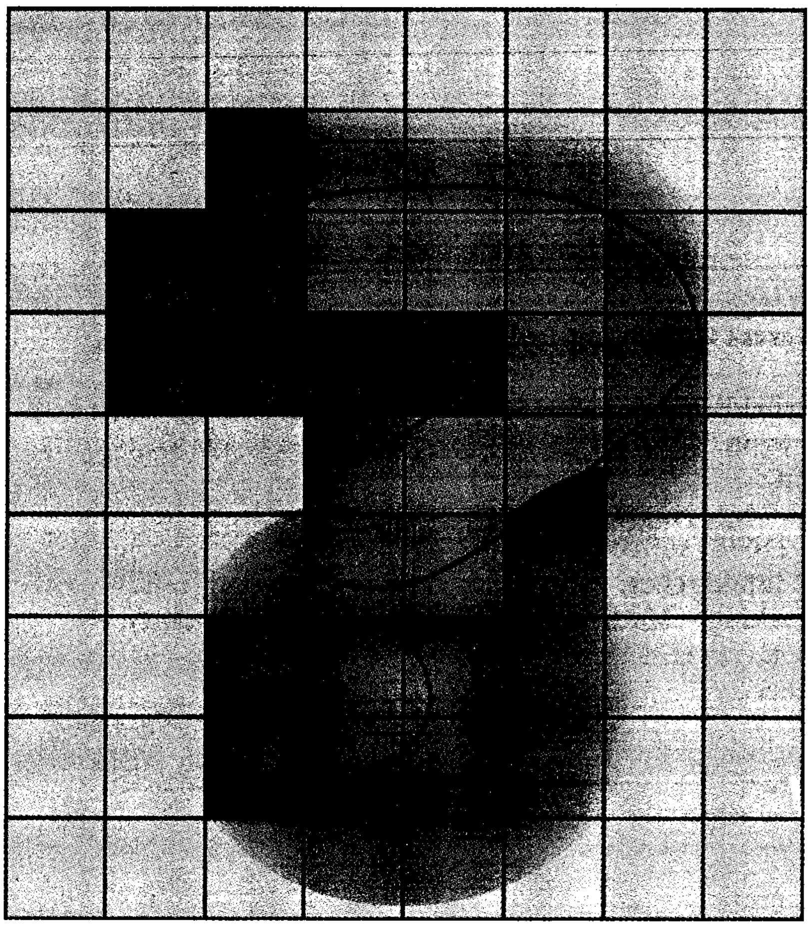
Rough Fuzzy Hybridization

A New Trend In Decision-Making



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Rough Sets, Rough Function and Rough Calculus

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1 Introduction

The concept of the rough set – a mathematical basis for reasoning about vagueness and uncertainty proved to be a natural instrument to inquire into many theoretical and practical problems related to data analysis. Although many serious real-life problems have been formulated and solved in the framework of rough set theory it seems that the extension of this theory to rough relations and rough functions is badly needed, for numerous applications can not be covered by the concepts of a rough set only.

The objective of this paper is to give some ideas concerning rough functions along the lines proposed by the author in [Pa1, Pa2, Pa4, Pa5, Pa6] and this paper is a modified version of [Pa6], where basic concepts of rough calculus have been proposed. Some similar concepts have been considered by Nakamura and Rosenfeld in [NR1].

It is interesting that ideas presented in this paper are not entirely new and their origin can be traced back to calculus of finite differences by George Boole (cf.[Bo1]).

Physical phenomena are usually described by differential equations. Solutions of these equations are real valued-functions, i.e., functions which are defined and valued on continuum of points. However, due to limited accuracy of measurements and computations, we are unable to observe (measure) or compute (simulate) exactly the abstract solutions. Consequently, we deal with approximate rather than exact solutions, i.e., we are using discrete and not continuous variables and functions.

Thus abstract mathematical models of physical systems are expressed in terms of real functions, whereas observed or computational models are described by data sets obtained as a result of measurements or computations - which use not real but rational numbers from a finite subset of ratinal numbers.

Hence an important question arises - what is the relationship between these two approaches, i.e., based on continuous or discrete mathematics philosophy?

