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ORTHODOX AND NON-ORTHODOX SETS - SOME PHILOSOPHICAL REMARKS

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Abstract. We outline the relationship between classical (orthodox) sets from one side, and fuzzy and rough (non-orthodox) sets from another side. The classical concept of a set used in mathematics leads to antinomies, i.e., it is contradictory. This deficiency has, however, rather philosophical than practical meaning. Antinomies are associated with very “artificial” sets constructed in logic but not found in sets used in mathematics. That is why one can use mathematics safely. Fuzzy set and rough set theory are two different approaches to vagueness and are not remedy for classical set theory difficulties. Fuzzy set theory addresses gradualness of knowledge, expressed by the fuzzy membership, whereas rough set theory addresses granularity of knowledge, expressed by the indiscernibility relation. From practical point of view both theories are not competing but are rather complementary.

1. Introduction

In this paper, the relationship between classical (orthodox) sets, and fuzzy and rough (non-orthodox) sets will be outlined and briefly discussed.

The concept of a set is fundamental for the whole mathematics. Modern set theory was formulated by George Cantor [1].

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Bertrand Russell [5] has discovered that the intuitive notion of a set proposed by Cantor leads to antinomies. Two kinds of remedy for this discontent have been proposed: improvements of Cantorian set theory and alternative set theories.

Another issue discussed in connection with the notion of a set is vagueness. According to Gottlob Frege [2] mathematics requires that all mathematical notions (including set) must be exact. However philosophers and, recently, computer scientists got interested in vague concepts.

The notion of a fuzzy set proposed in 1965 by Lotfi Zadeh [6] is a new approach to vagueness, in which sets are defined by partial membership, in contrast to crisp membership used in classical definition of a set.

Rough set theory [3], proposed by the author in 1982, expresses vagueness, not by means of membership, but employing a boundary region of a set. If the boundary region of a set is empty, it means that the set is crisp, otherwise the set is rough (inexact). Nonempty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely.

Interesting philosophical discussion of fuzzy and rough sets from the point of view of vagueness can be found in [4].

2. Orthodox (Classical) Sets

The notion of a set is not only basic for the whole mathematics but it also plays an important role in natural language. We often speak about sets (collections) of various objects of interest, e.g., collection of books, paintings, people etc.

Intuitive meaning of a set according to *The Oxford English Dictionary* is the following:
"Number of things of the same kind, that belong together because they are similar or complementary to each other."

Thus a set is a collection of things which are somehow related to each other but the nature of this relationship is not specified in these definitions.

In fact, these definitions are due to the original definition given by the creator of set theory, a German mathematician George Cantor [1], which reads as follows:
"Unter einer Mannigfaltigkeit oder Menge verstehe ich nämlich allgenein jedes Viele, welches sich als Eines denken lässt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann."

Thus, according to Cantor, a set is a collection of any objects, which according to some law can be considered as a whole.

All mathematical objects, e.g., relations, functions, numbers, etc., are some kind of sets. In fact, set theory is needed in mathematics to provide rigor.

Bertrand Russell discovered [5] that the intuitive notion of a set given by Cantor leads to *antinomies* (contradictions). One of the best known antinomies called the powerset antinomy goes as follows: consider (infinite) set X of all sets. Thus X is the greatest set. Let Y denote the set of all subsets of X. Obviously Y is greater then X, because the number of subsets of a set is always greater than the number of its elements.

Thus the basis concept of mathematics, the concept of a set, is contradictory. That means that a set cannot be a collection of arbitrary elements as was stipulated by Cantor.

As a remedy for this defect, several improvements of set theory have been proposed.

All these improvements consist in restrictions, put on objects which can form a set. The restrictions are expressed by properly chosen axioms, which say how the set can be build. They are called, in contrast to Cantors' intuitive set theory, axiomatic set theories. Instead of improvements of Cantors' set theory by its axiomatization, some mathematicians proposed some escapes from classical set theory by creating completely new idea of a set, which would free the theory from antinomies, but in fact these ideas have not been accepted by mathematicians.

3. Vagueness

In classical set theory, a set is uniquely determined by its elements. In other words, it means that every element must be uniquely classified as belonging to the set or not. That is to say the notion of a set is a *crisp* (precise) one. For example, the set of odd numbers is crisp because every number is either odd or even, in contrast to odd numbers, the notion of a beautiful painting is vague, because we are unable to classify uniquely all paintings into two classes: beautiful and not beautiful. Some paintings cannot be decided whether they are beautiful or not and thus they remain in the doubtful area. Thus *beauty* is not a precise but a *vague* concept. In mathematics we have to use crisp notions, otherwise precise reasoning would be impossible. However, almost all concepts we are using in natural language are vague. Therefore, common sense reasoning based on natural language must be based on vague concepts and not on classical logic. This is why vagueness is important for philosophers and recently also for computer scientists.

