has nonempty boundary region; otherwise the set is crisp (precise). This is exactly the idea of vagueness proposed by Frege.

Let us observe that the definition of rough sets refers to data (knowledge), and is *subjective*, in contrast to the definition of classical sets, which is in some sense an *objective* one.

A rough set can also be characterized numerically by the following coefficient

$$\alpha_B(X) = \frac{card(B_*(X))}{card(B^*(X))}, \qquad (7)$$

called the *accuracy of approximation*, where $card(X)$ denotes the cardinality of $X \neq \emptyset$. Obviously $0 \leq \alpha_B(X) \leq 1$. If $\alpha_B(X) = 1$ then X is *crisp* with respect to B (X is *precise* with respect to B), and otherwise, if $\alpha_B(X) < 1$ then X is *rough* with respect to B (X is *vague* with respect to B). The accuracy of approximation can be used to measure the quality of approximation of decision classes on the universe U. One can use another measure of accuracy defined by $1 - \alpha_B(X)$ or by $1 - \frac{card(BN_B(X))}{card(U)}$. Some other measures of approximation accuracy are also used, e.g., based on entropy or some more specific properties of boundary regions (see, e.g., [108, 122, 27]). The choice of a relevant accuracy of approximation depends on a particular data set. Observe that the accuracy of approximation of X can be tuned by B. Another approach to accuracy of approximation can be based on the Variable Precision Rough Set Model (VPRSM) [152] (see Section 3.1).

In the next section, we discuss decision rules (constructed over a selected set B of features or a family of sets of features) which are used in inducing classification algorithms (classifiers) making it possible to classify to decision classes unseen objects. Parameters which are tuned in searching for a classifier with the high quality are its description size (defined using decision rules) and its quality of classification (measured by the number of misclassified objects on a given set of objects). By selecting a proper balance between the accuracy of classification and the description size we expect to find the classifier with the high quality of classification also on unseen objects. This approach is based on the minimal description length principle [97, 98, 124].

2.3 Decision Systems and Decision Rules

Sometimes we distinguish in an information system $\mathcal{A} = (U, A)$ a partition of A into two classes $C, D \subseteq A$ of attributes, called *condition* and *decision* (*action*) attributes, respectively. The tuple $\mathcal{A} = (U, C, D)$ is called a *decision system*.

Let $V = \bigcup \{V_a | a \in C\} \bigcup \{V_d | d \in D\}$. Atomic formulae over $B \subseteq C \cup D$ and V are expressions $a = v$ called *descriptors* (*selectors*) over B and V, where $a \in B$ and $v \in V_a$. The set $\mathcal{F}(B, V)$ of formulae over B and V is the least set containing all atomic formulae over B and V and closed with respect to the propositional connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation).

By $\|\varphi\|_{\mathcal{A}}$ we denote the meaning of $\varphi \in \mathcal{F}(B, V)$ in the decision table $\mathcal{A}$ which is the set of all objects in U with the property φ. These sets are defined by $\|a = v\|_{\mathcal{A}} = \{x \in U \mid a(x) = v\}$, $\|\varphi \wedge \varphi'\|_{\mathcal{A}} = \|\varphi\|_{\mathcal{A}} \cap \|\varphi'\|_{\mathcal{A}}$; $\|\varphi \vee \varphi'\|_{\mathcal{A}} = \|\varphi\|_{\mathcal{A}} \cup \|\varphi'\|_{\mathcal{A}}$; $\|\neg\varphi\|_{\mathcal{A}} = U - \|\varphi\|_{\mathcal{A}}$ The formulae from $\mathcal{F}(C, V)$, $\mathcal{F}(D, V)$ are called *condition formulae of $\mathcal{A}$* and *decision formulae of $\mathcal{A}$*, respectively.

Any object $x \in U$ belongs to the *decision class* $\| \bigwedge_{d \in D} d = d(x) \|_{\mathcal{A}}$ of $\mathcal{A}$. All decision classes of $\mathcal{A}$ create a partition U/D of the universe U.

A *decision rule* for $\mathcal{A}$ is any expression of the form $\varphi \Rightarrow \psi$, where $\varphi \in \mathcal{F}(C, V)$, $\psi \in \mathcal{F}(D, V)$, and $\|\varphi\|_{\mathcal{A}} \neq \emptyset$. Formulae φ and ψ are referred to as the *predecessor* and the *successor* of decision rule $\varphi \Rightarrow \psi$. Decision rules are often called "$IF \dots THEN \dots$" rules. Such rules are used in machine learning (see, e.g., [25]).

Decision rule $\varphi \Rightarrow \psi$ is *true* in $\mathcal{A}$ if and only if $\|\varphi\|_{\mathcal{A}} \subseteq \|\psi\|_{\mathcal{A}}$. Otherwise, one can measure its *truth degree* by introducing some inclusion measure of $\|\varphi\|_{\mathcal{A}}$ in $\|\psi\|_{\mathcal{A}}$.

Given two unary predicate formulae $\alpha(x), \beta(x)$ where x runs over a finite set U, Lukasiewicz [53] proposes to assign to $\alpha(x)$ the value $\frac{card(\|\alpha(x)\|)}{card(U)}$, where $\|\alpha(x)\| = \{x \in U : x \text{ satisfies } \alpha\}$. The fractional value assigned to the implication $\alpha(x) \Rightarrow \beta(x)$ is then $\frac{card(\|\alpha(x) \wedge \beta(x)\|)}{card(\|\alpha(x)\|)}$ under the assumption that $\|\alpha(x)\| \neq \emptyset$. Proposed by Lukasiewicz, that fractional part was much later adapted by machine learning and data mining literature.

Each object x of a decision system determines a *decision rule*

$$\bigwedge_{a \in C} a = a(x) \Rightarrow \bigwedge_{d \in D} d = d(x). \tag{8}$$

For any decision table $\mathcal{A} = (U, C, d)$ one can consider a *generalized decision function* $\partial_A : U \longrightarrow \mathrm{Pow}(\times_{d \in D} V_d)$ defined by

$$\partial_A(x) = \left\{ i \; : \; \exists x' \in U \, [\, (x', x) \in I(A) \text{ and } d(x') = i] \right\}, \tag{9}$$

where $\mathrm{Pow}(V_d)$ is the powerset of the Cartesian product $\times_{d \in D} V_d$ of the family $\{V_d\}_{d \in D}$.

$\mathcal{A}$ is called *consistent (deterministic)*, if $card(\partial_A(x)) = 1$, for any $x \in U$. Otherwise $\mathcal{A}$ is said to be *inconsistent (non-deterministic)*. Hence, a decision table is inconsistent if it consists of some objects with different decisions but indiscernible with respect to condition attributes. Any set consisting of all objects with the same generalized decision value is called a *generalized decision class*. Now, one can consider certain (possible) rules (see, e.g. [31, 33]) for decision classes defined by the lower (upper) approximations of such generalized decision classes of $\mathcal{A}$. This approach can be extend, using the relationships of rough sets with the Dempster-Shafer theory (see, e.g., [108, 101]), by considering rules relative to decision classes defined by the lower approximations of unions of decision classes of $\mathcal{A}$.

Numerous methods have been developed for different decision rule generation that the reader can find in the literature on rough sets (see also Section 3.2). Usually, one is searching for decision rules (semi) optimal with respect to some optimization criteria describing quality of decision rules in concept approximations.

In the case of searching for concept approximation in an extension of a given universe of objects (sample), the following steps are typical. When a set of rules has been induced from a decision table containing a set of training examples, they can be inspected to see if they reveal any novel relationships between attributes that are worth pursuing for further research. Furthermore, the rules can be applied to a set of unseen cases in order to estimate their classificatory power. For a systematic overview of rule application methods the reader is referred to the literature (see, e.g., [56, 3] and also Section 3.2).

2.4 Dependency of Attributes

Another important issue in data analysis is discovering dependencies between attributes in a given decision system $\mathcal{A} = (U, C, D)$. Intuitively, a set of attributes D depends totally on a set of attributes C, denoted $C \Rightarrow D$, if the values of attributes from C uniquely determine the values of attributes from D. In other words, D depends totally on C, if there exists a functional dependency between values of C and D. Hence, $C \Rightarrow D$ if and only if the rule (8) is true on $\mathcal{A}$ for any $x \in U$. D can depend partially on C. Formally such a dependency can be defined in the following way.

We will say that D *depends on* C to a *degree* k $(0 \leq k \leq 1)$, denoted $C \Rightarrow_k D$, if

$$k = \gamma(C, D) = \frac{card(POS_C(D))}{card(U)} \tag{10}$$

where

$$POS_C(D) = \bigcup_{X \in U/D} C_*(X), \tag{11}$$

called a *positive region* of the partition U/D with respect to C, is the set of all elements of U that can be uniquely classified to blocks of the partition U/D, by means of C.

If $k = 1$ we say that D *depends totally* on C, and if $k < 1$, we say that D *depends partially* (to *degree* k) on C. If $k = 0$ then the *positive region* of the partition U/D with respect to C is empty.

The coefficient k expresses the ratio of all elements of the universe, which can be properly classified to blocks of the partition U/D, employing attributes C and will be called the *degree of the dependency.*

It can be easily seen that if D depends totally on C then $I(C) \subseteq I(D)$. It means that the partition generated by C is finer than the partition generated by D. Notice, that the concept of dependency discussed above corresponds to that considered in relational databases.

Summing up: D is *totally* (*partially*) dependent on C, if *all* (*some*) elements of the universe U can be uniquely classified to blocks of the partition U/D, employing C.

Observer, that (10) defines only one of possible measures of dependency between attributes (see, e.g., [122]). One also can compare the dependency discussed in this section with dependencies considered in databases [26].

2.5 Reduction of Attributes

We often face a question whether we can remove some data from a data-table preserving its basic properties, that is – whether a table contains some superfluous data.

Let us express this idea more precisely.

Let $C, D \subseteq A$, be sets of condition and decision attributes respectively. We will say that $C' \subseteq C$ is a *D-reduct* (reduct with *respect* to D) of C, if C' is a minimal subset of C such that

$$\gamma(C, D) = \gamma(C', D). \tag{12}$$

The intersection of all D-reducts is called a *D-core* (core with *respect* to D). Because the core is the intersection of all reducts, it is included in every reduct, i.e., each element of the core belongs to some reduct. Thus, in a sense, the core is the most important subset of attributes, since none of its elements can be removed without affecting the classification power of attributes. Certainly,

the geometry of reducts can be more compound. For example, the core can be empty but there can exist a partition of reducts into a few sets with non empty intersection.

Many other kinds of reducts and their approximations are discussed in the literature (see, e.g., [5, 59, 60, 102, 121, 123, 124]). It turns out that they can be efficiently computed using heuristics based, e.g., on the Boolean reasoning approach.

2.6 Discernibility and Boolean Reasoning

Methodologies devoted to data mining, knowledge discovery, decision support, pattern classification, approximate reasoning require tools aimed at discovering *templates (patterns)* in data and classifying them into certain *decision classes*. Templates are in many cases most frequent sequences of events, most probable events, regular configurations of objects, the decision rules of high quality, standard reasoning schemes. Tools for discovering and classifying of templates are based on *reasoning schemes* rooted in various paradigms [20]. Such patterns can be extracted from data by means of methods based, e.g., on Boolean reasoning and discernibility.

The discernibility relations are closely related to indiscernibility and belong to the most important relations considered in rough set theory.

