ADVANCES IN THE DEMPSTER-SHAFER THEORY OF EVIDENCE

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12 Rough membership functions

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Abstract: A variety of numerical approaches for reasoning with uncertainty have been investigated in the literature. We propose *rough membership functions* (or *rm-functions*, for short) as a basis for such a reasoning. These functions have values in the interval [0, 1] of the real numbers and they are computable on the basis of the observable information about the objects rather than on the basis of the objects themselves. We investigate properties of the *rm*-functions. In particular we show that our approach is intensional with respect to the class of all information systems (Pawlak, 1991). As a consequence we point out some differences between the *rm*-functions and the fuzzy membership functions (Zadeh, 1965); the *rm*-function values for $X \cup Y (X \cap Y)$ cannot be computed in general by applying the operation max (min) to the *rm*-function values for X and Y. We propose the algorithm for computing the *rm*-functions for the sets from a given field of sets.

Keywords: reasoning with incomplete information, rough sets, fuzzy sets, evidence theory.

1. INTRODUCTION

One of the fundamental problems studied in artificial intelligence is related to the object classification that is the problem of associating a particular object to one of many predefined sets. In studing that problem, our approach is based on the observation that the classification of objects is performed on the basis of the accessible information about them. Objects with the same accessible information will be considered as indiscernible (Pawlak, 1991). Therefore we are faced with the problem of determining whether or not an object belongs to a given set when only some properties (i.e., attribute values) of the object are accessible.

We introduce the concept of a *rough membership function* (*rm*-functions, for short), which allows us to measure the degree with which any object with given attribute values belongs to a given set X. The information about objects is stored in data tables called information systems (Pawlak, 1991). Any *rm*-function μ_X^A is defined relatively to a given information system A and a given set X of objects.

The paper is structured as follows. Section 2 contains a brief discussion of information systems (Pawlak, 1991), information functions (Skowron, 1991b), and rough sets (Pawlak, 1991). In Section 3 we define a partition of boundary regions (Skowron, 1991a) and we present some basic properties of this partition, which we apply later. In Section 4 we define the *rm*-functions and we study their basic properties. In Section 5 we present formulas for computing the *rm*-function values $\mu_{X \cup Y}^A(x)$ and $\mu_{X \cap Y}^A(x)$ from the values $\mu_X^A(x)$ and $\mu_Y^A(x)$ (when it is possible, i.e., when classified objects are not in a particular boundary region) if information encoded in the information system A is accessible. In the construction of those formulas we apply a partition of boundary regions related to X and Y defined in Section 3. One can interpret that result as follows: the computation of *rm*-function values $\mu_{X \cup Y}^A(x)$ and $\mu_{X \cap Y}^A(x)$ (if one excludes a particular boundary region!) is *extensional* under the condition that the information system is fixed.

We also show, in Section 5, that our approach is *intensional* with respect to the set of all information systems (with a universe including sets X and Y); namely it is not possible, in general, to compute the *rm*-function values $\mu_{X \cup Y}^A(x)$ and $\mu_{X \cap Y}^A(x)$ from the values $\mu_X^A(x)$ and $\mu_Y^A(x)$ when information about A is not accessible (Theorem 3). Also in Section 5, we specify the maximal classes of information systems such that the computation of *rm*-function values for union and intersection is extensional when related to those classes, and is defined by the operations *min* and *max* as in the fuzzy set approach (Zadeh, 1965; Duois and Prade, 1980), that is, the values $\mu_{X \cup Y}^A(x)$ and $\mu_{X \cap Y}^A(x)$ are obtained by applying the operation *min* and the operation *max* to the values $\mu_X^A(x)$ and $\mu_Y^A(x)$, respectively (if A belongs to those maximal classes).

In Section 6 we present an algorithm for computing the *rm*-function values $\mu_X^A(x)$ for $x \in X$, where X is any set generated by the set theoretical operations $\cup$, $\cap$, - from a given family of finite sets.

2. INFORMATION SYSTEMS AND ROUGH SETS

Information systems (sometimes called data tables, attribute-value systems, condition-action tables etc.) are used for representing knowledge. The information system notion presented here is due to Pawlak (1991) and was investigated by several researchers (see the references in Pawlak, 1991).

Rough sets have been introduced as a tool to deal with inexact, uncertain, or vague knowledge in artificial intelligence applications as, for example, knowledge-based systems in medicine, natural language processing, pattern recognition, decision systems, approximate reasoning. Rough sets have been intensively studied since 1982 and many practical applications based on the theory of rough sets have already been implemented.

In this section we present some basic notions related to information systems and rough sets that will be necessary for understanding our results.

An information system is a pair $\mathbf{A} = (U, A)$, where

U – a nonempty, finite set called the *universe* and
A – a nonempty, finite set of *attributes*, i.e.,
$a : U \to V_a$ for $a \in A$,
where V_a is called the *value set* of a.

With every subset of attributes $B \subseteq A$ we associate a binary relation $IND(B)$, called *B-indiscernibility relation*, and defined as:

$$IND(B) = \{(x, y) \in U^2 : \text{for every } a \in B,\ a(x) = a(y)\}$$

By $[x]_{IND(B)}$ or $[x]_B$ we denote the equivalence class of the equivalence relation $IND(B)$ generated by x, i.e., the set $\{y \in U : x IND(B) y\}$.

We have that

$$IND(B) = \bigcap_{a \in B} IND(a)$$

If $x IND(B) y$, then we say that the objects x and y are indiscernible with respect to attributes from B. In other words, we cannot distinguish x from y in terms of attributes in B.

Some subsets of objects in an information system cannot be expressed exactly in terms of the available attributes; they can be only roughly defined.

If $\mathbf{A} = (U, A)$ is an information system, $B \subseteq A$ and $X \subseteq U$, then the sets

$$\underline{B}X = \{x \in U : [x]_B \subseteq X\} \text{ and } \bar{B}X = \{x \in X : [x]_B \cap X \neq \emptyset\}$$

are called the *B-lower* and the *B-upper approximation of* X in $\mathbf{A}$, respectively.

The set $BN_B(X) = \bar{B}X - \underline{B}X$ will be called the *B-boundary of* X.

Clearly, $\underline{B}X$ is the set of all elements of U, which can be with certainty classified as elements of X with respect to the values of attributes from B; and $\bar{B}X$ is the set of those elements of U that can be possibly classified as elements of X with respect of the values of the attributes from B; finally, $BN_B(X)$ is the set of elements that can be classified neither in X nor in $-X$ on the basis of the values of attributes from B.

A set X is said to be *B-definable* if $\bar{B}X = \underline{B}X$. It is easy to observe that $\underline{B}X$ is the greatest B-definable set contained in X, whereas $\bar{B}X$ is the smallest B-definable set containing X. One can observe that a set is B-definable iff it is the union of some equivalence classes of the indiscernibility relation $IND(B)$.

By $\mathbf{P}(X)$ we denote the powerset of X.