Similar problems have been faced in image processing as perceived by Rosenfeld in [Ro3] and pursued by Nakamura and others in [NA1, NA2, NA3].

Another tool developed for discrete system analysis is the so called "cell-to-cell mapping theory" [Hs1], in which real numbers are replaced by intervals. Due to the lack of sound mathematical foundations, this method seems to be better suited to computer simulation than to prove theorems about discrete systems. It is worthwhile to mention that the idea of cell-to-cell mapping has found interesting application in the design and analysis of fuzzy controllers [PT1, Pa6, SC1].

Some aspects of the considered problems are also related to interval analysis first anticipated by Warmus in [Wa1, Wa2] and developed extensively by many authors recently.

Independently of practical problems caused by the "continuous versus discrete" antinomy, the philosophical question, of how to avoid the concept of infinity in mathematical analysis, has been tackled for a long time by logicians. Nonstandard analysis [Ro1], finistic analysis [My1] and infinitesimal analysis [CS1] provide various views on this topics.

In this paper we are going to investigate on the relationship between real and discrete functions based on the rough set philosophy. In particular we define rough (discrete) lower and upper representation of real functions and define and investigate some properties of these representations, such as rough continuity, rough derivatives, rough integral and rough differential equations - which can be viewed as discrete counterparts of real functions.

In particular we are interested how discretization of the real line effects basic properties of real functions, such as continuity, differentiability, etc. It turns out that some properties of real functions have counterparts in the case of discrete functions, but this is not always the case. The proposed approach is based on the rough set philosophy, in which the indiscernibility relation, defined in our case on the set of reals, is the starting point of our considerations.

The proposed approach differs essentially from numerical and approximation methods, even though we use, in some cases, similar terminology (e.g., approximation of function by another function) - for our attempt is based on functions defined and valued in the set of integers - however it has some overlaps with nonstandard, finistic and infinitesimal analysis, mentioned above.

Last but not least the proposed philosophy can be seen as a generalization of qualitative reasoning [Ku1, We1], where three-valued $(+, 0, -$, i.e., increasing, not changing, decreasing) qualitative derivatives are replaced by more general concept of multi-valued qualitative derivatives, so that expressions such as "slowly increasing", "fast increasing", "very fast increasing" etc. can be used instead of only "increasing".

Ideas shown in this paper have been presented at the International Conference on Intelligent Systems, Augustow, June 5-10, 1995, Poland and Joint Conference on Information Sciences (JCIS'95), Wrightsville Beach, Sept 28 - Oct 1, 1995, North Carolina, USA.

2 Scale, Discretization and Indiscernibility

This section introduces the basic concept of our approach - the indiscernibility relation. As mentioned in the introduction, real-valued parameters of a physical system can be exactly measured or computed with some approximation only. Therefore, we will introduce the concept of a scale, which is a finite set of integers $\{0, 1, \ldots, n\}$ and is intended to be used as a set of measurement units, like kg, km, hr, etc. - and a mapping of the scale into the set of real numbers. Elements of the scale, i.e., measurement units, are understood as approximations of real numbers, inaccessible due to our lack of infinite precision of measurement or computation. Notice that the concept of the scale is similar to that of the landmark, used in the qualitative reasoning methods, but both concepts are used differently.

Every scale determines uniquely a partition of the real line, or, in other words, defines an equivalence relation on reals, called in what follows an iniscernibility relation. Elements of the same equivalence class of the indiscernibility relation are said to be indiscernible with respect to the scale, and can be expressed approximately only by units of the scale. Thus, due to the use of the assumed scale real-valued parameters are replaced by approximate, integer-valued parameters.

A more formal presentation of the above ideas is given below [Ob1].

Let $[n] = \{0, 1, \ldots, n\}$ be a set of natural numbers. A strictly monotonic function $d : [n] \rightarrow \mathbf{R}$, i.e., such that for all $i, j \in [n]$, $i < j$ implies $d(i) < d(j)$ will be called a *scale*.