The ability to discern between perceived objects is important for constructing many entities like reducts, decision rules or decision algorithms. In the classical rough set approach the *discernibility relation $DIS(B) \subseteq U \times U$* is defined by $xDIS(B)y$ if and only if $non(xI(B)y)$. However, this is in general not the case for the generalized approximation spaces (one can define indiscernibility by $x \in I(y)$ and discernibility by $I(x) \cap I(y) = \emptyset$ for any objects x, y where $I(x) = B(x), I(y) = B(y)$ in the case of the indiscernibility relation defined in Section 2.2 and in more general case (see Section 3) $I(x), I(y)$ are neighborhoods of objects not necessarily defined by the equivalence relation.

The idea of Boolean reasoning is based on construction for a given problem P of a corresponding Boolean function f_P with the following property: the solutions for the problem P can be decoded from prime implicants of the Boolean function f_P. Let us mention that to solve real-life problems it is necessary to deal with Boolean functions having large number of variables.

A successful methodology based on the discernibility of objects and Boolean reasoning has been developed for computing of many important for applications. These include reducts and their approximations, decision rules, association rules, discretization of real value attributes, symbolic value grouping, searching for new features defined by oblique hyperplanes or higher order surfaces, pattern extraction from data as well as conflict resolution or negotiation.

Most of the problems related to generation of the above mentioned entities are NP-complete or NP-hard. However, it is possible to develop efficient

heuristics returning suboptimal solutions of the problems. The results of experiments on many data sets are very promising. They show very good quality of solutions generated by the heuristics in comparison with other methods reported in literature (e.g., with respect to the classification quality of unseen objects). Moreover, they are very efficient from the point of view of time necessary for computing of the solution. Many of these methods are based on discernibility matrices. Note, that it is possible to compute the necessary information about these matrices using directly[4] information or decision systems (e.g., sorted in preprocessing [3, 62, 144]) which significantly improves the efficiency of algorithms.

It is important to note that the methodology makes it possible to construct heuristics having a very important *approximation property* which can be formulated as follows: expressions generated by heuristics (i.e., implicants) *close* to prime implicants define approximate solutions for the problem.

2.7 Rough Membership

Let us observe that rough sets can be also defined employing the rough membership function (see Eq. 13) instead of approximation [77]. That is, consider

$$\mu_X^B : U \to < 0, 1 >,$$

defined by

$$\mu_X^B(x) = \frac{card(B(x) \cap X)}{card(X)}, \tag{13}$$

where $x \in X \subseteq U$.

The value $\mu_X^B(x)$ can be interpreted as the degree that x belongs to X in view of knowledge about x expressed by B or the degree to which the elementary granule $B(x)$ is included in the set X. This means that the definition reflects a subjective knowledge about elements of the universe, in contrast to the classical definition of a set.

The rough membership function can also be interpreted as the conditional probability that x belongs to X given B. This interpretation was used by several researchers in the rough set community (see, e.g., [4, 32, 123, 140, 143]). Note also that the ratio on the right hand side of the equation (13) is known as the confidence coefficient in data mining [25, 37]. It is worthwhile to mention that set inclusion to a degree has been considered by Łukasiewicz [53] in studies on assigning fractional truth values to logical formulas.

It can be shown that the rough membership function has the following properties [77]:

1) $\mu_X^B(x) = 1$ *iff* $x \in B_*(X)$;
2) $\mu_X^B(x) = 0$ *iff* $x \in U - B^*(X)$;
3) $0 < \mu_X^B(x) < 1$ *iff* $x \in BN_B(X)$;

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[4] i.e., without the necessity of generation and storing of the discernibility matrices

4) $\mu_{U-X}^B(x) = 1 - \mu_X^B(x)$ *for any* $x \in U$;

5) $\mu_{X \cup Y}^B(x) \geq max(\mu_X^B(x), \ \mu_Y^B(x))$ *for any* $x \in U$;

6) $\mu_{X \cap Y}^B(x) \leq min(\mu_X^B(x), \ \mu_Y^B(x))$ *for any* $x \in U$.

From the properties it follows that the rough membership differs essentially from the fuzzy membership [149], for properties 5) and 6) show that the membership for union and intersection of sets, in general, cannot be computed – as in the case of fuzzy sets – from their constituents membership. Thus formally the rough membership is more general from fuzzy membership. Moreover, the rough membership function depends on an available knowledge (represented by attributes from B). Besides, the rough membership function, in contrast to fuzzy membership function, has a probabilistic flavor.

Let us also mention that rough set theory, in contrast to fuzzy set theory, clearly distinguishes two very important concepts, vagueness and uncertainty, very often confused in the AI literature. Vagueness is the property of sets and can be described by approximations, whereas uncertainty is the property of objects considered as elements of a set and can be expressed by the rough membership function.

Both fuzzy and rough set theory represent two different approaches to vagueness. Fuzzy set theory addresses *gradualness* of knowledge, expressed by the fuzzy membership, whereas rough set theory addresses *granularity* of knowledge, expressed by the indiscernibility relation. A nice illustration of this difference has been given by Dider Dubois and Henri Prade [19] in the following example. In image processing fuzzy set theory refers to gradualness of gray level, whereas rough set theory is about the size of pixels.

Consequently, both theories are not competing but are rather complementary. In particular, the rough set approach provides tools for approximate construction of fuzzy membership functions. The rough-fuzzy hybridization approach proved to be successful in many applications (see, e.g., [68, 71]).

Interesting discussion of fuzzy and rough set theory in the approach to vagueness can be found in [96]. Let us also observe that fuzzy set and rough set theory are not a remedy for classical set theory difficulties.

One of the consequences of perceiving objects by information about them is that for some objects one cannot decide if they belong to a given set or not. However, one can estimate the degree to which objects belong to sets. This is a crucial observation in building foundations for approximate reasoning. Dealing with imperfect knowledge implies that one can only characterize satisfiability of relations between objects to a degree, not precisely. One of the fundamental relations on objects is a rough inclusion relation describing that objects are parts of other objects to a degree. The rough mereological approach [70, 87, 88, 90] based on such a relation is an extension of the Leśniewski mereology [45].

3 Extensions

The rough set concept can be defined quite generally by means of topological operations, *interior* and *closure*, called *approximations* [84]. It was observed in [73] that the key to the presented approach is provided by the exact mathematical formulation of the concept of approximative (rough) equality of sets in a given approximation space. In [74], an approximation space is represented by the pair (U, R), where U is a universe of objects, and $R \subseteq U \times U$ is an indiscernibility relation defined by an attribute set (i.e., $R = I(A)$ for some attribute set A). In this case R is an equivalence relation. Let $[x]_R$ denote an equivalence class of an element $x \in U$ under the indiscernibility relation R, where $[x]_R = \{y \in U : xRy\}$.

In this context, R-approximations of any set $X \subseteq U$ are based on the exact (crisp) containment of sets. Then set approximations are defined as follows:

- $x \in U$ belongs with certainty to the R-lower approximation of $X \subseteq U$, if $[x]_R \subseteq X$.
- $x \in U$ belongs with certainty to the complement set of $X \subseteq U$, if $[x]_R \subseteq U - X$.
- $x \in U$ belongs with certainty to the R-boundary region of $X \subseteq U$, if $[x]_R \cap X \neq \oslash$ and $[x]_R \cap (U - X) \neq \oslash$.

Several generalizations of the above approach have been proposed in the literature (see, e.g., [29, 70, 113, 118, 131, 152]). In particular, in some of these approaches, set inclusion to a degree is used instead of the exact inclusion.

Different aspects of vagueness in the rough set framework are discussed, e.g., in [55, 65, 66, 96, 107].

Our knowledge about the approximated concepts is often partial and uncertain [30]. For example, concept approximation should be constructed from examples and counter examples of objects for the concepts [25]. Hence, the concept approximations constructed from a given sample of objects is extended, using inductive reasoning, on unseen so far objects. The rough set approach for dealing with concept approximation under such partial knowledge is presented, e.g., in [118]. Moreover, the concept approximations should be constructed under dynamically changing environments [107]. This leads to a more complex situation when the boundary regions are not crisp sets what is consistent with the postulate of the higher order vagueness, considered by philosophers (see, e.g., [36]). It is worthwhile to mention that a rough set approach to the approximation of compound concepts has been developed and at this time no traditional method is able directly to approximate compound concepts [11, 141]. The approach is based on hierarchical learning and ontology approximation [8, 61, 70, 111]). Approximation of concepts in distributed environments is discussed in [105]. A survey of algorithmic methods for concept approximation based on rough sets and Boolean reasoning in presented, e.g., in [103].

3.1 Generalizations of Approximation Spaces

Several generalizations of the classical rough set approach based on approximation spaces defined as pairs of the form (U, R), where R is the equivalence relation (called indiscernibility relation) on the set U, have been reported in the literature. Let us mention two of them.

A generalized approximation space[5] can be defined by a tuple $AS = (U, I, \nu)$ where I is the *uncertainty function* defined on U with values in the powerset $\mathrm{Pow}(U)$ of U ($I(x)$ is the *neighboorhood* of x) and ν is the *inclusion function* defined on the Cartesian product $\mathrm{Pow}(U) \times \mathrm{Pow}(U)$ with values in the interval $[0, 1]$ measuring the degree of inclusion of sets. The lower AS_* and upper AS^* approximation operations can be defined in AS by

$$AS_*(X) = \{x \in U : \nu(I(x), X) = 1\}, \tag{14}$$
$$AS^*(X) = \{x \in U : \nu(I(x), X) > 0\}. \tag{15}$$

In the standard case $I(x)$ is equal to the equivalence class $B(x)$ of the indiscernibility relation $I(B)$; in case of tolerance (similarity) relation $\tau \subseteq U \times U$ [95] we take $I(x) = \{y \in U : x\tau y\}$, i.e., $I(x)$ is equal to the tolerance class of τ defined by x. The standard inclusion relation ν_{SRI} is defined for $X, Y \subseteq U$ by

$$\nu_{SRI}(X, Y) = \begin{cases} \frac{card(X \cap Y)}{card(X)}, & \text{if } X \text{ is non} - \text{empty,} \\ 1, & \text{otherwise.} \end{cases} \tag{16}$$

For applications it is important to have some constructive definitions of I and ν.

One can consider another way to define $I(x)$. Usually together with AS we consider some set F of formulae describing sets of objects in the universe U of AS defined by semantics $\| \cdot \|_{AS}$, i.e., $\|\alpha\|_{AS} \subseteq U$ for any $\alpha \in F$. Now, one can take the set

$$N_F(x) = \{\alpha \in F : x \in \|\alpha\|_{AS}\}, \tag{17}$$

and $I(x) = \{\|\alpha\|_{AS} : \alpha \in N_F(x)\}$. Hence, more general uncertainty functions having values in $\mathrm{Pow}(\mathrm{Pow}(U))$ can be defined and in the consequence different definitions of approximations are considered. For example, one can consider the following definitions of approximation operations in AS:

$$AS_{\circ}(X) = \{x \in U : \nu(Y, X) = 1 \text{ for some } Y \in I(x)\}, \tag{18}$$
$$AS^{\circ}(X) = \{x \in U : \nu(Y, X) > 0 \text{ for any } Y \in I(x)\}. \tag{19}$$

There are also different forms of rough inclusion functions. Let us consider some examples.

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[5] Some other generalizations of approximation spaces are also considered in the literature (see, e.g., [47, 49, 104, 146, 147, 145, 148]).