Every information system $\mathbf{A} = (U, A)$ determines an *information function*

$$Inf_{\mathbf{A}} : U \rightarrow \mathbf{P}(A \times \bigcup_{a \in A} V_a)$$

defined as

$$Inf_{\mathbf{A}}(x) = \{(a, a(x)) : a \in A\}$$

Hence, $xIND(A)y$ iff $Inf_{\mathbf{A}}(x) = Inf_{\mathbf{A}}(y)$.

We restrict our considerations in the paper to the information functions related to information systems but our results can be extended to the case of more general information functions (Skowron, 1991b). One can consider as *information function* an arbitrary function f defined on the set of objects U with values in some computable set C.

For example, one may take as the set U of objects the set $Tot_{\mathbf{A}}$ of total elements in the Scott information system $\mathbf{A}$ (Scott, 1982) and as

C a computable (an accessible) subset of the set D of sentences in A. The information function f related to C can be defined as follows:

$$f(x) = x \cap C \quad \text{for} \quad x \in Tot_{\mathbf{A}}$$

Every such general information function f defines the indiscernibility relation $IND(f) \subseteq U \times U$ as follows:

$$x\, IND(f)\, y \quad \text{iff} \quad f(x) = f(y)$$

3. AN APPROXIMATION OF CLASSIFICATIONS

In this section we introduce and study the notion of approximation of classification. It was preliminarily considered in Skowron (1991a) and Skowron and Grzymała-Busse (1991). The main idea is based on observation that it is possible to classify boundary regions corresponding to sets from a given classification, that is, a partition of object universe.

Let $\mathbf{A} = (U, A)$ be an information system and let $\mathbf{X}$ and $\mathbf{Z}$ be families of subsets of U such that $\mathbf{Z} \subseteq X$ and $|\mathbf{Z}| > 1$, where $|\mathbf{Z}|$ denotes the cardinality of $\mathbf{Z}$. The set

$$\bigcap_{X \in \mathbf{Z}} BN_A(X) \cap \bigcap_{X \in \mathbf{X} - \mathbf{Z}} (U - BN_A(X))$$

is said to be the $\mathbf{Z}$-boundary region defined by $\mathbf{X}$ and $\mathbf{A}$ and is denoted by $Bd_{\mathbf{A}}(\mathbf{Z}, \mathbf{X})$.

By $CLASS\_APPR_{\mathbf{A}}(\mathbf{X})$ we denote the set family

$$\{\underline{A}X : X \in \mathbf{X}\} \cup \{Bd_{\mathbf{A}}(\mathbf{Z}, \mathbf{X}) : \mathbf{Z} \subseteq \mathbf{X} \text{ and } |\mathbf{Z}| > 1\}$$

From the above definitions we get the following proposition (Skowron, 1991):

Proposition 1. Let $\mathbf{A} = (U, A)$ be an information system and let $\mathbf{X}$ be a family of pairwise disjoint subsets of U such that $\bigcup \mathbf{X} = U$. Let $\mathbf{Z} \subseteq \mathbf{X}$ and $|\mathbf{Z}| > 1$. Then

(i) The set $Bd_{\mathbf{A}}(\mathbf{Z}, \mathbf{X})$ is definable in A;
(ii) $CLASS\_APPR_{\mathbf{A}}(\mathbf{X}) - \{\emptyset\}$ is a partition of U;
(iii) If $x \in Bd_{\mathbf{A}}(\mathbf{Z}, \mathbf{X})$ then $[x]_A \subseteq \bigcup \mathbf{Z}$;
(iv) If $x \in Bd_{\mathbf{A}}(\mathbf{Z}, \mathbf{X})$ then for every $X \in \mathbf{X}$ the following equivalence is true:
$[x]_A \cap X \neq \emptyset \quad \text{iff} \quad X \in \mathbf{Z}$;

(v) The following equality holds:

$$\underline{A}(\bigcup \mathbf{Y}) = \bigcup_{X \in \mathbf{Y}} \underline{A}X \cup \bigcup_{|\mathbf{Z}|>1, \mathbf{Z} \subseteq \mathbf{Y}} Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X}), \text{ where } \mathbf{Y} \subseteq \mathbf{X}.$$

Proof. (i) If $x \in Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X})$ then $x \in BN_{\mathrm{A}}(X)$ for any $X \in \mathbf{Z}$ and $x \in U - BN_{\mathrm{A}}(X)$ for any $X \in \mathbf{X} - \mathbf{Z}$. From the definability in A of sets $BN_{\mathrm{A}}(X)$ and $U - BN_{\mathrm{A}}(X)$ for $X \subseteq U$ we have $[x]_A \subseteq BN_{\mathrm{A}}(X)$ for any $X \in \mathbf{Z}$ and $[x]_A \subseteq U - BN_{\mathrm{A}}(X)$ for any $X \in \mathbf{X} - \mathbf{Z}$. Hence $[x]_A \subseteq Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X})$. We proved that $Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X}) \subseteq \underline{A}(Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X}))$. Since $Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X}) \supseteq \underline{A}(Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X})$ we get $Bd_A(\mathbf{Z}, \mathbf{X}) = \underline{A}(Bd_A(\mathbf{Z}, \mathbf{X}))$.

(ii) It is easy to observe that $CLASS\_APPR_{\mathrm{A}}(\mathbf{X})$ is a family of pairwise disjoint sets. We prove that $\bigcup CLASS\_APPR_{\mathrm{A}}(\mathbf{X}) = U$. If $x \in U$ then $x \in X$ for some $X \in \mathbf{X}$. If $x \in \underline{A}X$ then $x \in CLASS\_APPR_{\mathrm{A}}(\mathbf{X})$, otherwise $x \in \bar{A}X - \underline{A}X$. In the latter case let $\mathbf{Z}_x = \{X \in \mathbf{X} : [x]_A \cap X \neq \emptyset\}$. Then we have $|\mathbf{Z}_x| > 1$ and $x \in Bd_{\mathrm{A}}(\mathbf{Z}_x, \mathbf{X})$.

(iii) Let $x \in Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X})$. Suppose that $y \notin \bigcup \mathbf{Z}$ for some $y \in [x]_A$. Since $\bigcup \mathbf{X} = U$, we have $y \in \bigcup \mathbf{X} - \bigcup \mathbf{Z}$. Hence $y \in X_o$ for some $X_o \in \mathbf{X} - \mathbf{Z}$. In the consequence $X_o \cap [y]_A = X_o \cap [x]_A \neq \emptyset$. If $x \in Bd_A(\mathbf{Z},\mathbf{X})$ then $x \in U - BN_A(X)$ for $X \in \mathbf{Z} - \mathbf{Z}$. Since $U - BN_A(X)$ is definable in A we obtain $[x]_A \subseteq U - BN_A(X) = (U - \bar{A}X) \cup \underline{A}X$. Hence $[x]_A \subseteq \underline{A}X_o$ or $[x]_A \subseteq U - \bar{A}X_o$. Since $X_o \cap [x]_A \neq \emptyset$ we get

(*) $[x]_A \subseteq \underline{A}X_o$

From the assumption $x \in Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X})$ we have also $x \in BN_A(X)$ for any $X \in \mathbf{Z}$, so

(**) $[x]_A \cap X \neq \emptyset$ for any $X \in \mathbf{Z}$

From (*) and (**) we would have $X \cap X_o \neq \emptyset$ for any $X \in \mathbf{Z}$ but this contradicts the assumption that $\mathbf{X}$ is a family of pairwise disjoint sets.