Any scale $d : [n] \rightarrow \mathbf{R}$ is a finite increasing sequence of reals $x_0, x_1, \ldots, x_n$, such that $x_i = d(i)$, for very $i \in [n]$ - thus it can be seen as a *discretization* of the closed interval $R_n = \langle d(0), d(n) \rangle = \langle x_0, x_n \rangle$.

Given a scale $d : [n] \rightarrow \mathbf{R}$ then one can define two functions

$$d_*(x) = max\{i \in [n] : x_i \leq x\}$$

$$d^*(x) = min\{i \in [n] : x_i \geq x\}$$

for every $x \in R_n$.

On the interval $R_n = \langle x_0, x_n \rangle$ we define an equivalence relation I_d, called the *indiscernibility* relation, and defined thus

$$xI_dy \text{ iff } d_*(x) = d_*(y) \text{ and } d^*(x) = d^*(x).$$

The family of all equivalence classes of the relation I_d, or the partition of the interval R_n, is given below

$$\{x_0\}, (x_0, x_1), \{x_1\}, (x_1, x_2), \{x_2\}, \ldots, (x_{n-1}, x_n), \{x_n\}$$

where each equivalence classe $[x]_d$ is an interval such that $[x]_d = (x_i, x_{i+1})$ whenever $x_i < x < x_{i+1}$, and $[x_i]_d = \{x_i\}$ for all $i \in [n]$.

If $x_i < x < x_{i+1}$, then $I_{*d}(x) = d(d_*(x)) = x_i$ and $I_{*d}(x) = d(d^*(x)) = x_{i+1}$, i.e., $I^*_d(x)$ and $I^*_d(x)$ are the ends of the interval $\langle x_i, x_{i+1} \rangle$; if $x = x_i$, then $I_{*d}(x) = I^*_d(x) = x_i$.

The ends of the interval $\langle x_i, x_{i+1} \rangle$ are called the *lower* and the *upper d-approximation* of x, respectively.