The idea of vagueness can be traced back to ancient Greek philosophers Eubulides (ca. 400 BC) who first formulated so called sorites (Bald Man or Heap) paradox. The paradox goes as follows: suppose a man has 100 000 hair on his head. Removing one hair from his head surely cannot make him bald. Repeating this step we arrive at the conclusion that a man without any hair is not bald. Similar reasoning can be applied to a hip of stones.

Nowadays vagueness is usually associated with the boundary region approach (i.e., existing of objects which cannot be uniquely classified to the set or its complement) which was first formulated in 1893 by the father of modern logic, German logician, Gottlob Frege [2]. He wrote:

“Der Begrieff muss scharf begrenzt sein. Einem unscharf begrenzten Begriff würde ein Bezirk ensprechen, der nicht überall ein scharfe Grentzlinie hätte, sondern stellenweise ganz verschwimmend in die Umgebung übergine“.

Thus according to Frege

“The concept must have a sharp boundary. To the concept without a sharp boundary there would correspond an area that had not a sharp boundary-line all around.”

I.e., mathematics must use crisp, not vague concepts, otherwise it would be impossible to reason precisely.

Summing up, vagueness is

- not allowed in mathematics,
- interesting for philosophy,
- necessary for computer science.

4. Fuzzy sets

In 1965, Professor of the University of Berkeley, Lotfi Zadeh proposed a very interesting approach to vagueness called *fuzzy set theory* [5]. In his approach an element can belong to a set to a degree k ($0 \leq k \leq 1$), in contrast to classical set theory where an element must definitely belong or not to a set. E.g., in classical set theory one can be definitely ill or healthy, whereas in fuzzy set theory we can say that someone is ill (or healthy) in 60 percent (i.e. in the degree 0.6). Of course, at once the question arises where we get the value of degree from. This issue raised a lot of discussion, but we will refrain from considering this problem here.

Thus, fuzzy membership function can be presented as

$$\mu_X(x) \in <0,1>$$

where, X is a set and x is an element.

Let us observe that the definition of fuzzy set involves more advanced mathematical concepts, real numbers and functions, whereas in classical set theory the notion of a set is used as a fundamental notion of whole mathematics and is used to derive any other mathematical concepts, e.g., numbers and functions. Consequently, fuzzy set theory cannot replace classical set theory, because, in fact, the theory is needed to define fuzzy sets.

Fuzzy membership function has the following properties:

a) $\mu_{U-X}^B(x) = 1 - \mu_X^B(x)$ for any $x \in U$,

b) $\mu_{X \cup Y}^B(x) = \max(\mu_X^B(x), \mu_Y^B(x))$ for any $x \in U$,

c) $\mu_{X \cap Y}^B(x) = \min(\mu_X^B(x), \mu_Y^B(x))$ for any $x \in U$.

That means that the membership of an element to the union and intersection of sets is uniquely determined by its membership to constituent sets. This is a very nice property and allows very simple operations on fuzzy sets, which is a very important feature both theoretically and practically.

Fuzzy set theory and its applications developed very extensively over last years and attracted attention of practitioners, logicians and philosophers worldwide.

5. Rough sets

Rough set theory [3] is still another approach to vagueness, proposed by the author in 1982. Similarly to fuzzy set theory, it is not an alternative to classical set theory but it is embedded in it. Rough set theory can be viewed as a specific implementation of Frege's idea of vagueness, i.e., imprecision in this approach is expressed by a boundary region of a set, and not by a partial membership, like in fuzzy set theory.

Because rough set theory is relatively less known, in comparison to fuzzy set theory, we will discuss this concept more exactly.

Rough set concept can be defined quite generally by means of topological operations, *interior* and *closure*, called *approximations.*

However, from practical point of view it is better to define basic concepts of this theory in terms of data. Therefore, we will start our considerations from a data set called an *information system.* An information system is a data table rows of which are labeled by objects of interest, columns – by attributes and entries of the table are attribute values. For example, the data table can describe a set of patients in a hospital. The patients can be characterized by some attributes, like *age*, *sex*, *blood pressure*, *body temperature*, etc. With every attribute a set of its values is associated, e.g., values of the attribute age can be *young*, *middle*, and *old*. Attribute values can also be numerical. Basic problem we are interested in, in data analysis, is to find patterns in data, i.e., to find relationships between some sets of attributes, e.g., we might be interested whether *blood* pressure depends on age and sex.