(iv) Let $x \in Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X})$ and $X \in \mathbf{X}$. Suppose that $[x]_A \cap X \neq \emptyset$, i.e., $x \in BN_A(X)$. Hence from the definition of $Bd_A(\mathbf{Z}, \mathbf{X})$ we have $X \in \mathbf{Z}$. If $X \in \mathbf{Z}$ then we have $x \in BN_A(X)$. Hence $[x]_A \cap X \neq \emptyset$.

(v) ($\subseteq$) If $x \in Bd_{\mathrm{A}}(\mathbf{Z},\mathbf{X})$ we have from (iii) that $x \in \underline{A} \bigcup \mathbf{Z} \subseteq \underline{A} \bigcup \mathbf{Y}$. We also have $\underline{A}X \subseteq \underline{A} \bigcup \mathbf{Y}$ for any $X \in \mathbf{Y}$.

($\supseteq$) Let $x \in \underline{A} \bigcup \mathbf{Y}$, i.e., $[x] \subseteq \bigcup \mathbf{Y}$. If $x \notin \underline{A}X$ for a certain $X \in \mathbf{Y}$ then let $\mathbf{Z}_x = \{X \in \mathbf{Y} : [x]_A \cap X \neq \emptyset\}$. Hence $|\mathbf{Z}_x| > 1$ and $[x]_A \subseteq \bigcup \mathbf{Z}_x$. Thus, we have $x \in Bd_{\mathrm{A}}(\mathbf{Z}_x, \mathbf{X})$.

4. ROUGH MEMBERSHIP FUNCTIONS: DEFINITION AND BASIC PROPERTIES

One of the fundamental notions of set theory is the membership relation, usually denoted by $\in$. When one considers subsets of a given universe it is possible to apply the characteristic functions for expressing the fact whether or not a given element belongs to a given set. We discuss the case when only partial information about objects is accessible. In this section we show it is possible to extend characteristic function notion to that case.

Let $\mathbf{A} = (U, A)$ be an information system and let $\emptyset \neq X \subseteq U$. The *rough* $\mathbf{A}$*-membership function of the set* X (or *rm*-function, for short) denoted by $\mu_X^{\mathbf{A}}$, is defined as follows:

$$\mu_X^{\mathbf{A}}(x) = \frac{|[x]_A \cap X|}{|[x]_A|}, \text{ for } x \in U, \ \mu_\emptyset \equiv 0.$$

The above definition is illustrated in Figure 12.1.

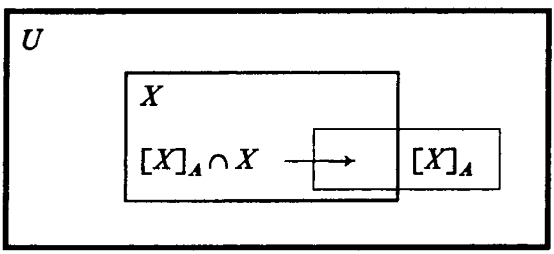

Figure 12.1

One can observe a similarity between the expression on the right-hand side of the above definition and that expression used to define the conditional probability.

From the definition of $\mu_X^{\mathbf{A}}$ we have the following proposition characterizing some basic properties of *rm*-functions.

Proposition 2. Let $\mathbf{A} = (U, A)$ be an information system and let $X, Y \subseteq U$.

The rm-**functions have the following properties:**

(i) $\mu_X^{\mathbf{A}}(x) = 1$ iff $x \in \underline{A}X$;
(ii) $\mu_X^{\mathbf{A}}(x) = 0$ iff $x \in U - \bar{A}X$;

(iii) $0 < \mu_X^A(x) < 1$ iff $x \in BN_A(X)$;
(iv) If $IND(A) = \{(x,x): x \in U\}$ then μ_X^A is the characteristic function of X;
(v) If $xIND(A)y$ then $\mu_X^A(x) = \mu_X^A(y)$;
(vi) $\mu_{U-X}^A(x) = 1 - \mu_X^A(x)$ for any $x \in X$;
(vii) $\mu_{X \cup Y}^A(x) \geqslant \max(\mu_X^A(x), \mu_Y^A(x))$ for any $x \in U$;
(viii) $\mu_{X \cap Y}^A(x) \leqslant \min(\mu_X^A(x), \mu_Y^A(x))$ for any $x \in U$;
(ix) If **X** is a family of pairwise disjoint subsets of U then

$$\mu_{\cup \mathbf{X}}^A(x) = \sum_{X \in \mathbf{X}} \mu_X^A(x) \text{ for any } x \in U.$$

Proof.

(i) We have $x \in \underline{A}X$ iff $[x]_A \subseteq X$ iff $\mu_X^A(x) = 1$.
(ii) We have $x \in U - \bar{A}X$ iff $[x]_A \cap X = \emptyset$ iff $\mu_X^A(x) = 0$.
(iii) We have $x \in BN_A(X)$ iff ($[x]_A \cap X \neq \emptyset$ and $[x]_A \cap (U - X) \neq \emptyset$) iff ($\mu_X^A(x) > 0$ and $\mu_X^A(x) < 1$).
(iv) If $IND(A) = \{(x,x): x \in U\}$ then $|[x]_A| = 1$ for any $x \in X$. Moreover $|[x]_A \cap X| = 1$ if $x \in X$ and $|[x]_A \cap X| = 0$ if $x \in U - X$.
(v) Since $[x]_A = [y]_A$ we have $\mu_X^A(x) = \mu_X^A(y)$.
(vi) $\mu_{U-X}^A(x) = \dfrac{|[x]_A \cap (U - X)|}{|[x]_A|} = 1 - \dfrac{|[x]_A \cap X|}{|[x]_A|} = 1 - \mu_X^A(x)$.
(vii) $\mu_{X \cup Y}^A(x) = \dfrac{|[x]_A \cap (X \cup Y)|}{|[x]_A|} \geqslant \dfrac{|[x]_A \cap X)|}{|[x]_A|} = \mu_X^A(x)$. In a similar way one can obtain $\mu_{X \cup Y}^A(x) \geqslant \mu_Y^A(x)$.
(viii) Proof runs as in case (vi).
(ix) $\mu_{\cup \mathbf{X}}^A(x) = \dfrac{|[x]_A \cap \bigcup \mathbf{X}|}{|[x]_A|} = \dfrac{|\bigcup\{[x]_A \cap X : X \in \mathbf{X}|}{|[x]_A|} = \sum_{X \in \mathbf{X}} \mu_X^A(x)$

The last equality follows from the assumption that **X** is a family of pairwise disjoint sets.

The set $\{Inf_A(x): x \in U\}$ is called the A-information set and it is denoted by INF(A). For every $X \subseteq U$ we define the rough A-information function, denoted by $\hat{\mu}_X^A$, as:

$$\hat{\mu}_X^A(u) = \mu_X^A(x), \text{ where } u \in \text{INF(A) and } Inf_A(x) = u$$

The correctness of the above definition follows from (v) in Proposition 1.