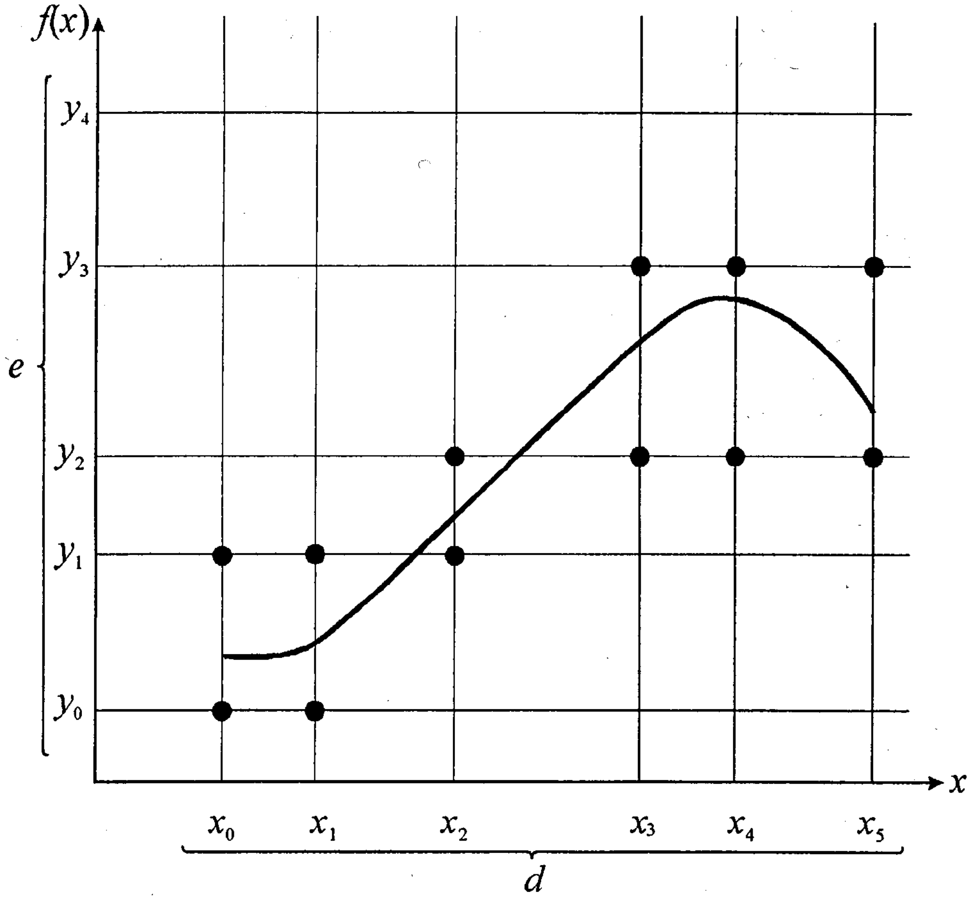

Fig. 1. The lower and upper approximation of a real function

The above discussed ideas are illustrated in Fig. 1.

Suppose we are given two scales $d : [n] \to \mathbf{R}$ and $e : [m] \to \mathbf{R}$, and let $f : R_n \to R_m$ be a function, where R_n, R_m denote the both side closed intervals $\langle x_0, x_n \rangle$, $\langle y_0, y_m \rangle$ respectively. We define its *lower rough representation* f_* with respect to d and e and its *upper rough representation* f^* with respect to d and e defined on $[n]$ and valued in $[m]$, as

$$f_*(i) = e_*(f(x))$$

$$f^*(i) = e^*(f(x))$$

where $d_*(x) = x_i$, for all $i \in [n]$ (see Fig. 1).

Thus with every real function one can associate two discrete functions; its lower and upper approximation. These approximations are uniquely determined by indiscernibility relations superimposed on the domain and range of the real function.

Let us observe that the just-defined approximations of real functions are different from those considered in approximation theory.

In what follows we are going to give some properties of discrete functions, defined and valued in the set of integers - mimicking some properties of real functions. It turns out that for this class of functions one can define concepts similar to that of real function, like continuity, derivatives, integrals, etc. These concepts display similar properties to those of real functions, and consequently discrete functions obtained as a result of measurements can be treated similarly to real functions.

We will start our consideration by defining rough (approximate) continuity for discrete functions.

3 Roughly Continuous Discrete Functions

The concept of continuity is strictly connected with real functions. Intuitively a function is continuous if a small change of its argument causes a small change of its value, or in other words - it cannot "vary too fast" [CJ1]. A similar idea can be employed also in the case of discrete functions, and we will say that a discrete function is roughly (approximately) continuous if a small change of its argument causes a small change of its value. In fact the concept of continuity of discrete functions has been used for a long time in qualitative reasoning [Ku1, We1] and others (cf. [Ch1, Pa1, Ro2]). Below the formal definition of roughly continuous function is given and some elementary properties of these functions are presented.

A discrete function $f : [n] \to [m]$ is *roughly continuous* iff for all $i, j \in [n]$, $|i - j| = 1$ implies $|f(i) - f(j)| \leq 1$.

The intermediate value property is valid for roughly continuous discrete functions as shown by the following proposition.

Proposition 1. *A discrete function $f : [n] \to [m]$ is roughly continuous iff for all $i, j \in [n], i \neq j$, and for every q between $f(i)$ and $f(j)$ there exist $p \in [n]$ between i and j for which $f(p) = q$.*

Thus the basic property of continuous real functions, the intermediate value theorem, after slight modifications is also valid for discrete functions. Hence it seems that the idea of continuity need not be necessarily attributed to real functions only, and can be extended to discrete functions.

4 Rough Derivatives and Rough Integrals of Discrete Functions

Now we are going to define two basic concepts in our approach to discrete functions, namely the rough derivative and the rough integral. It turns out that they display similar properties to "classical" derivatives and integrals. Let us observe that they are defined not on reals but on integers (representing finite set of data).

For a discrete function $f : [n] \to [m]$ we define the *rough derivative* f' as

$$f'(i) = \Delta f(i) = f(i + 1) - f(i), \text{ for all } i \in [n - 1].$$

We say that $f : [n] \to [m]$ has Darboux property if for every $i \in [n-1]$ we have that $f'(i) \in \{-1, 0, 1\}$. Thus for $f : [n] \to [m]$ having rough Darboux property and $i \in [n-1]$ the value $f'(i)$ is that $\alpha_i \in \{-1, 0, 1\}$ which makes $f(i+1) = f(i) + \alpha_i$.