In the first example of rough inclusion function a a threshold $t \in (0, 0.5)$ is used to relax the degree of inclusion of sets. The rough inclusion function ν_t is defined by

$$\nu_t(X, Y) = \begin{cases} 1, & \text{if} \quad \nu_{SRI}(X, Y) \geq 1 - t, \\ \frac{\nu_{SRI}(X,Y) - t}{1 - 2t}, & \text{if} \quad t < \nu_{SRI}(X, Y) < 1 - t, \\ 0, & \text{if} \quad \nu_{SRI}(X, Y) \leq t. \end{cases} \quad (20)$$

One can obtain approximations considered in the variable precision rough set approach (VPRSM) [152] by substituting in (14)-(15) the rough inclusion function ν_t defined by (20) instead of ν, assuming that Y is a decision class and $N(x) = B(x)$ for any object x, where B is a given set of attributes.

Another example of application of the standard inclusion was developed by using probabilistic decision functions. For more detail the reader is referred to [122, 123].

The rough inclusion relation can be also used for function approximation [118] and relation approximation [133]. In the case of function approximation the inclusion function ν^* for subsets $X, Y \subseteq U \times U$, where $U \subseteq \mathcal{R}$ and $\mathcal{R}$ is the set of reals, is defined by

$$\nu^*(X, Y) = \begin{cases} \frac{card(\pi_1(X \cap Y))}{card(\pi_1(X))}, & \text{if } \pi_1(X) \neq \emptyset, \\ 1, & \text{if } \pi_1(X) = \emptyset, \end{cases} \quad (21)$$

where π_1 is the projection operation on the first coordinate. Assume now, that X is a cube and Y is the graph $G(f)$ of the function $f : \mathcal{R} \longrightarrow \mathcal{R}$. Then, e.g., X is in the lower approximation of f if the projection on the first coordinate of the intersection $X \cap G(f)$ is equal to the projection of X on the first coordinate. This means that the part of the graph $G(f)$ is "well" included in the box X, i.e., for all arguments that belong to the box projection on the first coordinate the value of f is included in the box X projection on the second coordinate.

The approach based on inclusion functions has been generalized to the *rough mereological approach* [70, 87, 88, 90] (see also Section 3.6). The inclusion relation $x\mu_r y$ with the intended meaning *x is a part of y to a degree at least r* has been taken as the basic notion of the rough mereology being a generalization of the Leśniewski mereology [45, 46]. Research on rough mereology has shown importance of another notion, namely *closeness* of complex objects (e.g., concepts). This can be defined by $xcl_{r,r'} y$ if and only if $x\mu_r y$ and $y\mu_{r'} x$.

Rough mereology offers a methodology for synthesis and analysis of objects in a distributed environment of intelligent agents, in particular, for synthesis of objects satisfying a given specification to a satisfactory degree or for control in such a complex environment. Moreover, rough mereology has been recently used for developing the foundations of the *information granule calculi*, aiming at formalization of the Computing with Words paradigm, recently formulated by Lotfi Zadeh [150]. More complex information granules

are defined recursively using already defined information granules and their measures of inclusion and closeness. Information granules can have complex structures like classifiers or approximation spaces. Computations on information granules are performed to discover relevant information granules, e.g., patterns or approximation spaces for complex concept approximations.

Usually there are considered families of approximation spaces labeled by some parameters. By tuning such parameters according to chosen criteria (e.g., minimal description length) one can search for the optimal approximation space for concept description (see, e.g., [3]).

3.2 Concept Approximation

In this section, we consider the problem of approximation of concepts over a universe U^∞ (concepts that are subsets of U^∞). We assume that the concepts are perceived only through some subsets of U^∞, called samples. This is a typical situation in the machine learning, pattern recognition, or data mining approaches [25, 37]. We explain the rough set approach to induction of concept approximations using the generalized approximation spaces of the form $AS = (U, I, \nu)$ defined in Section 3.1.

Let $U \subseteq U^\infty$ be a finite sample. By Π_U we denote a perception function from $P(U^\infty)$ into $P(U)$ defined by $\Pi_U(C) = C \cap U$ for any concept $C \subseteq U^\infty$. Let $AS = (U, I, \nu)$ be an approximation space over the sample U.

The problem we consider is how to extend the approximations of $\Pi_U(C)$ defined by AS to approximation of C over U^∞. We show that the problem can be described as searching for an extension $AS_C = (U^\infty, I_C, \nu_C)$ of the approximation space AS, relevant for approximation of C. This requires to show how to extend the inclusion function ν from subsets of U to subsets of U^∞ that are relevant for the approximation of C. Observe that for the approximation of C it is enough to induce the necessary values of the inclusion function ν_C without knowing the exact value of $I_C(x) \subseteq U^\infty$ for $x \in U^\infty$.

Let AS be a given approximation space for $\Pi_U(C)$ and let us consider a language L in which the neighborhood $I(x) \subseteq U$ is expressible by a formula $pat(x)$, for any $x \in U$. It means that $I(x) = \|pat(x)\|_U \subseteq U$, where $\|pat(x)\|_U$ denotes the meaning of $pat(x)$ restricted to the sample U. In case of rule based classifiers patterns of the form $pat(x)$ are defined by feature value vectors.

We assume that for any new object $x \in U^\infty \setminus U$ we can obtain (e.g., as a result of sensor measurement) a pattern $pat(x) \in L$ with semantics $\|pat(x)\|_{U^\infty} \subseteq U^\infty$. However, the relationships between information granules over U^∞ such as sets $\|pat(x)\|_{U^\infty}$ and $\|pat(y)\|_{U^\infty}$, for different $x, y \in U^\infty$, are, in general, known only if they can be expressed by relationships between the restrictions of these sets to the sample U, i.e., between sets $\Pi_U(\|pat(x)\|_{U^\infty})$ and $\Pi_U(\|pat(y)\|_{U^\infty})$.

The set of patterns $\{pat(x) : x \in U\}$ is usually not relevant for approximation of the concept $C \subseteq U^\infty$. Such patterns are too specific or not enough

general, and can directly be applied only to a very limited number of new objects. However, by using some generalization strategies, one can search, in a family of patterns definable from $\{pat(x) : x \in U\}$ in L, for such new patterns that are relevant for approximation of concepts over U^{∞}. Let us consider a subset $PATTERNS(AS, L, C) \subseteq L$ chosen as a set of pattern candidates for relevant approximation of a given concept C. For example, in case of rule based classifier one can search for such candidate patterns among sets definable by subsequences of feature value vectors corresponding to objects from the sample U. The set $PATTERNS(AS, L, C)$ can be selected by using some quality measures checked on meanings (semantics) of its elements restricted to the sample U (like the number of examples from the concept $\Pi_U(C)$ and its complement that support a given pattern). Then, on the basis of properties of sets definable by these patterns over U we induce approximate values of the inclusion function ν_C on subsets of U^{∞} definable by any of such pattern and the concept C.

Next, we induce the value of ν_C on pairs (X, Y) where $X \subseteq U^{\infty}$ is definable by a pattern from $\{pat(x) : x \in U^{\infty}\}$ and $Y \subseteq U^{\infty}$ is definable by a pattern from $PATTERNS(AS, L, C)$.

Finally, for any object $x \in U^{\infty} \setminus U$ we induce the approximation of the degree $\nu_C(\|pat(x)\|_{U^{\infty}}, C)$ applying a conflict resolution strategy $Conflict\_res$ (a voting strategy, in case of rule based classifiers) to two families of degrees:

$$\{\nu_C(\|pat(x)\|_{U^{\infty}}, \|pat\|_{U^{\infty}}) : pat \in PATTERNS(AS, L, C)\}, \qquad (22)$$

$$\{\nu_C(\|pat\|_{U^{\infty}}, C) : pat \in PATTERNS(AS, L, C)\}. \qquad (23)$$

Values of the inclusion function for the remaining subsets of U^{∞} can be chosen in any way – they do not have any impact on the approximations of C. Moreover, observe that for the approximation of C we do not need to know the exact values of uncertainty function I_C – it is enough to induce the values of the inclusion function ν_C. Observe that the defined extension ν_C of ν to some subsets of U^{∞} makes it possible to define an approximation of the concept C in a new approximation space AS_C.

Observe that one can also follow principles of Bayesian reasoning and use degrees of ν_C to approximate C (see, e.g., [76, 125, 128]).

In this way, the rough set approach to induction of concept approximations can be explained as a process of inducing a relevant approximation space.

3.3 Higher Order Vagueness

In [36], it is stressed that vague concepts should have non-crisp boundaries. In the definition presented in Section 2.2, the notion of boundary region is defined as a crisp set $BN_B(X)$. However, let us observe that this definition is relative to the subjective knowledge expressed by attributes from B. Different sources of information may use different sets of attributes for concept approximation. Hence, the boundary region can change when we consider these different

views. Another aspect is discussed in [107, 117] where it is assumed that information about concepts is incomplete, e.g., the concepts are given only on samples (see, e.g., [25, 37, 56]). From [107, 117] it follows that vague concepts cannot be approximated with satisfactory quality by *static* constructs such as induced membership inclusion functions, approximations or models derived, e.g., from a sample. Understanding of vague concepts can be only realized in a process in which the induced models are adaptively matching the concepts in a dynamically changing environment. This conclusion seems to have important consequences for further development of rough set theory in combination with fuzzy sets and other soft computing paradigms for adaptive approximate reasoning.

3.4 Information Granulation

Information granulation can be viewed as a human way of achieving data compression and it plays a key role in the implementation of the strategy of divide-and-conquer in human problem-solving [150]. Objects obtained as the result of granulation are information granules. Examples of elementary information granules are indiscernibility or tolerance (similarity) classes (see Section 2.2). In reasoning about data and knowledge under uncertainty and imprecision many other more compound information granules are used (see, e.g., [92, 94, 104, 114, 115]). Examples of such granules are decision rules, sets of decision rules or classifiers. More compound information granules are defined by means of less compound ones. Note that inclusion or closeness measures between information granules should be considered rather than their strict equality. Such measures are also defined recursively for information granules.

Let us discuss shortly an example of information granulation in the process of modeling patterns for compound concept approximation (see, e.g., [6, 7, 8, 9, 10, 61, 134]. We start from a generalization of information systems. For any attribute $a \in A$ of an information system (U, A) we consider together with the value set V_a of a a relational structure $\mathcal{R}_a$ over the universe V_a (see, e.g., [119]). We also consider a language $\mathcal{L}_a$ of formulas (of the same relational signature as $\mathcal{R}_a$). Such formulas interpreted over $\mathcal{R}_a$ define subsets of Cartesian products of V_a. For example, any formula α with one free variable defines a subset $\|\alpha\|_{\mathcal{R}_a}$ of V_a. Let us observe that the relational structure $\mathcal{R}_a$ induces a relational structure over U. Indeed, for any k-ary relation r from $\mathcal{R}_a$ one can define a k-ary relation $g_a \subseteq U^k$ by $(x_1, \ldots, x_k) \in g_a$ if and only if $(a(x_1), \ldots, a(x_k)) \in r$ for any $(x_1, \ldots, x_k) \in U^k$. Hence, one can consider any formula from $\mathcal{L}_a$ as a constructive method of defining a subset of the universe U with a structure induced by $\mathcal{R}_a$. Any such a structure is a new information granule. On the next level of hierarchical modeling, i.e., in constructing new information systems we use such structures as objects and attributes are properties of such structures. Next, one can consider similarity between new constructed objects and then their similarity neighborhoods will correspond to clusters of relational structures. This process is usually more

complex. This is because instead of relational structure $\mathcal{R}_a$ we usually consider a fusion of relational structures corresponding to some attributes from A. The fusion makes it possible to describe constraints that should hold between parts obtained by composition from less compound parts. Examples of relational structures can be defined by indiscernibility, similarity, intervals obtained in discretization or symbolic value grouping, preference or spatio-temporal relations (see, e.g., [29, 37, 113]). One can see that parameters to be tuned in searching for relevant[6] patterns over new information systems are, among others, relational structures over value sets, the language of formulas defining parts, and constraints.