If $\mathbf{A} = (U, A)$ is an information system then we define rough A-inclusion of subsets of U in the standard way:

$$X \leqslant_{\mathbf{A}} Y \quad \text{iff} \quad \mu_X^{\mathbf{A}}(x) \leqslant \mu_Y^{\mathbf{A}}(x) \text{ for any } x \in U$$

Proposition 3. If $X \leqslant_{\mathbf{A}} Y$ then $\underline{A}X \subseteq \underline{A}Y$ and $\bar{A}X \subseteq \bar{A}Y$.

Proof. Follows from Proposition 2 (see (i) and (ii)).

The above definition of the rough A-inclusion is not equivalent to the one in Pawlak (1991). Indeed, in Pawlak (1991) the reverse implication to that formulated in Proposition 2 is not valid.

One can show that they are equivalent for any information system A only if $\bar{A}X \subseteq \underline{A}Y$. This is a consequence of our definition taking into account some additional information about objects from the boundary regions.

5. ROUGH MEMBERSHIP FUNCTIONS FOR UNION AND INTERSECTION

Now we present some results obtained as a consequence of our assumption that objects are observable by means of partial information about them represented by attribute values. In this section we prove that the inequalities in (vii) and (viii) of Proposition 1 cannot be in general substituted by the equalities.

We also prove that for some boundary regions it is not possible to compute the values of the *rm*-functions for union $X \cup Y$ and intersection $X \cap Y$ knowing the values of *rm*-functions for X and Y only (if information about information systems is not accessible and does not hold some special relations between sets X and Y). These results show that the assumptions about properties of the fuzzy membership functions (Dubois and Prade, 1980, p. 11) related to the union and intersection should be modified if one would like to take into account that objects are classified on the basis of a partial information about them. We present also the necessary and sufficient conditions for the following equalities (which are the ones used in fuzzy set theory) to be true for any $x \in U$.

$$\mu_{X \cup Y}^{\mathbf{A}}(x) = \max(\mu_X^{\mathbf{A}}(x), \mu_Y^{\mathbf{A}}(x))$$

$$\mu_{X \cap Y}^{\mathbf{A}}(x) = \min(\mu_X^{\mathbf{A}}(x), \mu_Y^{\mathbf{A}}(x))$$

These conditions are expressed by means of the boundary regions of a partition of U defined by sets X and Y or by means of some relationships that should hold for the sets X and Y. In particular we show that the above equalities are true for arbitrary information system **A** iff $X \subseteq Y$ or $Y \subseteq X$.

First we prove the following two lemmas.

Lemma 1. Let $\mathbf{A} = (U, A)$ be an information system, $X, Y \subseteq U$ and $\mathbf{X} = \{X \cap Y, X \cap -Y, -X \cap Y, -X \cap -Y\}$.

If $x \in U - Bd_A(\mathbf{X}, \mathbf{X})$, then

$\mu^{\mathbf{A}}_{X \cap Y}(x) =$

if $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) \cup Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X})$

then 0

else if $x \in Bd_{\mathbf{A}}(\{X \cap Y, X \cap -Y, -X \cap Y)$ then $\mu^{\mathbf{A}}_X(x) + \mu^{\mathbf{A}}_Y(x) - 1$

else $\min(\mu^{\mathbf{A}}_X(x), \mu^{\mathbf{A}}_Y(x))$

Proof. In the proof we apply property (iii) from Proposition 1. Let $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\} \cup Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}$. Hence $[x]_A \subseteq (X \cap -Y) \cup (-X \cap Y) \cup (-X \cap -Y)$, so $[x]_A \cap (X \cap Y) = \emptyset$ and $\mu^{\mathbf{A}}_{X \cap Y}(x) = 0$.

If $x \in Bd_{\mathbf{A}}(\{X \cap Y, X \cap -Y, -X \cap Y\}, \mathbf{X})$ then $[x]_A \subseteq (X \cap Y \cup X \cap -Y \cup -X \cap Y)$. Hence, $[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y)$, so $[x]_A = [x]_A \cap X \cup [x]_A \cap Y$.

We obtain $|[x]_A| = |[x]_A \cap X| + |[x]_A \cap Y| - |[x]_A \cap (X \cap Y)|$. Hence $\mu^{\mathbf{A}}_{X \cap Y}(x) = \mu^{\mathbf{A}}_X(x) + \mu^{\mathbf{A}}_Y(x) - 1$.

If $x \in \underline{A}(X \cap Y)$, then $[x]_A \subseteq X \cap Y$. Hence $\mu^{\mathbf{A}}_{X \cap Y}(x) = 1$. We have also $[x]_A \subseteq X$ and $[x]_A \subseteq Y$ because $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. Hence $\mu^{\mathbf{A}}_X(x) = \mu^{\mathbf{A}}_Y(x) = 1$.

If $x \in \underline{A}(X \cap -Y)$ then $[x]_A \subseteq X \cap -Y$. Hence $[x]_A \cap (X \cap Y) = \emptyset$ and $[x]_A \cap Y \subseteq (X \cap -Y) \cap Y = \emptyset$, so $\mu^{\mathbf{A}}_{X \cap Y}(x) = \min(\mu^{\mathbf{A}}_X(x), \mu^{\mathbf{A}}_Y(x))$.

If $x \in \underline{A}(-X \cap Y)$ the proof is analogous to the latter case.

If $x \in \underline{A}(-X \cap -Y)$ we obtain $\mu^{\mathbf{A}}_{X \cap Y}(x) = \mu^{\mathbf{A}}_X(x) = \mu^{\mathbf{A}}_Y(x) = 0$.

If $x \in Bd_{\mathbf{A}}(\{X \cap Y, X \cap -Y\}, \mathbf{X})$ we have $[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y)$. Hence $[x]_A \cap (X \cap Y) = [x]_A \cap Y$ and $[x]_A = [x]_A \cap X \subseteq X$. Hence $\mu^{\mathbf{A}}_{X \cap Y}(x) = \mu^{\mathbf{A}}_Y(x) \leqslant \mu^{\mathbf{A}}_X(x) = 1$.

If $x \in Bd_{\mathbf{A}}(\{X \cap Y, -X \cap Y\}, \mathbf{X})$ the proof is analogous to the latter case.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap -Y), \mathbf{X})$ one can calculate $\mu^{\mathbf{A}}_{X \cap Y}(x) = \mu^{\mathbf{A}}_Y(x) = 0 \leqslant \mu^{\mathbf{A}}_X(x)$. Similarly, in the case when $x \in Bd_{\mathbf{A}}(\{-X \cap Y, -X \cap -Y\}, \mathbf{X})$ one can calculate that $\mu^{\mathbf{A}}_{X \cap Y}(x) = \mu^{\mathbf{A}}_X(x) = 0 \leqslant \mu^{\mathbf{A}}_Y(x)$.

If $x \in Bd_{\mathbf{A}}(\{X \cap Y, -X \cap -Y\}, \mathbf{X})$ we have $\mu^{\mathbf{A}}_{X \cap Y}(x) = \mu^{\mathbf{A}}_{X}(x) = \mu^{\mathbf{A}}_{Y}(x)$.