Proposition 2. *A discrete function $f : [n] \to [m]$ is roughly continuous iff f has Darboux property.*

Directly from the definition of the rough derivative for discrete functions, we obtain the following counterpart of the well known theorem of differential calculus (cf. [Pa6]).

Proposition 3. *Let f and g be discrete function with domain $[n]$ and range $[m]$ respectively. Than for $f+g, fg$ and f/g we have*

a)
$$(f+g)'(i) = f'(i) + g'(i),$$

b)
$$(fg)'(i) = f'(i)g(i) + f(i)g'(i) + f'(i)g'(i),$$

c)
$$(f/g)'(i) = \frac{f'(i)g(i) - f(i)g'(i)}{g^2(i) + g(i)g'(i)}.$$

¿From the definition of the rough derivative of discrete function and Proposition 3 we get the following proposition.

Proposition 4. *1) The rough derivative of a constant discrete function is equal to zero.*
2) If $f(i) = i + k$, where k is an integer constant, then $f'(i) = 1$.
3) If $f(i) = ki$, then $f'(i) = k$.
4) If $f(i) = k^i$, then $f'(i) = (k-1)k^i$; for $k = 2$ we have $f'(i) = 2^i$.
5) If $f(i) = i^k$, then $f'(i) = \sum_{j=0}^{k} \binom{k}{j} i^{k-j} - i^k$.
In particular, if $k = 2$ we get $f'(i) = 2i+1$; for $k = 3$ we have $f'(i) = 3i^2+3i+1$, etc.

Higher order derivatives can be also defined in the same manner. In general, k-th rough derivative $f^{(k)}$ of a discrete function f is defined by the following well known formula in the difference calculus

$$f^{(k)}(i) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(i+k-j).$$

The following egxample illustrates application of the above formula.

i	0	1	2	3	4	5
$f(i)$	1	1	3	4	2	1
$f^{(1)}(i)$	0	2	1	-2	-1	
$f^{(2)}(i)$	2	-1	-3	1		
$f^{(3)}(i)$	-3	-2	4			
$f^{(4)}(i)$	1	6				
$f^{(5)}(i)$	5					

Notice that f is a discrete function $f : [n] \to [m]$ defined on $n+1$ points, i.e., on the set $\{0, 1, \ldots, n\}$, and $f^{(k)} : [n-k] \to [m]$ is defined on $n-k+1$ points. Thus each discrete function $f : [n] \to [m]$ has at most derivatives up to the n-th order.
Consequently each discrete function $f : [n] \to [m]$ is uniquely defined by the set of the following initial conditions $f^{(n)}(0), f^{(n-1)}(0), \ldots, f^{(1)}(0), f^{(0)}(0)$, where $f^{(0)}(0) = f(0)$.

Some important properties of real functions are not valid for discrete functions, as shown by the following two propositions.

Proposition 5. *Assume that a discrete function $f : [n] \to [m]$ has a maximum (minimum) at $i \in (n)$, where $(n) = \{1, 2, \ldots, n-1\}$. Then not necessarily $f'(i) = 0$.*

Rolle's theorem does not hold for discrete functions, as shown by the proposition below.

Proposition 6. *Let $f : [n] \to [m]$ be a discrete, function, such that $f(0) = f(n) = 0$. Then not necessarily there exists $i \in (n)$ such that $f'(i) = 0$.*

We say that a discrete function f is *roughly smooth* if its first rough derivative is roughly continuous. It can be easily seen that for roughly smooth functions the above two propositions are valid, provided that they are slightly modified. Detailed discussion of this problem is left to the reader.

Next we define integration of discrete functions.

Let $f : [n] \to [m]$ be a discrete function. By a *rough integral* of f we mean the function

$$\int_{j=0}^{i} f(j)\Delta(j) = \sum_{j=0}^{i} f(j)\Delta(j)$$

where $\Delta(j) = (j+1) - j = 1$.

The following important property holds.

Proposition 7.

$$\int_{j=0}^{i} f'(j)\Delta(j) = f(i) + k$$

where k is an integer constant.