3.5 Ontological Framework for Approximation

In a number of papers (see, e.g., [116]) the problem of ontology approximation has been discussed together with possible applications to approximation of compound concepts or to knowledge transfer (see, e.g., [63, 99, 116, 106]).

In the ontology [132] (vague) concepts and local dependencies between them are specified. Global dependencies can be derived from local dependencies. Such derivations can be used as hints in searching for relevant compound patterns (information granules) in approximation of more compound concepts from the ontology. The ontology approximation problem is one of the fundamental problems related to approximate reasoning in distributed environments. One should construct (in a given language that is different from the ontology specification language) not only approximations of concepts from ontology but also vague dependencies specified in the ontology. It is worthwhile to mention that an ontology approximation should be induced on the basis of incomplete information about concepts and dependencies specified in the ontology. Information granule calculi based on rough sets have been proposed as tools making it possible to solve this problem. Vague dependencies have vague concepts in premises and conclusions. The approach to approximation of vague dependencies based only on degrees of closeness of concepts from dependencies and their approximations (classifiers) is not satisfactory for approximate reasoning. Hence, more advanced approach should be developed. Approximation of any vague dependency is a method which allows us for any object to compute the arguments "for" and "against" its membership to the dependency conclusion on the basis of the analogous arguments relative to the dependency premises. Any argument is a compound information granule (compound pattern). Arguments are fused by local schemes (production rules) discovered from data. Further fusions are possible through composition of local schemes, called approximate reasoning schemes (AR schemes) (see, e.g., [9, 92, 70]). To estimate the degree to which (at least) an object belongs to concepts from ontology the arguments "for" and "against" those concepts are collected and next a conflict resolution strategy is applied to them to predict the degree.

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[6] for target concept approximation

3.6 Mereology and Rough Mereology

This section introduces some basic concepts of rough mereology (see, e.g., [85, 86, 88, 92, 93, 94]).

Exact and rough concepts can be characterized by a new notion of an element, alien to naive set theory in which this theory has been coded until now. For an information system $\mathcal{A}=(U, A)$, and a set B of attributes, the mereological element $el^{\mathcal{A}_B}$ is defined by letting

$$xel_B^A X \text{ if and only if } B(x) \subseteq X. \tag{24}$$

Then, a concept X is B-exact if and only if either $xel_B^A X$ or $xel_B^A U \setminus X$ for each $x \in U$, and the concept X is B–rough if and only if for some $x \in U$ neither $xel_B^A X$ nor $xel_B^A U \setminus X$.

Thus, the characterization of the dichotomy exact–rough cannot be done by means of the element notion of naive set theory, but requires the notion of containment ($\subseteq$), i.e., a notion of mereological element.

The Leśniewski Mereology (theory of parts) is based on the notion of a part [45, 46]. The relation π of part on the collection U of objects satisfies

$$\text{1. if } x\pi y \text{ then not } y\pi x, \tag{25}$$

$$\text{2. if } x\pi y \text{ and } y\pi z \text{ then } x\pi z. \tag{26}$$

The notion of mereological element el_π is introduced as

$$xel_\pi y \text{ if and only if } x\pi y \text{ or } x = y. \tag{27}$$

In particular, the relation of proper inclusion $\subset$ is a part relation π on any non–empty collection of sets, with the element relation $el_\pi =\subseteq$.

Formulas expressing, e.g., rough membership, quality of decision rule, quality of approximations can be traced back to a common root, i.e., $\nu(X, Y)$ defined by equation (16). The value $\nu(X, Y)$ defines the degree of *partial containment* of X into Y and naturally refers to the Leśniewski Mereology. An abstract formulation of this idea in [88] connects the mereological notion of element el_π with the partial inclusion by introducing a *rough inclusion* as a relation $\nu \subseteq U \times U \times [0, 1]$ on a collection of pairs of objects in U endowed with part π relation, and such that

$$\text{1. } \nu(x, y, 1) \text{ if and only if } xel_\pi y, \tag{28}$$

$$\text{2. if } \nu(x, y, 1) \text{ then (if } \nu(z, x, r) \text{ then } \nu(z, y, r)), \tag{29}$$

$$\text{3. if } \nu(z, x, r) \text{ and } s < r \text{ then } \nu(z, x, s). \tag{30}$$

Implementation of this idea in information systems can be based on *Archimedean* t–norms [88]; each such norm T is represented as $T(r, s) = g(f(r)+f(s))$ with f, g pseudo–inverses to each other, continuous and decreasing on $[0, 1]$. Letting for (U, A) and $x, y \in U$

$$DIS(x, y) = \{a \in A : a(x) \neq a(y)\} \tag{31}$$

and

$$\nu(x, y, r) \text{ if and only if } g(\frac{card(DIS(x, y))}{card(A)}) \geq r \tag{32}$$

defines a rough inclusion that satisfies additionally the transitivity rule

$$\frac{\nu(x, y, r), \nu(y, z, s)}{\nu(x, z, T(r, s))}. \tag{33}$$

Simple examples here are: the Menger rough inclusion in the case $f(r) = -\ln r$, $g(s) = e^{-s}$ yields $\nu(x, y, r)$ if and only if $e^{-\frac{card(DIS(x,y))}{card(A)}} \geq r$ and it satisfies the transitivity rule:

$$\frac{\nu(x, y, r), \nu(y, z, s)}{\nu(x, y, r \cdot s)}, \tag{34}$$

i.e., the t–norm T is the Menger (product) t–norm $r \cdot s$, and, the Łukasiewicz rough inclusion with $f(x) = 1 - x = g(x)$ yielding $\nu(x, y, r)$ if and only if $1 - \frac{card(DIS(x,y))}{card(A)} \geq r$ with the transitivity rule:

$$\frac{\nu(x, y, r), \nu(y, z, s)}{\nu(x, y, max\{0, r + s - 1\})}, \tag{35}$$

i.e., with the Lukasiewicz t–norm.

Rough inclusions [88] can be used in *granulation of knowledge* [150]. Granules of knowledge are constructed as aggregates of indiscernibility classes close enough with respect to a chosen measure of closeness. In a nutshell, a granule $g_r(x)$ about x of radius r can be defined as the aggregate of all y with $\nu(y, x, r)$. The aggregating mechanism can be based on the class operator of mereology (cf. rough mereology [88]) or on set theoretic operations of union.

Rough mereology [88] combines rough inclusions with methods of mereology. It employs the operator of mereological class that makes collections of objects into objects. The class operator Cls satisfies the requirements, with any non–empty collection M of objects made into the object $Cls(M)$

if $x \in M$ then $xel_\pi Cls(M)$, $\tag{36}$

if $xel_\pi Cls(M)$ then there exist y, z such that $yel_\pi x, yel_\pi z, z \in M$. $\tag{37}$

In case of the part relation $\subset$ on a collection of sets, the class $Cls(M)$ of a non–empty collection M is the union $\bigcup M$.

Granulation by means of the class operator Cls consists in forming the granule $g_r(x)$ as the class $Cls(y : \nu(y, x, r))$. One obtains a granule family with regular properties (see [142]).

4 Conflicts

Knowledge discovery in databases considered in the previous sections reduces to searching for functional dependencies in the data set.

In this section, we will discuss another kind of relationship in the data - not dependencies, but conflicts.

Formally, the conflict relation can be seen as a negation (not necessarily, classical) of indiscernibility relation which was used as a basis of rough set theory. Thus indiscernibility and conflict are closely related from logical point of view.

It turns out that the conflict relation can be used to the conflict analysis study.

Conflict analysis and resolution play an important role in business, governmental, political and lawsuits disputes, labor-management negotiations, military operations and others. To this end many mathematical formal models of conflict situations have been proposed and studied, e.g., [12, 15, 16, 24, 41, 42, 43, 54, 58, 64, 75, 136].

Various mathematical tools, e.g., graph theory, topology, differential equations and others, have been used to that purpose.

Needless to say that game theory can be also considered as a mathematical model of conflict situations.

In fact there is no, as yet, "universal" theory of conflicts and mathematical models of conflict situations are strongly domain dependent.

We are going to present in this paper still another approach to conflict analysis, based on some ideas of rough set theory – along the lines proposed in [75] and extended in this paper.

The considered model is simple enough for easy computer implementation and seems adequate for many real life applications but to this end more research is needed.

4.1 Basic Concepts of Conflict Theory

In this section, we give after [75] definitions of basic concepts of the proposed approach.

Let us assume that we are given a finite, non-empty set Ag called the *universe*. Elements of Ag will be referred to as *agents*. Let a *voting function* $v : Ag \rightarrow \{-1, 0, 1\}$, or in short $\{-, 0, +\}$, be given assigning to every agent the number $-1, 0$ or 1, representing his opinion, view, voting result, etc. about some discussed issue, and meaning *against, neutral* and *favorable*, respectively.

Voting functions correspond to situations. Hence, let us assume there is given a set U of situations and a set $Voting\_Fun$ of voting functions as well as a *conflict function $Conflict : U \longrightarrow Voting\_Fun$*. Any pair $S = (s, v)$ where $s \in U$ and $v = Conflict(s)$ will be called a *conflict situation*.

In order to express relations between agents from Ag defined by a given voting function v we define three basic binary relations in Ag^2 : *conflict, neutrality,* and *alliance*.

To this end we first define the following auxiliary function:

$$\phi_v(ag, ag') = \begin{cases} 1, & \text{if } v(ag)v(ag') = 1 \text{ or } ag = ag' \\ 0, & \text{if } v(ag)v(ag') = 0 \text{ and } ag \neq ag' \\ -1, & \text{if } v(ag)v(ag') = -1. \end{cases} \quad (38)$$

This means that, if $\phi_v(ag, ag') = 1$, then agents ag and ag' have the same opinion about issue v (are *allied* on v); $\phi_v(ag, ag') = 0$ means that at least one agent ag or ag' has neutral approach to issue v (is *neutral* on v), and $\phi_v(ag, ag') = -1$, means that both agents have different opinions about issue v (are in *conflict* on v).

In what follows we will define three basic binary relations $R_v^+, R_v^0, R_v^- \subseteq Ag^2$ called *alliance, neutrality* and *conflict* relations respectively, and defined by

$$R_v^+(ag, ag') \text{ iff } \phi_v(ag, ag') = 1, \quad (39)$$
$$R_v^0(ag, ag') \text{ iff } \phi_v(ag, ag') = 0,$$
$$R_v^-(ag, ag') \text{ iff } \phi_v(ag, ag') = -1.$$

It is easily seen that the alliance relation has the following properties:

$$R_v^+(ag, ag), \quad (40)$$
$$R_v^+(ag, ag') \text{ implies } R_v^+(ag', ag),$$
$$R_v^+(ag, ag') \text{ and } R_v^+(ag', ag'') \text{ implies } R_v^+(ag, ag''),$$

i.e., R_v^+ is an *equivalence* relation. Each equivalence class of alliance relation will be called a *coalition* with respect to v. Let us note that the last condition in (40) can be expressed as "a friend of my friend is my friend".