Let us describe this problem more precisely. Suppose we are given a set of objects *U* called the *universe* and the set of *attributes A*, describing objects of the universe in terms of *attribute values*. Let *X* be a subset of *U* and *B* a subset of *A*. We want to characterize the set *X* in terms of attributes *B*. To this end we will need the basic concepts of rough set theory given below.

- The *lower approximation* of a set *X* with respect to *B* is the set of all objects, which can be for *certain* classified as *X* using *B* (are *certainly X* in view of *B*).
- The *upper approximation* of a set *X* with respect to *B* is the set of all objects which can be *possibly* classified as *X* using *B* (are *possibly X* in view of *B*).

The *boundary region* of a set *X* with respect to *B* is the set of all objects, which can be classified neither as *X* nor as not-*X* using *B*.

Now we are ready to give the definition of rough sets.

- Set *X* is *crisp* (exact with respect to *B*), if the boundary region of *X* is empty.
- Set *X* is *rough* (inexact with respect to *B*), if the boundary region of *X* is nonempty.

Thus a set is *rough* (imprecise) if it has nonempty boundary region; otherwise the set is *crisp* (precise). This is exactly the idea of vagueness proposed by Frege.

Let us observe that the definition of rough sets refers to data (knowledge), and is *subjective*, in contrast to the definition of classical sets, which is in some sense *objective* one. The same remark applies to fuzzy sets.

The approximations and the boundary region can be defined more precisely. To this end we need some additional notation.

It is obvious that every subset of attributes *B* determines an equivalence relation on *U*. This relation will be referred to as an *indiscernibility relation*. The equivalence class determined by element *x* and the set of attributes *B* will be denoted *B*(*x*). The indiscernibility relation in certain sense describes our lack of knowledge about the universe. Equivalence classes of the indiscernibility relation, called *granules* generated by the set of attributes *B*, represent elementary portion of knowledge we are able to perceive in terms of available data. Thus, in view of the data we are unable, in general, to observe individual objects but we are forced to reason only about the accessible granules of knowledge.

Formal definitions of approximations and the boundary region are as follows:

- *B-lower approximation* of *X*

$$B_*(x) = \bigcup_{x \in U} \{B(x) : B(x) \subseteq X\},$$

- *B-upper approximation* of *X*

$$B^*(x) = \bigcup_{x \in U} \{B(x) : B(x) \cap X \neq \varnothing\},$$

- *B-boundary region* of *X*

$$BN_B(X) = B^*(X) - B_*(X).$$

As we can see from the definition approximations are expressed in terms of granules of knowledge. The lower approximation of a set is a union of all granules determined by the set of attributes *B* which are entirely included in the set; the upper approximation – is a union of all granules which have non-empty intersection with the set; the boundary region of the set is the difference between the upper and the lower approximation.

This definition is clearly depicted in Figure 1.