In other words

$$f(i) = f(0) + \sum_{j=0}^{i-1} f'(j)$$

or in recursive form

$$f(i+1) = f(i) + f'(i)$$

with the initial condition

$$f(0) = k.$$

This proposition can be used for solving rough differential equations, and will be discused in the next section.

The reader is advised to compare the concept of the rough derivative and the rough integral with corresponding concepts considered in [Bo1].

5 Rough Differential Equations

Starting from the notion of a rough derivative for discrete functions one can define a concept of differential equation for discrete functions, called in what follows a rough differential equation [Ob2] (see also [Bol]). Rough differential equation, together with initial condition can be solved inductively by employing Proposition 7, which gives the relationship between initial condition, rough derivative and the solution.

Ordinary 1-st order differential equation is shown below

$$f'(x) = \Phi(x, f(x))$$

where Φ is a real valued function on the Cartesian product of reals.

Similarly one can define a *rough differential equation*, for discrete functions as

$$(*) \qquad f'(i) = \Phi(i, f(i))$$

where Φ is an integer valued function defined on the Cartesian product $[n] \times [m]$.

Because $f'(i) = f(i+1) - f(i)$, the rough differential equation can be presented as

$$f(i+1) = \Phi(i, f(i)) + f(i)$$

which together with an initial condition

$$f(0) = j_0,\ j_0 \in [m]$$

defines uniquely the solution of the rough differential equation $(*)$.

Example 1. Consider a very simple rough differential equation given by the formula

$$(**) \qquad f'(i) = 4i + 1$$

with the initial condition $f(0) = 2$.

By employing Proposition 3 one can easily show that the solution of this equation has the form

$$f(i) = f(0) + 2i^2 - i$$

We can also solve this equation by using Proposition 7. Suppose we are given the rough differential equation $(**)$ in tabular form, and we do not know its analytical presentation. In this case, by Proposition 7 we have

$$f(i+1) = f(i) + f'(i)$$

with $f(0) = 2$, which yields

$$f(0) = 2$$

$$f(1) = f(0) + f'(0) = 3$$

$$f(2) = f(1) + f'(1) = 8$$

$$f(3) = f(2) + f'(2) = 17$$

$$f(4) = f(3) + f'(3) = 30$$

$$f(5) = f(4) + f'(4) = 47$$

etc.

Thus we have two ways of solving rough differential equations. The first one is similar to that used in analysis, and it boils down to symbolic manipulation on formulas, whereas the second is suitable to functions presented in tabular form.

Ideas presented in this section can be easily extended for two-dimensional case (cf. [Gr1, Wa1]).

6 Conclusion

In this paper we have defined and investigated notions of rough (approximate) continuity, rough derivatives, rough integrals and rough differential equations for discrete functions, i.e., functions defined and valued on the set of integers. We have shown that the introduced concepts mirror some basic properties of calculus, and that discrete functions display properties similar to those of real functions, however this is not always the case.

However, it should be noted that the porposed approach essentialy differs from numerical methods because: firstly, our domains are finite hence we do not consider method convergence typical to numerical methods; secondly, rough differential equations should be derived from finite date sets in contrast to numerical methods obtained from given differential equations.

Many problems connected with the proposed approach still remain open. We did not cover much of material needed a serious consideration in connection with "rough (approximate) calculus". Nevertheless we hope that some fundamental notions have been clarified and sound foundations for further research and applications have been laid down.

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References

[Bo1] Boole, G.: (1860). Treatise on the Calculus of Finite Differences. J. F. Moulton (ed.), Third Edition, G. E. Stechert and Co., New York, Printed in 1946 by the Murray Printing Company, Cambridge, Massachusets

[Bo2] Boxer, L.: Digitally Continuous Functions. Pattern Recognition Letters **15** (1994) 833–839

[Ch1] Chen, Li: The Necessary and Sufficient Condition and the Efficient Algorithms for Gradually Varied Fill. Chinese Science Bulletin **35/10** (1990) 870–873

[CJ1] Courant, R., John, F.: Introduction to Calculus and Analysis, 1, Interscience Publishers, A Division of John Wiley and Sons, Inc. New York, London, Sydney (1965)