For the conflict relation we have the following properties:

$$\text{not } R_v^-(ag, ag), \quad (41)$$
$$R_v^-(ag, ag') \text{ implies } R_v^-(ag', ag),$$
$$R_v^-(ag, ag') \text{ and } R_v^-(ag', ag'') \text{ implies } R_v^+(ag, ag''),$$
$$R_v^-(ag, ag') \text{ and } R_v^+(ag', ag'') \text{ implies } R_v^-(ag, ag'').$$

The last two conditions in (41) refer to well known sayings "an enemy of my enemy is my friend" and "a friend of my enemy is my enemy".

For the neutrality relation we have:

$$\text{not } R_v^0(ag, ag), \quad (42)$$
$$R_v^0(ag, ag') = R_v^0(ag', ag).$$

Let us observe that in the conflict and neutrality relations there are no coalitions.

We have $R_v^+ \cup R_v^0 \cup R_v^- = Ag^2$ because if $(ag, ag') \in Ag^2$ then $\Phi_v(ag, ag') = 1$ or $\Phi_v(ag, ag') = 0$ or $\Phi_v(ag, ag') = -1$ so $(ag, ag') \in R_v^+$ or $(ag, ag') \in R_v^-$ or $(ag, ag') \in R_v^-$. All the three relations R_v^+, R_v^0 , R_v^- are pairwise disjoint, i.e., every pair of objects (ag, ag') belongs to exactly one of the above defined relations (is in conflict, is allied or is neutral).

With every conflict situation $S = (s, v)$ we will associate a *conflict graph*

$$G_S = (R_v^+, R_v^0, R_v^-). \tag{43}$$

An example of a conflict graph is shown in Figure 2. Solid lines are denoting
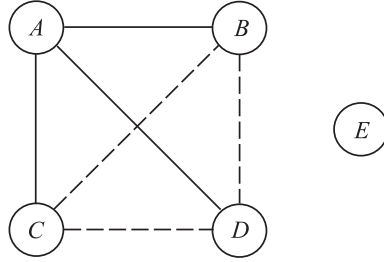
Fig. 2. Exemplary conflict graph

conflicts, doted line – alliance, and neutrality, for simplicity, is not shown explicitly in the graph. Of course, B, C, and D form a coalition.

A conflict degree $Con(S)$ of the conflict situation $S = (s, v)$ is defined by

$$Con(S) = \frac{\sum_{\{(ag, ag'):\ \phi_v(ag, ag') = -1\}} |\phi_v(ag, ag')|}{2\lceil \frac{n}{2} \rceil \times (n - \lceil \frac{n}{2} \rceil)} \tag{44}$$

where $n = Card(Ag)$.

One can consider a more general definition of conflict function $Conflict :$ $U \longrightarrow Voting\_Fun^k$ where k is a positive integer. Then, a conflict situation is any pair $S = (s, (v_1, ..., v_k))$ where $(v_1, ..., v_k) = Conflict(s)$ and the conflict degree in S can be defined by

$$Con(S) = \frac{\sum_{i=1}^{k} Con(S_i)}{k} \tag{45}$$

where $S_i = (s, v_i)$ for $i = 1, \ldots, k$. Each function v_i is called a voting function on the $i - th$ issue in s.

4.2 An Example

In this section, we will illustrate the above presented ideas by means of a very simple tutorial example using concepts presented in the previous section. We consider a conflict situation $S = (s, v)$ where the domain ag of

the voting function v is defined by $Ag = \{(1, A), \ldots, (240, A), (241, B), \ldots,$
$(280, B), (281, C), \ldots, (340,\ C), (341, D), \ldots, (500, D)\}$ and $v(1, A) = \ldots =$
$v(200, A) = +, v(201, A) = \ldots = v(230, A) = 0, v(231, A) = \ldots =$
$v(240, A) = -, v(241, B) = \ldots = v(255, B) = +, v(256, B) = \ldots =$
$v(280, B) = -, v(281, C) = \ldots = v(300, C) = 0, v(301, C) = \ldots =$
$v(340, C) = -, v(341, D) = \ldots = v(365, D) = +, v(366, D) = \ldots =$
$v(400, D) = 0, v(401, D) = \ldots = v(500, D) = -.$

This conflict situation is presented in Table 1. The maximal coalitions in this conflict situations are $v^{-1}(+)$ and $v^{-1}(-)$.

Table 1. Conflict situation with agents (*Member,Party*) and the voting function *Voting*

(*Member, Party*)	*Voting*	(*Member, Party*)	*Voting*
(1,A)	+	(281,C)	0
...	...	...	...
(200,A)	+	(300,C)	0
(201,A)	0	(301,C)	−
...	...	...	...
(230,A)	0	(340,C)	−
(231,A)	−	(341,D)	+
...	...	...	...
(240,A)	−	(365,D)	+
(241,B)	+	(366,D)	0
...	...	...	...
(255,B)	+	(400,D)	0
(256,B)	−	(401,D)	−
...	...	...	...
(280,B)	−	(500,D)	−

If one would like to keep only party name then Table 1 can be represented as it is shown in Table 2. This table presents a decision table in which the only condition attribute is *Party*, whereas the decision attribute is *Voting*. The table describes voting results in a parliament containing 500 members grouped in four political parties denoted A, B, C and D. Suppose the parliament discussed certain issue (e.g., membership of the country in European Union) and the voting result is presented in column *Voting*, where +, 0 and − denoted *yes*, *abstention* and *no* respectively. The column *support* contains the number of voters for each option.

The strength, certainty and the coverage factors for Table 2 are given in Table 3. The certainty and coverage factors have now a natural interpretation for the considered conflict situation.

From the certainty factors we can conclude, for example, that:

Table 2. Decision table with one condition attribute *Party* and the decision *Voting*

Fact	Party	Voting	Support
1	A	+	200
2	A	0	30
3	A	−	10
4	B	+	15
5	B	−	25
6	C	0	20
7	C	−	40
8	D	+	25
9	D	0	35
10	D	−	100

Table 3. Certainty and the coverage factors for Table 2

Fact	Strength	Certainty	Coverage
1	0.40	0.83	0.83
2	0.06	0.13	0.35
3	0.02	0.04	0.06
4	0.03	0.36	0.06
5	0.05	0.63	0.14
6	0.04	0.33	0.23
7	0.08	0.67	0.23
8	0.05	0.16	0.10
9	0.07	0.22	0.41
10	0.20	0.63	0.57

- 83.3% of party *A* voted *yes*,
- 12.5% of party *A* *abstained*,
- 4.2% of party *A* voted *no*.

From the coverage factors we can get, for example, the following explanation of voting results:

- 83.3% *yes* votes came from party *A*,
- 6.3% *yes* votes came from party *B*,
- 10.4% *yes* votes came from party *C*.

4.3 Conflicts and Rough Sets

There are strong relationships between the approach to conflicts and rough sets presented in Section 4.1. In this section, we discuss examples of such relationships. The presented in this section approach seems to be very promising

for solving problems related to conflict resolution and negotiations (see, e.g., [41, 42, 43, 136]).

The application of rough sets can bring new results in the area related to conflict resolution and negotiations between agents because this make it possible to introduce approximate reasoning about vague concepts into the area.

Now, we would like to outline this possibility.

First, let us observe that any conflict situation $S = (s, V)$ where $V = (v_1, \ldots, v_k)$ and each v_i is defined on the set of agents $Ag = \{ag_1, \ldots, ag_n\}$ can be treated as an information system $\mathcal{A}(S)$ with the set of objects Ag and the set of attributes $\{v_1, \ldots, v_k\}$. The discernibility between agents ag and ag' in S can be defined by

$$disc_S(ag, ag') = \frac{\sum_{\{i:\ \phi_{v_i}(ag, ag') = -1\}} |\phi_{v_i}(ag, ag')|}{k}, \tag{46}$$

where $ag, ag' \in Ag$.

Now, one can consider reducts of $\mathcal{A}(S)$ relative to the discernibility defined by $disc_S$. For example, one can consider agents ag, ag' as discernible if

$$disc_S(ag, ag') \geq tr,$$

where tr a given threshold.[7] Any reduct $R \subseteq V$ of S is a minimal set of voting functions preserving all discernibility in voting between agents that are at least equal to tr. All voting functions from $V - R$ are dispensable with respect to preserving such discernibility between objects.

In an analogous way can be considered reducts of the information system $\mathcal{A}_T(S)$ with the universe of objects equal to $\{v_1, \ldots, v_k\}$ and attributes defined by agents and voting functions by $ag(v) = v(ag)$ for $ag \in Ag$ and $v \in V$. The discernibility between voting functions can be defined, e.g., by

$$disc(v, v') = |Con(S_v) - Con(S_{v'})|, \tag{47}$$

and makes it possible to measure the difference between voting functions v and v', respectively.

Any reduct R of $\mathcal{A}_T(S)$ is a minimal set of agents that preserves the differences between voting functions that are at least equal to a given threshold tr.

In our next example we extend a model of conflict by adding a set A of (condition) attributes making it possible to describe the situations in terms of values of attributes from A. The set of given situations is denoted by U. In this way we have defined an information system (U, A). Let us assume that there

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[7] To compute such reducts one can follow a method presented in [112] assuming that any entry of the discernibility matrix corresponding to (ag, ag') with $disc(ag, ag') < tr$ is empty and the remaining entries are families of all subsets of V on which the discernibility between (ag, ag') is at least equal to tr [17].

is also given a set of agents Ag. Each agent $ag \in Ag$ has access to a subset $A_{ag} \subseteq A$ of condition attributes. Moreover we assume that $Ag = \bigcup_{ag \in AG} A_{ag}$. We also assume that there is also defined a decision attribute d on U such that $d(s)$ is a conflict situation $S = (s, V)$, where $V = (v_1, \dots, v_k)$. Observe that $S = (s, V)$ can be represented by a matrix

$$[v_i(ag_j)]_{i=1,\dots,n; j=1,\dots,k}$$

where $v_i(ag_j)$ is the result of voting by jth agent on the ith issue. Such a matrix is a compound decision[8] corresponding to s. For the constructed decision system (U, A, d) one can use, e.g., the introduced above function (44) to measure the discernibility between compound decision values which correspond to conflict situations in the constructed decision table. The reducts of this decision table relative to decision have a natural interpretation with respect to conflicts.

The described decision table can also be used in conflict resolution. We would like to illustrate this possibility. First, let us recall some notation. For $B \subseteq A$ we denote by $Inf_B(s)$ the B-signature of the situation s, i.e., the set $\{(a, a(s)) : a \in A\}$. Let $INF(B) = \{Inf_B(s) : s \in U\}$. Let us also assume that for each agent $ag \in Ag$ there is given a similarity relation $\tau_{ag} \subseteq INF(A_{ag}) \times INF(A_{ag})$. In terms of these similarity relations one can consider a problem of conflict resolution relative to a given threshold tr in a given situation s described by $Inf_A(s)$. This is the searching problem for a situation s', if such a situation exists, satisfying the following conditions:

1. $Inf_A(s')|A_{ag} \in \tau_{ag}(Inf_{A_{ag}}(s))$, where $\tau_{ag}(Inf_{A_{ag}}(s))$ is the tolerance class of $Inf_{A_{ag}}(s)$ with respect to τ_{ag} and $Inf_A(s')|A_{ag}$ denotes the restriction of $Inf_A(s')$ to A_{ag}.
2. $Inf_A(s')$ satisfies given local constraints (e.g., specifying coexistence of local situations, see, e.g., [17, 83, 135]) and given global constraints (e.g., specifying quality of global situations, see, e.g., [17]).
3. The conflict degree in the conflict situation $d(s')$[9] measured by means of the chosen conflict measure[10] is at most tr.