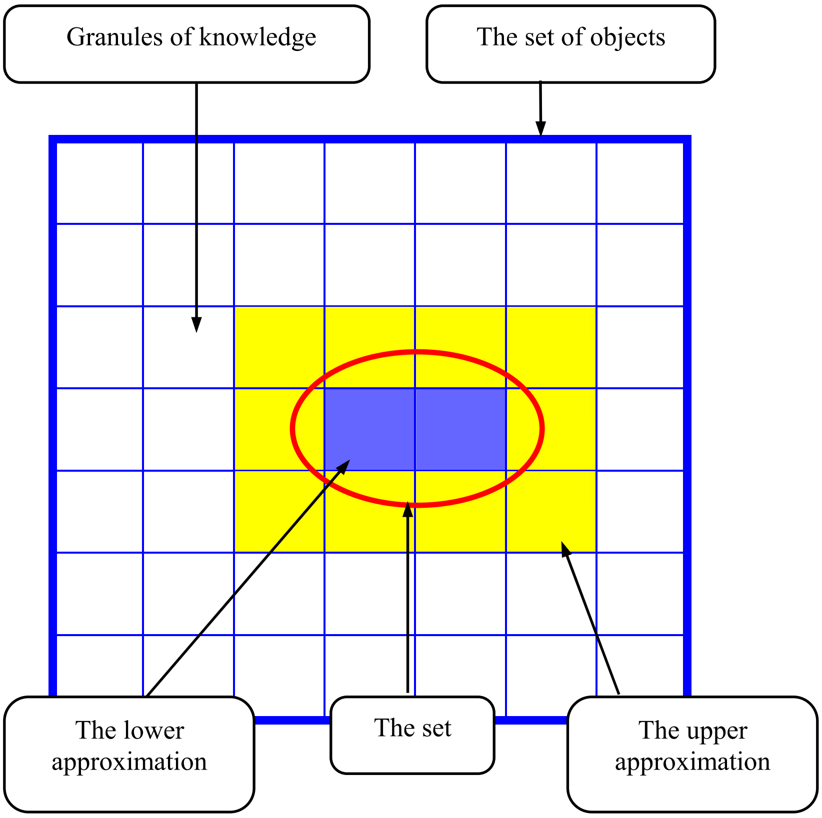

Figure 1. Graphical representation of the definition of approximations

It is interesting to compare definitions of classical sets, fuzzy sets and rough sets. Classical set is a primitive notion and is defined intuitively or axiomatically. Fuzzy sets are defined by employing the fuzzy membership function, which involves advanced mathematical structures, numbers and functions. Rough sets are defined by approximations. Thus this definition also requires advanced mathematical concepts.

It is easily seen that approximations are in fact interior and closure operations in a topology generated by the data. Thus fuzzy set theory and rough set theory require completely different mathematical setting.

Rough sets can be also defined employing, instead of approximation, rough membership function

$$\mu_X^B : U \to < 0,1 >$$

where

$$\mu_X^B(x) = \frac{| X \cap B(x)|}{| B(x)|}$$

and $|X|$ denotes the cardinality of X.

The rough membership function expresses a conditional probability that *x* belongs to *X* given *B* and can be interpreted as a degree that *x* belongs to *X* in view of information about *x* expressed by *B*.

It can be shown that the membership function has the following properties:

1. $\mu_X^B(x) = 1$ iff $x \in B_*(X)$,
2. $\mu_X^B(x) = 0$ iff $x \in U - B^*(X)$,
3. $0 < \mu_X^B(x) < 1$ iff $x \in BN_B(X)$,
4. $\mu_{U-X}^B(x) = 1 - \mu_X^B(x)$ for any $x \in U$,
5. $\mu_{X \cup Y}^B(x) \geq max\ (\mu_X^B(x), \mu_Y^B(x))$ for any $x \in U$,
6. $\mu_{X \cap Y}^B(x) \leq min\ (\mu_X^B(x), \mu_Y^B(x))$ for any $x \in U$.

From the properties it follows that the rough membership differs essentially from the fuzzy membership, for properties 5) and 6) show that the membership for union and intersection of sets, in general, cannot be computed – as in the case of fuzzy sets – from their constituent memberships. Thus, formally, the rough membership is a generalization of the fuzzy membership. Besides, the rough membership function, in contrast to fuzzy membership function, has a probabilistic flavor.

Let us also observe that rough set theory clearly distinguishes two very important concepts, vagueness and uncertainty, very often confused in the AI literature. Vagueness is the property of sets and can be described by approximations, whereas uncertainty is the property of elements of a set and can be expressed by the rough membership function.

6. Conclusions

Basic concept of mathematics, the set, leads to antinomies, i.e., it is contradictory.

The deficiency of sets, has rather philosophical than practical meaning, for sets used in mathematics are free from the above discussed faults. Antinomies are associated with very "artificial" sets constructed in logic but not found in sets used in mathematics. That is why we can use mathematics safely.

Fuzzy set and rough set theory are two different approaches to vagueness and are not remedy for classical set theory difficulties.

Fuzzy set theory addresses gradualness of knowledge, expressed by the fuzzy membership – whereas rough set theory addresses granularity of knowledge, expressed by the indiscernibility relation.

From practical point of view both theories are not competing but are rather complementary.

Summing up:

- The notion of orthodox (classical) set is fundamental for whole mathematics and is necessary to provide rigor in mathematics.
- Nonorthodox sets (fuzzy and rough) cannot replace orthodox sets – for their definitions need orthodox set theory (i.e. more advanced mathematical concepts, real numbers, functions and relations).
- The orthodox sets, lead to antinomies.
- The deficiency of orthodox sets, has rather philosophical than practical meaning, for sets used in everyday mathematics are free from antinomies.

Nonorthodox sets (fuzzy and rough) are not remedy for classical set theory difficulties but are two different approaches to vagueness.

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