Lemma 2. Let $\mathbf{A} = (U, A)$ be an information system, $X, Y \subseteq U$ and $\mathbf{X} = \{X \cap Y, X \cap -Y, -X \cap Y, -X \cap -Y\}$. If $x \in U - Bd_{\mathbf{A}}(\mathbf{X}, \mathbf{X})$ then

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$\mu^{\mathbf{A}}_{X \cup Y}(x) =$
if $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) \cup Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X})$
then $\mu^{\mathbf{A}}_{X}(x) + \mu^{\mathbf{A}}_{Y}(x)$
else if $x \in Bd_{\mathbf{A}}(\{X \cap Y, X \cap -Y, -X \cap Y\}, \mathbf{X})$ **then** 1
else $\max(\mu^{\mathbf{A}}_{X}(x), \mu^{\mathbf{A}}_{Y}(x))$

Proof. In the proof we apply property (iii) from Proposition 1.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}$ then $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y)$. Hence $[x]_A \cap X = [x]_A \cap X \cap -Y$, $[x]_A \cap Y = [x]_A \cap -X \cap Y$.

Since $[x]_A \cap (X \cup Y) = ([x]_A \cap X) \cup ([x]_A \cap Y)$ and $([x]_A \cap X) \cap ([x]_A \cap Y) = [x]_A \cap X \cap -Y \cap -X \cap Y = \emptyset$, we get $\mu^{\mathbf{A}}_{X \cup Y}(x) = \mu^{\mathbf{A}}_{X}(x) + \mu^{\mathbf{A}}_{Y}(x)$.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X})$, then $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (-Y \cap -Y)$.

Since $[x]_A \cap (X \cup Y) = ([x]_A \cap X) \cup ([x]_A \cap Y)$ and $([x]_A \cap X) \cap ([x]_A \cap Y) = [x]_A \cap X \cap -Y \cap -X \cap Y = \emptyset$, we get $\mu^{\mathbf{A}}_{X \cup Y}(x) = \mu^{\mathbf{A}}_{X}(x) + \mu^{\mathbf{A}}_{Y}(x)$.

If $x \in Bd_{\mathbf{A}}(\{X \cap Y, X \cap -Y, -X \cap Y\}, \mathbf{X})$ then $[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y)$. Hence $[x]_A \cap (X \cup Y) = [x]_A$, so $\mu^{\mathbf{A}}_{X \cup Y}(x) = 1$.

If $x \in \underline{A}(-X \cap -Y)$ then $[x]_A = [x]_A \cap (-X \cap -Y)$. Hence $[x]_A \cap (X \cup Y) = [x]_A \cap X = [x]_A \cap Y = \emptyset$.

If $x \in \underline{A}(X \cap Y)$, then $[x]_A = [x]_A \cap X \cap Y$. Hence $[x]_A \cap (X \cup Y) = [x]_A = [x]_A \cap X = [x]_A \cap Y$.

If $x \in \underline{A}(-X \cap Y)$, then $[x]_A = [x]_A \cap (-X \cap Y)$. Hence $[x]_A \cap (X \cup Y) = [x]_A \cap Y \neq \emptyset$ and $[x]_A \cap X = \emptyset$.

If $x \in \underline{A}(-X \cap Y)$, the proof is analogous as in the latter case.

If $x \in Bd_{\mathbf{A}}(\{X \cap Y, X \cap -Y\}, \mathbf{X})$ then $[x]_A = [x]_A \cap (X \cap Y) \cup [x]_A \cap (X \cap -Y)$. Hence $[x]_A \cap (X \cup Y) = [x]_A \cap X \supseteq [x]_A \cap (X \cap Y) = [x]_A \cap Y$.

If $x \in Bd_{\mathbf{A}}(\{X \cap Y, -X \cap Y\}, \mathbf{X})$, then the proof is analogous as in the latter case.

If $x \in Bd_A(\{X \cap Y, -X \cap -Y\}, \mathbf{X})$, then $\mu^{A}_{X \cup Y}(x) = \mu^{A}_{X}(x) = \mu^{A}_{Y}(x)$.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap -Y\}, \mathbf{X})$, then $\mu^{\mathbf{A}}_{X \cup Y}(x) = \mu^{\mathbf{A}}_{X}(x)$ and $\mu^{\mathbf{A}}_{Y}(x) = 0$.

If $x \in Bd_{A}(\{-X \cap Y, -X \cap -Y\}, \mathbf{X})$, then $\mu^{A}_{X \cup Y}(x) = \mu^{A}_{X}(x)$ and $\mu^{\mathbf{A}}_{Y}(x) = 0$.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, X \cap Y, -X \cap -Y\}, \mathbf{X})$, then $\mu^{\mathbf{A}}_{X \cup Y}(x) = \mu^{\mathbf{A}}_{X}(x) \geqslant \mu^{\mathbf{A}}_{Y}(x)$.

If $x \in Bd_{\mathbf{A}}(\{-X \cap Y, X \cap Y, -X \cap -Y\}, \mathbf{X})$, then $\mu^{\mathbf{A}}_{X \cup Y}(x) = \mu^{\mathbf{A}}_{Y}(x) \geqslant \mu^{\mathbf{A}}_{X}(x)$.

Theorem 1. Let $\mathbf{Z}$ be a (nonempty) class of information systems with the universe including sets X and Y. The following conditions are equivalent:

(i) $\mu^{\mathbf{A}}_{X \cap Y}(x) = \min(\mu^{\mathbf{A}}_{X}(x), \mu^{\mathbf{A}}_{Y}(x))$ for any $x \in U$ and $\mathbf{A} = (U, A) \in \mathbf{Z}$.

(ii) $Bd_{\mathbf{A}}(\mathbf{Y}, \mathbf{X}) = \varnothing$ for any $\mathbf{X} \supseteq \mathbf{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ and $\mathbf{A} = (U, A) \in \mathbf{Z}$, where $\mathbf{X} = \{X \cap Y, -X \cap Y, X \cap -Y, -X \cap -Y\}$.

Proof. (ii) $\rightarrow$ (i) Follows from Lemma 1.

(i) $\rightarrow$ (ii) Suppose that $Bd_{\mathbf{A}}(\mathbf{Y}, \mathbf{X}) \neq \varnothing$ for some $\mathbf{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ and $\mathbf{A} \in \mathbf{Z}$.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) \neq \varnothing$ for some $\mathbf{A} \in \mathbf{Z}$, then $[x]_A \cap (X \cap -Y) \neq \varnothing$ and $[x]_A \cap (-X \cap Y) \neq \varnothing$. Hence $\mu^{\mathbf{A}}_{X}(x) > 0$ and $\mu^{\mathbf{A}}_{Y}(x) > 0$. We also have from Lemma 1 $\mu^{\mathbf{A}}_{X \cap Y}(x) = 0$. Thus we have $\mu^{\mathbf{A}}_{X \cap Y}(x) \neq \min\ (\mu^{\mathbf{A}}_{X}(x), \mu^{\mathbf{A}}_{Y}(x))$, i.e., a contradiction with (i).