In searching for conflict resolution one can apply methods based on Boolean reasoning (see, e.g., [17, 112]).

We have proposed that changes to the acceptability of an issue by agents can be expressed by similarity relations. Observe that in real-life applications these similarities can be more compound than it was suggested above, i.e., they are not defined directly by sensory concepts describing situations. However, they are often specified by high level concepts (see, e.g., [41, 116] and also

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[8] For references to other papers on compound decision the reader is referred, e.g., to [4].

[9] Let us observe that s' is not necessarily in U. In such a case the value $d(s')$ should be predicted by the induced classifier from (U, A, d).

[10] For example, one can consider (45).

Section 3.5). These high level concepts can be vague and are linked with the sensory concepts describing situations by means of a hierarchy of other vague concepts. Approximation of vague concepts in the hierarchy and dependencies between them (see Section 3.5) makes it possible to approximate the similarity relations. This allows us to develop searching methods for acceptable value changes of sensory concepts preserving similarities (constraints) specified over high level vague concepts. One can also introduce some costs of changes of local situations into new ones by agents and search for new situations accessible with minimal or sub-minimal costs.

4.4 Negotiations for Conflict Resolution

In the previous section we have presented an outline to conflict resolution assuming that the acceptable changes of current situations of agents are known (in the considered example they were described by similarities). However, if the required solution does not exists in the current searching space then the negotiations between agents should start. Using the rough set approach to conflict resolution by negotiations between agents one can consider more advanced models in which actions and plans [28] performed by agents or their teams are involved in negotiations and conflict resolution.

We would like to outline such a model. Assume that each agent $ag \in Ag$ is able to perform actions from a set $Action_{ag}$. Each action $ac \in Action(ag)$ is specified by the input condition $in(ac)$ and the output condition $out(ac)$, representing conditions making it possible to perform the action and the result of action, respectively. We assume that $in(ac) \in INF(IN_{ag})$ and $out(ac) \in INF(OUT_{ag})$ for some sets $IN_{ag}, OUT_{ag} \subseteq A_{ag}$. The action ac can be applied to an object s if and only if $in(ac) \subseteq Inf_{A_{ag}}(s)$ and the result of performing of ac in s is described by $(Inf_{A_{ag}}(s) - Inf_{OUT_{ag}}(s)) \cup out(ac)$. Now, the conflict resolution task (see Section 4.3) in a given object (state) s can be formulated as searching for a plan, i.e., a sequence $ac_1, \ldots, ac_m$ of actions from $\bigcup_{ag \in Ag} Action(ag)$ transforming the objects s into object s' satisfying the requirements formulated in Section 4.3.[11]

In searching for the plan, one can use a Petri net constructed from the set $\bigcup_{ag \in Ag} Action(ag)$ of actions.[12] In this net places correspond to descriptors, i.e., pairs $(attribute, value)$ from $in(ac)$ and $out(ac)$ for $ac \in Action(ac)$ and $ag \in Ag$, transitions correspond to actions, and any transition is linked in a natural way with places corresponding to input and outputs conditions. Such a Petri net can be enriched by an additional control making it possible to preserve dependencies between local states (see, e.g., [83, 135]) or constraints

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[11] To illustrate possible arising problems let us consider an example. Assume that two vague concepts are given with classifiers for them. For any state satisfying the first concept we would like to find a (sub-)optimal plan transforming the given state to the state satisfying the second concept.

[12] For applications of Petri nets in planning the reader is referred, e.g., to [23]

related to similarities. Next, a cooperation protocol between actions performed by different agents should be discovered and the Petri net should be extended by this protocol. Finally, markings corresponding to objects s' with the conflict degree at most tr, if such states exist, are reachable in the resulting Petri net from a given marking corresponding to the state s. Searching for such protocols is a challenge.

One of the very first possible approaches in searching for sequences of actions (plans) transforming a given situation into another one with the required decreasing level of conflict degree is to create a Petri net from specification of actions and perform experiments with such a net. The examples of markings reachable in the net are stored in an information system. This system is next extended to a decision system by adding a decision describing the conflict degree for each situation corresponding to the marking. From such a decision table one can induce a classifier for different levels of conflicts. Next, this classifiers can be used to create a new decision table. In this new decision table any object consists of a pair of situations together with a sequence of actions transforming the first situation to the second. The decision for a given object is equal to the difference in conflict degrees of situations from the object. Then, condition attributes which make it possible to induce rules for prediction differences in conflict degrees are discovered. These condition attributes express properties of the first situation in the pair and properties of the sequence of actions (of a given length) performed starting from this situation. Such rules specify the additional constraints for the net and they can be embedded into the net as an additional control. The resulting net makes it possible to select only such plans (i.e., sequences of actions) which decrease conflict degrees. Certainly, to make the task feasible one should consider a relevant length of the sequences of actions and next to develop a method for composing plans. In turn, this will require to use hierarchical modelling with of concept ontology and actions on different levels of hierarchy between them. In our current project we are developing the outlined above methods.

5 Summary

In this article, we have presented basic concepts of rough set theory and also some extensions of the basic approach. We have also discussed relationships of rough sets with conflict analysis which is of great importance for e-service intelligence. In our further study we would like to develop the approach based to conflict analysis outlined in the paper.

There are numerous active research directions on rough set theory, and applications of rough sets also in combination with other approaches to soft computing. For more details the reader is referred to the bibliography on rough sets and web pages cited in this paper.

References

1. J. J. Alpigini, J. F. Peters, A. Skowron, N. Zhong (Eds.). Third International Conference on Rough Sets and Current Trends in Computing (RSCTC'2002), Malvern, PA, October 14-16, 2002, Lecture Notes in Artificial Intelligence, vol. 2475. Springer-Verlag, Heidelberg, Germany, 2002.

2. R. Ariew, D. Garber (Eds.). Leibniz, G. W., Philosophical Essays. Hackett Publishing Company, Indianapolis, 1989.

3. J. Bazan, H. S. Nguyen, S. H. Nguyen, P. Synak, J. Wróblewski. Rough set algorithms in classification problems. In: Polkowski et al. [87], pp. 49–88.

4. J. Bazan, A. Osmólski, A. Skowron, D. Ślęzak, M. Szczuka, J. Wróblewski. Rough set approach to the survival analysis. In: Alpigini et al. [1], pp. 522–529.

5. J. G. Bazan. A comparison of dynamic and non-dynamic rough set methods for extracting laws from decision tables. In: Polkowski and Skowron [90], pp. 321–365.

6. J. G. Bazan, H. S. Nguyen, J. F. Peters, A. Skowron, M. Szczuka. Rough set approach to pattern extraction from classifiers. In: Skowron and Szczuka [120], pp. 20–29.
   URL http://www.elsevier.nl/locate/entcs/volume82.html

7. J. G. Bazan, H. S. Nguyen, A. Skowron, M. Szczuka. A view on rough set concept approximation. In: Wang et al. [142], pp. 181–188.

8. J. G. Bazan, J. F. Peters, A. Skowron. Behavioral pattern identification through rough set modelling. In: Ślęzak et al. [127], pp. 688–697.

9. J. G. Bazan, A. Skowron. Classifiers based on approximate reasoning schemes. In: Dunin-Kęplicz et al. [21], pp. 191–202.

10. S. Behnke. Hierarchical Neural Networks for Image Interpretation, Lecture Notes in Computer Science, vol. 2766. Springer, Heidelberg, Germany, 2003.

11. L. Breiman. Statistical modeling: The two cultures. Statistical Science 16(3) (2001) 199–231.

12. J. L. Casti. Alternate Realities? Mathematical Models of Nature and Man. John Wiley & Sons, New York, NY, 1989.

13. N. Cercone, A. Skowron, N. Zhong (Eds.). (Special issue), Computational Intelligence: An International Journal, vol. 17(3). 2001.

14. K. Cios, W. Pedrycz, R. Swiniarski. Data mining methods for knowledge discovery. Kluwer, Norwell, MA, 1998.

15. C. H. Coombs, G. S. Avruin. The Structure of Conflicts. Lawrence Erlbaum, London, 1988.

16. R. Deja. Conflict analysis, rough set methods and applications. In: Polkowski et al. [87], pp. 491–520.

17. R. Deja, A. Skowron. On some conflict models and conflict resolution. Romanian Journal of Information Science and Technology 5(1-2) (2002) 69–82.
18. S. Demri, E. Orłowska. Incomplete Information: Structure, Inference, Complexity. Monographs in Theoretical Cpmputer Sience, Springer-Verlag, Heidelberg, Germany, 2002.
19. D. Dubois, H. Prade. Foreword. In: Rough Sets: Theoretical Aspects of Reasoning about Data [74].
20. R. Duda, P. Hart, R. Stork. Pattern Classification. John Wiley & Sons, New York, NY, 2002.
21. B. Dunin-Kęplicz, A. Jankowski, A. Skowron, M. Szczuka (Eds.). Monitoring, Security, and Rescue Tasks in Multiagent Systems (MSRAS'2004). Advances in Soft Computing, Springer, Heidelberg, Germany, 2005.
22. I. Düntsch, G. Gediga. Rough set data analysis: A road to non-invasive knowledge discovery. Methodos Publishers, Bangor, UK, 2000.
23. A. E. Fallah-Seghrouchni. Multi-agent planning for autonomous agents coordination. In: Dunin-Kęplicz et al. [21], pp. 53–68.
24. M. Fedrizzi, J. Kacprzyk, H. Nurmi. How different are social choice functions: A rough sets approach. Quality and Quantity 30 (1996) 87–99.
25. J. H. Friedman, T. Hastie, R. Tibshirani. The Elements of Statistical Learning: Data Mining, Inference, and Prediction. Springer-Verlag, Heidelberg, Germany, 2001.
26. H. Garcia-Molina, J. Ullman, J. Widom. Database Systems: The Complete Book. Prentice Hall, Upper Saddle River, New Jersey, 2002.
27. G. Gediga, I. Düntsch. Rough approximation quality revisited. Artificial Intelligence 132 (2001) 219–234.
28. M. Ghallab, D. Nau, P. Traverso (Eds.). Automated Planning Theory and Practice. Morgan Kaufmann, San Francisco, 2004.
29. S. Greco, B. Matarazzo, R. Słowiński. Rough set theory for multicriteria decision analysis. European Journal of Operational Research 129(1) (2001) 1–47.
30. J. W. Grzymała-Busse. Managing Uncertainty in Expert Systems. Kluwer Academic Publishers, Norwell, MA, 1990.
31. J. W. Grzymała-Busse. LERS - A system for learning from examples based on rough sets. In: Słowiński [129], pp. 3–18.
32. J. W. Grzymala-Busse. A new version of the rule induction system LERS. Fundamenta Informaticae 31(1) (1997) 27–39.
33. J. W. Grzymala-Busse. LERS - A data mining system. In: The Data Mining and Knowledge Discovery Handbook. 2005, pp. 1347–1351.
34. S. Hirano, M. Inuiguchi, S. Tsumoto (Eds.). Proceedings of International Workshop on Rough Set Theory and Granular Computing (RSTGC'2001), Matsue, Shimane, Japan, May 20-22, 2001, Bulletin of the International Rough Set Society, vol. 5(1-2). International Rough Set Society, Matsue, Shimane, 2001.
35. M. Inuiguchi, S. Hirano, S. Tsumoto (Eds.). Rough Set Theory and Granular Computing, Studies in Fuzziness and Soft Computing, vol. 125. Springer-Verlag, Heidelberg, Germany, 2003.
36. R. Keefe. Theories of Vagueness. Cambridge Studies in Philosophy, Cambridge, UK, 2000.
37. W. Kloesgen, J. Żytkow (Eds.). Handbook of Knowledge Discovery and Data Mining. Oxford University Press, Oxford, 2002.
38. J. Komorowski, Z. Pawlak, L. Polkowski, A. Skowron. Rough sets: A tutorial. In: Pal and Skowron [71], pp. 3–98.