[CS1] Chuaqui, R. Suppes, P.: Free-variable Axiomatic Foundations of Infinitesemal Analysis: a Fragment with finitary Consistency Proof. Journal of Symbolic Logic **60/1** (1995) 122–159

[Gr1] Grodzki, Z.: Rough Two-Dimentional Calculus. (manustript) (1997)

[Hs1] Hsu, C. S.: Cell-to-cell Mapping - A Metheod of Global Analysis for Non-linear Systems. Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo (1987)

[Ku1] Kuipers, B.: Qualitative Simulation. Artificial Intelligence **29** (1986) 289–338

[My1] Mycielski, J.: Analysis without Actual Infinity. Journal of Symbolic Logic **46** (1981) 625–633

[NA1] Nakamura, A., Aizawa, K.: Digital Circles, Computer Vision. Garphics and Image Processing **29** (1984) 242–255

[NA2] Nakamura, A., Aizawa, K.: Digital Images and Geometric Pictures, Digital Circles. Computer Vision, Garphics and Image Processing **30** (1985) 107–120

[NA3] Nakamura, A., Aizawa, K.: On the Recognition of Digital Pictures. IEEE Transaction on Pattern Analysis and Machine Intelligence Vol. PAMI-7 **6** (1985) 708–713

[NR1] Nakamura, A., Rosenfeld, A.: Digital Calculus. Information Sciences **98** (1997) 63–98

[Ob1] Obtułowicz, A.: Some Remarks on Rough Real Functions. ICS WUT Report 9/95 (1995)

[Ob2] Obtułowicz, A.: Differential Equations for Discrete Functions. ICS WUT Report 28/95 (1995)

[PT1] Papa, M., Tai, H. M., Shenoi, S.: (1994) Cell Mappings and Fuzzy Controller Design. Proceedings JCIS'94 (1994) 361

[Pa1] Pawlak, Z.: Rough Functions. Bull. PAS, Tech. Ser. **35/5-6** (1987) 249–251

[Pa2] Pawlak, Z.: Rough Sets, Rough Real Functions. ICS WUT Report 50/94 (1994)

[Pa3] Pawlak, Z.: Rough Real Functions and Rough Controllers. ICS WUT Report 1/95 (1994)

[Pa4] Pawlak, Z.: On Some Issues Connected with Roughly Continuous Functions. ICS WUT Report 21/95 (1995)

[Pa5] Pawlak, Z.: On Rough Dervatives, Rough Integrals and Rough Differential Equations. ICS WUT Report 41/95 (1995)

[Pa6] Pawlak, Z.: Rough Calculus. ICS WUT Raport, 58/95, also in: Proc. 2nd Annu. Joint Conf. on Information Sci. (1995) 34–35 (abstract)

[RS1] Rizk, Ch. G., Smith, S. M.: Automatic Design and Analysis of Fuzzy Logic Controllers Using Cell Mapping and Least Mean Square Algorithm. Proceedings JCIS'95 (1995) 328–431

[Ro1] Robinson, A.: Non-Standard Analysis, North-Holland Publishing Company (1970)

[Ro2] Rosenfeld, A.: Continous Functions on Digital Pictures. Pattern Rseognition Letters **4** (1986) 177–184

[Ro3] Rosenfeld, A.: Picture Languages. New York, Academic Press (1979)

[SC1] Smith, S. M., Comer, D. J., Nokleby, B.: A Computational Approach to Fuzzy Logic Controllers Design and Analysis Using Cell State Space Methods. In: A. Kandel, G. Langoltz (eds.), Fuzzy Control Systems, CRC Press (1994) 397–428

[Wa1] Warmus, M.: Calculus of Approximations. Bull. Acad, Pol. Sci., Cl. III **4(5)** (1956) 253–259

[Wa2] Warmus, M.: Approximations and Inequilities in the Calculus of Approximations: Classifications of Approximate Numbers. Bull. Acad, Pol. Sci., Cl. III **9(4)** (1961) 241–245

[We1] Werthner, H.: Qualitative Reasoning – Modeling and the Generation of Behavior. Springer-Verlag, Wien, New York (1994)