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X})$ for some $\mathbf{A} \in \mathbf{Z}$ and $x \in U$ then one can see that it contradicts (i) in the same manner as before.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbf{X}) \neq \varnothing$ for some $\mathbf{A} \in \mathbf{Z}$ then we have $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y)$. Hence $[x]_A \cap X = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (X \cap Y)$ and $[x]_A \cap Y = [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y)$.

Since $[x]_A \cap (X \cap -Y) \neq \varnothing$ and $[x]_A \cap (-X \cap Y) \neq \varnothing$, we would have $\mu^{\mathbf{A}}_{X}(x) > \mu^{\mathbf{A}}_{X \cap Y}(x)$ and $\mu^{\mathbf{A}}_{Y}(x) > \mu^{\mathbf{A}}_{X \cap Y}(x)$ but this contradicts assumption (i).

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbf{X})$ for some $\mathbf{A} \in \mathbf{Z}$, then $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y) \cup [x]_A \cap (-X \cap -Y)$.

Again we would have $[x]_A \cap X = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (X \cap Y)$ and $[x]_A \cap Y = [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y)$.

Since $[x]_A \cap (X \cap -Y) \neq \varnothing$ and $[x]_A \cap (-X \cap Y) \neq \varnothing$, we would have $\mu^{\mathbf{A}}_{X}(x) > \mu^{\mathbf{A}}_{X \cap Y}(x)$ and $\mu^{\mathbf{A}}_{Y}(x) > \mu^{\mathbf{A}}_{X \cap Y}(x)$ but this contradicts assumption (i).

This completes the proof of (i) $\rightarrow$ (ii).

Theorem 2. Let $\mathbf{Z}$ be a (nonempty) class of information systems with the set of objects including sets X and Y. The following conditions are equivalent:

(i) $\mu^{\mathbf{A}}_{X\cup Y}(x) = \max(\mu^{\mathbf{A}}_X(x), \mu^{\mathbf{A}}_Y(x))$ for any $x \in U$ and $\mathbf{A} = (U, A) \in \mathbf{Z}$.

(ii) $Bd_{\mathbf{A}}(\mathbf{Y}, \mathbf{X}) = \emptyset$ for any $\mathbf{X} \supseteq \mathbf{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ and $\mathbf{A} = (U, A) \in \mathbf{Z}$, where $\mathbf{X} = \{X \cap Y, -X \cap Y, X \cap -Y, -X \cap -Y\}$.

Proof. (ii) $\rightarrow$ (i) Follows from Lemma 2.

(i) $\rightarrow$ (ii) Suppose that $Bd_{\mathbf{A}}(\mathbf{Y}, \mathbf{X}) \neq \emptyset$ for some $\mathbf{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ and $\mathbf{A} \in \mathbf{Z}$.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) \neq \emptyset$ for some $\mathbf{A} \in \mathbf{Z}$, then $[x]_A \cap (X \cap -Y) \neq \emptyset$ and $[x]_A \cap (-X \cap Y) \neq \emptyset$. Hence $\mu^{\mathbf{A}}_X(x) > 0$ and $\mu^{\mathbf{A}}_Y(x) > 0$. We have also from Lemma 2 that $\mu^{\mathbf{A}}_{X\cup Y}(x) = \mu^{\mathbf{A}}_X(x) + \mu^{\mathbf{A}}_Y(x)$. This gives $\mu^{\mathbf{A}}_{X\cup Y}(x) > \mu^{\mathbf{A}}_X(x)$ and $\mu^{\mathbf{A}}_{X\cup Y}(x) > \mu^{\mathbf{A}}_Y(x)$, contrary to (i).

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X})$ for some $\mathbf{A} \in \mathbf{Z}$ and $x \in U$ then one can see that it contradicts (i) in the same manner as before.

If $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbf{X}) \neq \emptyset$ for some $\mathbf{A} \in \mathbf{Z}$ then we have $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y)$ and $[x]_A \cap Z \neq \emptyset$ for $Z \in \{X \cap -Y, -X \cap Y, X \cap Y\}$. Hence $|[x]_A| > |[x]_A \cap X|$ and $|[x]_A > |[x]_A \cap Y|$. Thus $\mu^{\mathbf{A}}_X(x) < 1$ and $\mu^{\mathbf{A}}_Y(x) < 1$. However $\mu^{\mathbf{A}}_{X\cup Y}(x) = 1$ from Lemma 2. This contradicts our assumption (i).

Now let us assume that $x \in Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbf{X})$ for some $\mathbf{A} \in \mathbf{Z}$. Then $[x]_A = [x]_A \cap (X \cap -Y) \cup [x]_A \cap (-X \cap Y) \cup [x]_A \cap (X \cap Y) \cup [x]_A(-X \cap -Y)$ and $[x]_A \cap Z \neq \emptyset$ for $Z \in \{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}$. Hence $[x]_A \cap (X \cup Y) = [x]_A \cap X \cup [x]_A \cap (-X \cap Y)$ and $[x]_A \cap (X \cup Y)$ $[x]_A \cap Y \cup [x]_A \cap (X \cap -Y)$.

Consequently $\mu^{\mathbf{A}}_{X\cup Y}(x) > \mu^{\mathbf{A}}_X(x)$ and $\mu^{\mathbf{A}}_{X\cup Y}(x) > \mu^{\mathbf{A}}_Y(x)$. This contradicts our assumption (i), which completes the proof of (i) $\rightarrow$ (ii).

Now we would like to characterize the conditions related to the boundary regions occurring in Theorems 1 and 2.

Lemma 3. Let $\mathbf{Z}$ be a class of information systems with the set of objects including sets X and Y. The following conditions are equivalent for arbitrary $\mathbf{A} = (U, A) \in \mathbf{Z}$:

(i) $Bd_{\mathbf{A}}(\mathbf{Y}, \mathbf{X}) = \emptyset$ for any $\mathbf{X} \supseteq \mathbf{Y} \supseteq \{X \cap -Y, -X \cap Y\}$, where $\mathbf{X} = \{X \cap Y, -X \cap Y, X \cap -Y, -X \cap -Y\}$;

(ii) $\alpha \vee \beta \vee \gamma \vee \delta \vee \varepsilon$ where

$\alpha := (X \subseteq Y \text{ or } Y \subseteq X)$;

$\beta := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y = U \text{ and } X \cap Y = \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) = \emptyset)$;

$\gamma := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y = U \text{ and } X \cap Y = \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) = \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbf{X}) = \emptyset)$;

$\delta := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y \neq U \text{ and } X \cap Y \neq \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) = \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X}) = \emptyset)$;

$\varepsilon := (X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset \text{ and } X \cup Y \neq U \text{ and } X \cap Y \neq \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y\}, \mathbf{X}) = \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X}) = \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbf{X}) = \emptyset \text{ and } Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, X \cap Y, -X \cap -Y\}, \mathbf{X}) = \emptyset)$.