39. B. Kostek. Soft Computing in Acoustics, Applications of Neural Networks, Fuzzy Logic and Rough Sets to Physical Acoustics, Studies in Fuzziness and Soft Computing, vol. 31. Physica-Verlag, Heidelberg, Germany, 1999.

40. B. Kostek. Perception-Based Data Processing in Acoustics. Applications to Music Information Retrieval and Psychophysiology of Hearing, Studies in Computational Intelligence, vol. 3. Springer, Heidelberg, Germany, 2005.

41. R. Kowalski. A logic-based approach to conflict resolution. Report, Department of Computing, Imperial College (2003) 1–28.
URL http://www.doc.ic.ac.uk/~rak/papers/conflictresolution.pdf

42. S. Kraus. Strategic Negotiations in Multiagent Environments. The MIT Press, Cambridge, MA, 2001.

43. G. Lai, C. Li, K. Sycara, J. A. Giampapa. Literature review on multi-attribute negotiations. Technical Report CMU-RI-TR-04-66 (2004) 1–35.

44. G. W. Leibniz. Discourse on metaphysics. In: Ariew and Garber [2], pp. 35–68.

45. S. Leśniewski. Grungzüge eines neuen Systems der Grundlagen der Mathematik. Fundamenta Mathematicae 14 (1929) 1–81.

46. S. Leśniewski. On the foundations of mathematics. Topoi 2 (1982) 7–52.

47. T. Y. Lin. Neighborhood systems and approximation in database and knowledge base systems. In: M. L. Emrich, M. S. Phifer, M. Hadzikadic, Z. W. Ras (Eds.), Proceedings of the Fourth International Symposium on Methodologies of Intelligent Systems (Poster Session), October 12-15, 1989, Oak Ridge National Laboratory, Charlotte, NC. 1989, pp. 75–86.

48. T. Y. Lin (Ed.). Special issue, Journal of the Intelligent Automation and Soft Computing, vol. 2(2). 1996.

49. T. Y. Lin. The discovery, analysis and representation of data dependencies in databases. In: L. Polkowski, A. Skowron (Eds.), Rough Sets in Knowledge Discovery 1: Methodology and Applications, Physica-Verlag, Heidelberg, Germany, Studies in Fuzziness and Soft Computing, vol. 18. 1998, pp. 107–121.

50. T. Y. Lin, N. Cercone (Eds.). Rough Sets and Data Mining - Analysis of Imperfect Data. Kluwer Academic Publishers, Boston, USA, 1997.

51. T. Y. Lin, A. M. Wildberger (Eds.). Soft Computing: Rough Sets, Fuzzy Logic, Neural Networks, Uncertainty Management, Knowledge Discovery. Simulation Councils, Inc., San Diego, CA, USA, 1995.

52. T. Y. Lin, Y. Y. Yao, L. A. Zadeh (Eds.). Rough Sets, Granular Computing and Data Mining. Studies in Fuzziness and Soft Computing, Physica-Verlag, Heidelberg, Germany, 2001.

53. J. Łukasiewicz. Die logischen Grundlagen der Wahrscheinlichkeitsrechnung, 1913. In: L. Borkowski (Ed.), Jan Łukasiewicz - Selected Works, North Holland Publishing Company, Amsterdam, London, Polish Scientific Publishers, Warsaw. 1970, pp. 16–63.

54. Y. Maeda, K. Senoo, H. Tanaka. Interval density function in conflict analysis. In: Skowron et al. [109], pp. 382–389.

55. S. Marcus. The paradox of the heap of grains, in respect to roughness, fuzziness and negligibility. In: Polkowski and Skowron [89], pp. 19–23.

56. T. M. Mitchel. Machine Learning. McGraw-Hill Series in Computer Science, Boston, MA, 1999.

57. S. Mitra, T. Acharya. Data mining. Multimedia, Soft Computing, and Bioinformatics. John Wiley & Sons, New York, NY, 2003.

58. A. Nakamura. Conflict logic with degrees. In: Pal and Skowron [71], pp. 136–150.
    URL `http://citeseer.nj.nec.com/komorowski98rough.html`

59. H. S. Nguyen, S. H. Nguyen. Rough sets and association rule generation. Fundamenta Informaticae 40(4) (1999) 383–405.

60. H. S. Nguyen, D. Ślęzak. Approximate reducts and association rules - correspondence and complexity results. In: Skowron et al. [109], pp. 137–145.

61. S. H. Nguyen, J. Bazan, A. Skowron, H. S. Nguyen. Layered learning for concept synthesis. In: Peters and Skowron [79], pp. 187–208.

62. S. H. Nguyen, H. S. Nguyen. Some efficient algorithms for rough set methods. In: Sixth International Conference on Information Processing and Management of Uncertainty on Knowledge Based Systems IPMU'1996. Granada, Spain, 1996, vol. III, pp. 1451–1456.

63. T. T. Nguyen, A. Skowron. Rough set approach to domain knowledge approximation. In: Wang et al. [142], pp. 221–228.

64. H. Nurmi, J. Kacprzyk, M. Fedrizzi. Theory and methodology: Probabilistic, fuzzy and rough concepts in social choice. European Journal of Operational Research 95 (1996) 264–277.

65. E. Orłowska. Semantics of vague conepts. In: G. Dorn, P. Weingartner (Eds.), Foundation of Logic and Linguistics, Plenum Press, New York. 1984, pp. 465–482.

66. E. Orłowska. Reasoning about vague concepts. Bulletin of the Polish Academy of Sciences, Mathematics 35 (1987) 643–652.

67. E. Orłowska (Ed.). Incomplete Information: Rough Set Analysis, Studies in Fuzziness and Soft Computing, vol. 13. Springer-Verlag/Physica-Verlag, Heidelberg, Germany, 1997.

68. S. K. Pal, P. Mitra. Pattern Recognition Algorithms for Data Mining. CRC Press, Boca Raton, Florida, 2004.

69. S. K. Pal, W. Pedrycz, A. Skowron, R. Swiniarski (Eds.). Special volume: Rough-neuro computing, Neurocomputing, vol. 36. 2001.

70. S. K. Pal, L. Polkowski, A. Skowron (Eds.). Rough-Neural Computing: Techniques for Computing with Words. Cognitive Technologies, Springer-Verlag, Heidelberg, Germany, 2004.

71. S. K. Pal, A. Skowron (Eds.). Rough Fuzzy Hybridization: A New Trend in Decision-Making. Springer-Verlag, Singapore, 1999.

72. Z. Pawlak. Information systems - theoretical foundations. Information Systems 6 (1981) 205–218.

73. Z. Pawlak. Rough sets. International Journal of Computer and Information Sciences 11 (1982) 341–356.

74. Z. Pawlak. Rough Sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving, vol. 9. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

75. Z. Pawlak. An inquiry into anatomy of conflicts. Journal of Information Sciences 109 (1998) 65–78.

76. Z. Pawlak. Decision rules, Bayes' rule and rough sets. In: Skowron et al. [109], pp. 1–9.

77. Z. Pawlak, A. Skowron. Rough membership functions. In: R. Yager, M. Fedrizzi, J. Kacprzyk (Eds.), Advances in the Dempster-Shafer Theory of Evidence. John Wiley & Sons, New York, NY, 1994, pp. 251–271.

78. J. Peters, A. Skowron (Eds.). Special issue on a rough set approach to reasoning about data, International Journal of Intelligent Systems, vol. 16(1). 2001.

79. J. F. Peters, A. Skowron (Eds.). Transactions on Rough Sets I: Journal Subline, Lecture Notes in Computer Science, vol. 3100. Springer, Heidelberg, Germany, 2004.

80. J. F. Peters, A. Skowron (Eds.). Transactions on Rough Sets III: Journal Subline, Lecture Notes in Computer Science, vol. 3400. Springer, Heidelberg, Germany, 2005.

81. J. F. Peters, A. Skowron (Eds.). Transactions on Rough Sets IV: Journal Subline, Lecture Notes in Computer Science, vol. 3700. Springer, Heidelberg, Germany, 2005.

82. J. F. Peters, A. Skowron, D. Dubois, J. W. Grzymała-Busse, M. Inuiguchi, L. Polkowski (Eds.). Transactions on Rough Sets II. Rough sets and fuzzy sets: Journal Subline, Lecture Notes in Computer Science, vol. 3135. Springer, Heidelberg, Germany, 2004.

83. J. F. Peters, A. Skowron, Z. Suraj. An application of rough set methods in control design. Fundamenta Informaticae 43(1-4) (2000) 269–290.

84. L. Polkowski. Rough Sets: Mathematical Foundations. Advances in Soft Computing, Physica-Verlag, Heidelberg, Germany, 2002.

85. L. Polkowski. Rough mereology: A rough set paradigm for unifying rough set theory and fuzzy set theory. Fundamenta Informaticae 54 (2003) 67–88.

86. L. Polkowski. Toward rough set foundations. mereological approach. In: Tsumoto et al. [139], pp. 8–25.

87. L. Polkowski, T. Y. Lin, S. Tsumoto (Eds.). Rough Set Methods and Applications: New Developments in Knowledge Discovery in Information Systems, Studies in Fuzziness and Soft Computing, vol. 56. Springer-Verlag/Physica-Verlag, Heidelberg, Germany, 2000.

88. L. Polkowski, A. Skowron. Rough mereology: A new paradigm for approximate reasoning. International Journal of Approximate Reasoning 15(4) (1996) 333–365.

89. L. Polkowski, A. Skowron (Eds.). First International Conference on Rough Sets and Soft Computing RSCTC'1998, Lecture Notes in Artificial Intelligence, vol. 1424. Springer-Verlag, Warsaw, Poland, 1998.

90. L. Polkowski, A. Skowron (Eds.). Rough Sets in Knowledge Discovery 1: Methodology and Applications, Studies in Fuzziness and Soft Computing, vol. 18. Physica-Verlag, Heidelberg, Germany, 1998.

91. L. Polkowski, A. Skowron (Eds.). Rough Sets in Knowledge Discovery 2: Applications, Case Studies and Software Systems, Studies in Fuzziness and Soft Computing, vol. 19. Physica-Verlag, Heidelberg, Germany, 1998.

92. L. Polkowski, A. Skowron. Towards adaptive calculus of granules. In: L. A. Zadeh, J. Kacprzyk (Eds.), Computing with Words in Information/Intelligent Systems. Physica-Verlag, Heidelberg, Germany, 1999, pp. 201–227.

93. L. Polkowski, A. Skowron. Rough mereology in information systems. a case study: Qualitative spatial reasoning. In: Polkowski et al. [87], pp. 89–135.

94. L. Polkowski, A. Skowron. Rough mereological calculi of granules: A rough set approach to computation. Computational Intelligence: An International Journal 17(3) (2001) 472–492.

95. L. Polkowski, A. Skowron, J. Zytkow. Rough foundations for rough sets. In: Lin and Wildberger [51], pp. 55–58.

96. S. Read. Thinking about Logic: An Introduction to the Philosophy of Logic. Oxford University Press, Oxford, New York, 1994.