Proof. We have the following equivalencies:

$Bd_A(\{X \cap -Y, -X \cap Y\}) = \emptyset$ iff $X \subseteq Y$ or $Y \subseteq X$ or $X - Y \neq \emptyset$ and $Y - X \neq \emptyset$ and $Bd_A(\{X \cap -Y, -X \cap Y\}, X) \neq \emptyset$;

$Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X}) = \emptyset$ iff $X \subseteq Y$ or $Y \subseteq X$ or $X \cup Y = U$ or $(X - Y \neq \emptyset$ and $Y - X \neq \emptyset$ and $X \cup Y \neq U$ and $Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y\}, \mathbf{X}) = \emptyset)$;

$Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbf{X}) = \emptyset$ iff $X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$ or $(X - Y \neq \emptyset$ and $Y - X \neq \emptyset$ and $X \cap Y \neq \emptyset$ and $Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, X \cap Y\}, \mathbf{X}) = \emptyset)$;

$Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbf{X}) = \emptyset$ iff $X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$ or $X \cup Y = U$ or $(X - Y \neq \emptyset$ and $Y - X \neq \emptyset$ and $X \cap Y \neq \emptyset$ and $X \cup Y \neq U$ and $Bd_{\mathbf{A}}(\{X \cap -Y, -X \cap Y, -X \cap -Y, X \cap Y\}, \mathbf{X}) = \emptyset)$.

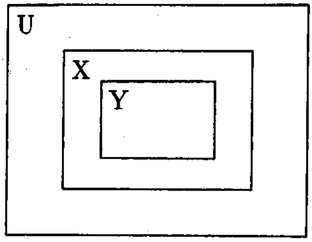
Hence, taking the conjunction of above equivalencies, we obtain: $Bd_{\mathbf{A}}(\mathbf{Y}, \mathbf{X}) = \emptyset$ for any $\mathbf{Y} \supseteq \{X \cap -Y, -X \cap Y\}$ iff one of the conditions α, β, γ, δ, ε from (ii) is satisfied.

Let us remark that only when condition α holds, that is, when $X \subseteq Y$ or $Y \subseteq X$, condition (ii) is independent from the properties of boundary regions in the information systems.

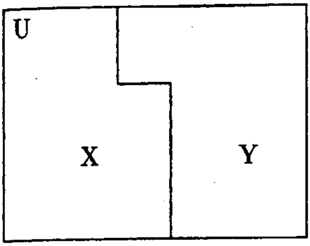
In Figure 12.2 we illustrate the conditions formulated in (ii) of Lemma 3.

Now we prove that the assumptions from Lemmas 1 and 2 related to the boundary region $Bd_{\mathbf{A}}(\mathbf{X}, \mathbf{X})$ cannot be removed because otherwise it will not be possible to compute the values of $\mu^{\mathbf{A}}_{X \cup Y}(x)$ and $\mu^{\mathbf{A}}_{X \cap Y}(x)$ knowing the values $\mu^{\mathbf{A}}_{X}(x)$ and $\mu^{\mathbf{A}}_{Y}(x)$ only.

α :

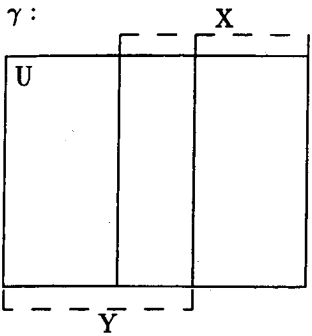


β :



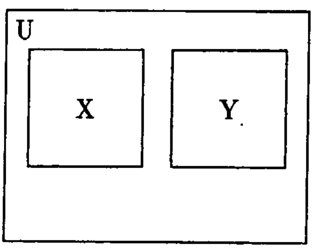
X and Y form a partition of U
The condition for the boundary regions is:
$Bd_{\mathbf{A}}(\{X\cap -Y, -X\cap Y\} \neq \emptyset)$

γ :



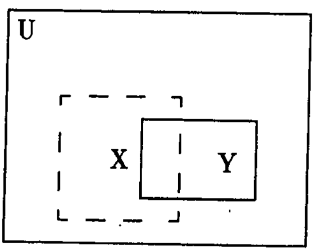
The conditions for the boundary regions are:
$Bd_{\mathbf{A}}(\{X\cap -Y, X\cap Y\}, \mathbf{X}) \neq \emptyset)$
$Bd_{\mathbf{A}}(\{X\cap -Y, X\cap Y, -X\cap Y\}, \mathbf{X}) \neq \emptyset)$

δ :



The conditions for the boundary regions are:
$Bd_{\mathbf{A}}(\{X\cap -Y, -X\cap Y\}, \mathbf{X}) \neq \emptyset)$
$Bd_{\mathbf{A}}(\{X\cap -Y, -X\cap Y, -X\cap -Y\}, \mathbf{X}) \neq \emptyset)$

ε :



The conditions for the boundary regions are:
$Bd_{\mathbf{A}}(\{X\cap -Y, -X\cap Y\}, \mathbf{X}) \neq \emptyset)$
$Bd_{\mathbf{A}}(\{X\cap -Y, -X\cap Y, X\cap Y\}, \mathbf{X}) \neq \emptyset)$
$Bd_{\mathbf{A}}(\{X\cap -Y, -X\cap Y, -X\cap -Y\}, \mathbf{X}) \neq \emptyset)$
$Bd_{\mathbf{A}}(\{X\cap -Y, -X\cap Y, X\cap Y, -X\cap -Y\}, \mathbf{X}) \neq \emptyset)$

Figure 12.2

Theorem 3. There is no function

$$F : [0,1] \times [0,1] \rightarrow [0,1]$$

such that for any finite sets X and Y and any information system $\mathbf{A} = (U, A)$ such that $X, Y \subseteq U$ the following equality holds:

$$\mu^{\mathbf{A}}_{X \cup Y}(x) = F(\mu^{\mathbf{A}}_{X}(x), \quad \mu^{\mathbf{A}}_{Y}(x)), \text{ for any } x \in U.$$

Proof. Let us take $X = \{1, 2, 3, 5\}$ and $Y = \{1, 2, 3, 4\}$. Let $U = \{1, \ldots, 8\}$. It is easy to construct atrribute sets A and A' such that $[1]_A = U$ and $[1]_{A'} = \{1, 4, 5, 6\}$. Thus we have $\mu^{\mathbf{A}}_{X}(1) = \mu^{\mathbf{A}}_{Y}(1) = 1/2$ and $\mu^{\mathbf{A}}_{X \cup Y}(1) = 5/8$, where $\mathbf{A} = (U, A)$ and $\mu^{\mathbf{B}}_{X}(1) = \mu^{\mathbf{B}}_{Y}(1) = 1/2$ and $\mu^{\mathbf{B}}_{X \cup Y}(1) = 3/4$, where $\mathbf{B} = (U, A')$.