97. J. Rissanen. Modeling by shortes data description. Automatica 14 (1978) 465–471.

98. J. Rissanen. Minimum-description-length principle. In: S. Kotz, N. Johnson (Eds.), Encyclopedia of Statistical Sciences, John Wiley & Sons, New York, NY. 1985, pp. 523–527.

99. A. Skowron. Rough sets in perception-based computing (keynote talk). In: First International Conference on Pattern Recognition and Machine Intelligence (PReMI'05) December 18-22, 2005, Indian Statistical Institute, Kolkata. pp. 21–29.

100. A. Skowron (Ed.). Proceedings of the 5th Symposium on Computation Theory, Zaborów, Poland, 1984, Lecture Notes in Computer Science, vol. 208. Springer-Verlag, Berlin, Germany, 1985.

101. A. Skowron. Boolean reasoning for decision rules generation. In: J. Komorowski, Z. W. Raś (Eds.), ISMIS'1993, Trondheim, Norway, June 15-18, 1993. Springer-Verlag, 1993, Lecture Notes in Artificial Intelligence, vol. 689, pp. 295–305.

102. A. Skowron. Extracting laws from decision tables. Computational Intelligence: An International Journal 11 (1995) 371–388.

103. A. Skowron. Rough sets in KDD - plenary talk. In: Z. Shi, B. Faltings, M. Musen (Eds.), 16-th World Computer Congress (IFIP'2000): Proceedings of Conference on Intelligent Information Processing (IIP'2000), Publishing House of Electronic Industry, Beijing. 2000, pp. 1–14.

104. A. Skowron. Toward intelligent systems: Calculi of information granules. Bulletin of the International Rough Set Society 5(1-2) (2001) 9–30.

105. A. Skowron. Approximate reasoning in distributed environments. In: Zhong and Liu [151], pp. 433–474.

106. A. Skowron. Perception logic in intelligent systems (keynote talk). In: S. Blair et al (Ed.), Proceedings of the 8th Joint Conference on Information Sciences (JCIS 2005), July 21-26, 2005, Salt Lake City, Utah, USA. X-CD Technologies: A Conference & Management Company, 15 Coldwater Road, Toronto, Ontario, M3B 1Y8, 2005, pp. 1–5.

107. A. Skowron. Rough sets and vague concepts. Fundamenta Informaticae 64(1-4) (2005) 417–431.

108. A. Skowron, J. W. Grzymala-Busse. From rough set theory to evidence theory. In: R. Yager, M. Fedrizzi, J. Kacprzyk (Eds.), Advances in the Dempster-Shafer Theory of Evidence, John Wiley & Sons, New York, NY. 1994, pp. 193–236.

109. A. Skowron, S. Ohsuga, N. Zhong (Eds.). Proceedings of the 7-th International Workshop on Rough Sets, Fuzzy Sets, Data Mining, and Granular-Soft Computing (RSFDGrC'99), Yamaguchi, November 9-11, 1999, Lecture Notes in Artificial Intelligence, vol. 1711. Springer-Verlag, Heidelberg, Germany, 1999.

110. A. Skowron, S. K. Pal (Eds.). Special volume: Rough sets, pattern recognition and data mining, Pattern Recognition Letters, vol. 24(6). 2003.

111. A. Skowron, J. Peters. Rough sets: Trends and challenges. In: Wang et al. [142], pp. 25–34. (plenary talk).

112. A. Skowron, C. Rauszer. The discernibility matrices and functions in information systems. In: Słowiński [129], pp. 331–362.

113. A. Skowron, J. Stepaniuk. Tolerance approximation spaces. Fundamenta Informaticae 27(2-3) (1996) 245–253.

114. A. Skowron, J. Stepaniuk. Information granules: Towards foundations of granular computing. International Journal of Intelligent Systems 16(1) (2001) 57–86.

115. A. Skowron, J. Stepaniuk. Information granules and rough-neural computing. In: Pal et al. [70], pp. 43–84.

116. A. Skowron, J. Stepaniuk. Ontological framework for approximation. In: Ślęzak et al. [126], pp. 718–727.

117. A. Skowron, R. Swiniarski. Rough sets and higher order vagueness. In: Ślęzak et al. [126], pp. 33–42.

118. A. Skowron, R. Swiniarski, P. Synak. Approximation spaces and information granulation. In: Peters and Skowron [80], pp. 175–189.

119. A. Skowron, P. Synak. Complex patterns. Fundamenta Informaticae 60(1-4) (2004) 351–366.

120. A. Skowron, M. Szczuka (Eds.). Proceedings of the Workshop on Rough Sets in Knowledge Discovery and Soft Computing at ETAPS 2003, April 12-13, 2003, Electronic Notes in Computer Science, vol. 82(4). Elsevier, Amsterdam, Netherlands, 2003.
URL http://www.elsevier.nl/locate/entcs/volume82.html

121. D. Ślęzak. Approximate reducts in decision tables. In: Sixth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems IPMU'1996. Granada, Spain, 1996, vol. III, pp. 1159–1164.

122. D. Ślęzak. Normalized decision functions and measures for inconsistent decision tables analysis. Fundamenta Informaticae 44 (2000) 291–319.

123. D. Ślęzak. Various approaches to reasoning with frequency-based decision reducts: a survey. In: Polkowski et al. [87], pp. 235–285.

124. D. Ślęzak. Approximate entropy reducts. Fundamenta Informaticae 53 (2002) 365–387.

125. D. Ślęzak. Rough sets and Bayes factor. In: Peters and Skowron [80], pp. 202–229.

126. D. Ślęzak, G. Wang, M. Szczuka, I. Düntsch, Y. Yao (Eds.). Proceedings of the 10th International Conference on Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing (RSFDGrC'2005), Regina, Canada, August 31-September 3, 2005, Part I, Lecture Notes in Artificial Intelligence, vol. 3641. Springer-Verlag, Heidelberg, Germany, 2005.

127. D. Ślęzak, J. T. Yao, J. F. Peters, W. Ziarko, X. Hu (Eds.). Proceedings of the 10th International Conference on Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing (RSFDGrC'2005), Regina, Canada, August 31-September 3, 2005, Part II, Lecture Notes in Artificial Intelligence, vol. 3642. Springer-Verlag, Heidelberg, Germany, 2005.

128. D. Ślęzak, W. Ziarko. The investigation of the Bayesian rough set model. International Journal of Approximate Reasoning 40 (2005) 81–91.

129. R. Słowiński (Ed.). Intelligent Decision Support - Handbook of Applications and Advances of the Rough Sets Theory, System Theory, Knowledge Engineering and Problem Solving, vol. 11. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

130. R. Słowiński, J. Stefanowski (Eds.). Special issue: Proceedings of the First International Workshop on Rough Sets: State of the Art and Perspectives, Kiekrz, Poznań, Poland, September 2–4 (1992), Foundations of Computing and Decision Sciences, vol. 18(3-4). 1993.

131. R. Słowiński, D. Vanderpooten. Similarity relation as a basis for rough approximations. In: P. Wang (Ed.), Advances in Machine Intelligence and Soft Computing Vol. 4, Duke University Press, Duke, NC. 1997, pp. 17–33.
132. S. Staab, R. Studer (Eds.). Handbook on Ontologies. International Handbooks on Information Systems, Springer, Heidelberg, Germany, 2004.
133. J. Stepaniuk. Approximation spaces, reducts and representatives. In: Polkowski and Skowron [91], pp. 109–126.
134. P. Stone. Layered Learning in Multi-Agent Systems: A Winning Approach to Robotic Soccer. The MIT Press, Cambridge, MA, 2000.
135. Z. Suraj. Rough set methods for the synthesis and analysis of concurrent processes. In: Polkowski et al. [87], pp. 379–488.
136. K. Sycara. Multiagent systems. AI Magazine (Summer 1998) 79–92.
137. T. Terano, T. Nishida, A. Namatame, S. Tsumoto, Y. Ohsawa, T. Washio (Eds.). New Frontiers in Artificial Intelligence, Joint JSAI'2001 Workshop Post-Proceedings, Lecture Notes in Artificial Intelligence, vol. 2253. Springer-Verlag, Heidelberg, Germany, 2001.
138. S. Tsumoto, S. Kobayashi, T. Yokomori, H. Tanaka, A. Nakamura (Eds.). Proceedings of the The Fourth Internal Workshop on Rough Sets, Fuzzy Sets and Machine Discovery, November 6-8, University of Tokyo , Japan. The University of Tokyo, Tokyo, 1996.
139. S. Tsumoto, R. Słowiński, J. Komorowski, J. G. Busse (Eds.). Proceedings of the 4th International Conference on Rough Sets and Current Trends in Computing (RSCTC'2004), Uppsala, Sweden, June 1-5, 2004, Lecture Notes in Artificial Intelligence, vol. 3066. Springer-Verlag, Heidelberg, Germany, 2004.
140. S. Tsumoto, H. Tanaka. PRIMEROSE: Probabilistic rule induction method based on rough sets and resampling methods. Computational Intelligence: An International Journal 11 (1995) 389–405.
141. V. Vapnik. Statistical Learning Theory. John Wiley & Sons, New York, NY, 1998.
142. G. Wang, Q. Liu, Y. Yao, A. Skowron (Eds.). Proceedings of the 9-th International Conference on Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing (RSFDGrC'2003), Chongqing, China, May 26-29, 2003, Lecture Notes in Artificial Intelligence, vol. 2639. Springer-Verlag, Heidelberg, Germany, 2003.
143. S. K. M. Wong, W. Ziarko. Comparison of the probabilistic approximate classification and the fuzzy model. Fuzzy Sets and Systems 21 (1987) 357–362.
144. J. Wróblewski. Theoretical foundations of order-based genetic algorithms. Fundamenta Informaticae 28 (1996) 423–430.
145. T. Y. Yao. On generalizing rough set theory. In: Wang et al. [142], pp. 44–51.
146. Y. Y. Yao. Generalized rough set models. In: Polkowski and Skowron [90], pp. 286–318.
147. Y. Y. Yao. Information granulation and rough set approximation. International Journal of Intelligent Systems 16 (2001) 87–104.
148. Y. Y. Yao, S. K. M. Wong, T. Y. Lin. A review of rough set models. In: Lin and Cercone [50], pp. 47–75.
149. L. A. Zadeh. Fuzzy sets. Information and Control 8 (1965) 338–353.
150. L. A. Zadeh. A new direction in AI: Toward a computational theory of perceptions. AI Magazine 22(1) (2001) 73–84.
151. N. Zhong, J. Liu (Eds.). Intelligent Technologies for Information Analysis. Springer, Heidelberg, Germany, 2004.

152. W. Ziarko. Variable precision rough set model. Journal of Computer and System Sciences 46 (1993) 39–59.

153. W. Ziarko (Ed.). Rough Sets, Fuzzy Sets and Knowledge Discovery: Proceedings of the Second International Workshop on Rough Sets and Knowledge Discovery (RSKD'93), Banff, Alberta, Canada, October 12–15 (1993). Workshops in Computing, Springer–Verlag & British Computer Society, London, Berlin, 1994.

154. W. Ziarko (Ed.). Special issue, Computational Intelligence: An International Journal, vol. 11(2). 1995.

155. W. Ziarko (Ed.). Special issue, Fundamenta Informaticae, vol. 27(2-3). 1996.

156. W. Ziarko, Y. Yao (Eds.). Proceedings of the 2nd International Conference on Rough Sets and Current Trends in Computing (RSCTC'2000), Banff, Canada, October 16-19, 2000, Lecture Notes in Artificial Intelligence, vol. 2005. Springer-Verlag, Heidelberg, Germany, 2001.