Similarly one can prove:

Theorem 4. There is no function

$$F : [0,1] \times [0,1] \rightarrow [0,1]$$

such that for any finite sets X and Y and any information system $\mathbf{A} = (U, A)$ such that $X, Y \subseteq U$ the following equality holds:

$$\mu^{\mathbf{A}}_{X \cap Y}(x) = F(\mu^{\mathbf{A}}_{X}(x), \quad \mu^{\mathbf{A}}_{Y}(x)), \text{ for any } x \in U.$$

6. AN ALGORITHM FOR COMPUTING THE ROUGH MEMBERSHIP FUNCTION VALUES

In the previous section we proved that it is not possible, in general, to construct a function such that it can be used for computing values of the *rm*-function corresponding to the $X \cup Y$ or $X \cap Y$ from the values of the *rm*-functions corresponding to X and Y. Hence any particular functions, for example, *min or max* applied for computing the values of *rm*-functions, will give incorrect values. This shows a major drawback of some approaches in fuzzy set theory.

We present an efficient algorithm for computing values of *rm*-functions based on the properties of the atomic components of the sets.

Let $\mathbf{X}$ be a (nonempty) family of subsets of a given finite set U. By $\mathbf{B}(\mathbf{X})$ we denote the field set generated by $\mathbf{X}$, that is, $\mathbf{B}(\mathbf{X})$ is the least family of sets satisfying the following two conditions:

(i) $\mathbf{X} \subseteq \mathbf{B}(\mathbf{X})$;
(ii) if $X, Y \in \mathbf{B}(\mathbf{X})$ then $X \cup Y$, $X \cap Y$, $-X \in \mathbf{B}(\mathbf{X})$.

If $X \subseteq U$ then we define $X^0 = X$ and $X^1 = U - X$. By $\mathrm{AT}(\mathbf{A},\mathbf{X})$ we denote the set of all nonempty atoms generated by $\mathbf{X} = \{X_1, \ldots, X_k\}$, i.e., $\mathrm{AT}(\mathbf{A},\mathbf{X}) = \{X_1^{i1} \cap \ldots \cap X_k^{ik} : i1, \ldots, ik \in (0,1\}$ and $X_1^{i1} \cap \ldots \cap X_k^{ik} \neq \emptyset\}$.

We will apply the well-known properties of atoms.

Proposition 4. Let $\mathbf{X}$ be a (nonempty) family of subsets of a given set U. The following properties hold:

(i) If $Y, Y' \in \mathrm{AT}(\mathbf{A},\mathbf{X})$ and $Y \neq Y'$ then $Y \cap Y' = \emptyset$.

(ii) If $\emptyset = Y \in \mathbf{B}(\mathbf{X})$ then there exists a uniquely determined set of (nonempty) atoms $\mathbf{Y} \subseteq \mathrm{AT}(\mathbf{A},\mathbf{X})$ such that $Y = \bigcup_{X \in \mathbf{Y}} X$.

Let $\mathbf{A} = (U, A)$ be an information system and let $\mathbf{X}$ be a family of subsets of U. For every $u \in \mathrm{INF}(\mathbf{A})$ we define the set $\mathrm{AT}(\mathbf{A},\mathbf{X},u)$ of all atoms $Y \in \mathrm{AT}(\mathbf{A},\mathbf{X})$ such that

$$Y \cap u_{\mathbf{A}} \neq \emptyset, \text{ where } u_{\mathbf{A}} = \{x \in U: \mathit{Inf}_{\mathbf{A}}(x) = u\}.$$

Moreover, let $f(\mathbf{A},\mathbf{X},u)$ be a function from $\mathrm{AT}(\mathbf{A},\mathbf{X})$ into nonnegative reals such that

$$f(\mathbf{A},\mathbf{X},u)(Y) = \frac{|u_{\mathbf{A}} \cap Y|}{|u_{\mathbf{A}}|} \text{ for any } Y \in \mathrm{AT}(\mathbf{A},\mathbf{X})$$

From the definition we have the following equality:

$$f(\mathbf{A},\mathbf{X},\mathit{Inf}_{\mathbf{A}}(x))(Y) = \mu_Y^{\mathbf{A}}(x), \text{ for any } x \in U \text{ and } Y \in \mathrm{AT}(\mathbf{A},\mathbf{X})$$

There is a simple method for computing all functions from the family $\{f(\mathbf{A},\mathbf{X},u)\}_{u \in \mathrm{INF}(\mathbf{A})}$ for a given information system $\mathbf{A}$. We represent the family $\{f(\mathbf{A},\mathbf{X},u)\}_{u \in \mathrm{INF}(\mathbf{A})}$ in a table $\mathrm{T}(\mathbf{A},\mathbf{X})$ in which rows correspond to different information $u \in \mathrm{INF}(\mathbf{A})$ and the columns correspond to different atoms from $\mathrm{AT}(\mathbf{A},\mathbf{X})$. In the table $\mathrm{T}(\mathbf{A},\mathbf{X})$ the position corresponding to an information u and to an atom $Y \in \mathrm{AT}(\mathbf{A},\mathbf{X})$ is empty if $Y \notin \mathrm{AT}(\mathbf{A},\mathbf{X},u)$ and contains the value $f(\mathbf{A},\mathbf{X},u)(Y)$ if $Y \in \mathrm{AT}(\mathbf{A},\mathbf{X},u)$.

Example 1. Let us consider the following information system. Let $U = \{1, \ldots, 20\}$, $A = \{a,b,c,d,e\}$, $\mathbf{X} = \{X_1, X_2\}$, $X_1 = (5, \ldots, 15)$, $X_2 = \{10, \ldots, 20\}$ and the attributes are defined as in Table 12.1.

Table 12.1

	a	*b*	*c*	*d*	*e*
1	1	1	0	0	0
2	0	0	1	0	1
3	1	0	1	0	1
4	1	1	1	1	1
5	0	0	1	0	1
6	1	1	1	1	1
7	1	0	1	0	1
8	1	1	0	0	0
9	0	1	0	1	0
10	0	0	1	1	1

	a	*b*	*c*	*d*	*e*
11	0	0	1	0	1
12	0	0	0	0	0
13	0	0	1	1	1
14	1	1	0	0	0
15	0	0	0	0	0
16	1	1	1	0	0
17	0	0	1	0	1
18	1	1	1	1	1
19	1	1	1	1	1
20	0	0	0	0	0

From the above definitions we get:

AT(A,X) $= \{Y_1, Y_2, Y_3, Y_4\}$, where $Y_1 = X_1 \cap X_2 = \{10,\ldots,15\}$, $Y_2 = X_1 \cap -X_2 = \{5,\ldots,9\}$, $Y_3 = -X_1 \cap X_2 = \{16,\ldots,20\}$, $Y_4 = -X_1 \cap -X_2 = \{1,\ldots,4\}$;

INF(A) = {11000, 00101, 10101, 11111, 01010, 00111, 0000, 11100};
$11000_A = \{1,8,14\}$; $00101_A = \{2,5,11,17\}$; $10101_A = \{3,7\}$;
$11111_A = \{4,6,18,19\}$; $01010_A = \{9\}$; $00111_A = \{10,13\}$;
$00000_A = \{12,15,20\}$; $11100_A = \{16\}$;
AT(A,X,11000) $= \{Y_1, Y_2, Y_4\}$; AT(A,X,00101) $= \{Y_1, Y_2, Y_3, Y_4\}$;
AT(A,X,10101) $= \{Y_2, Y_4\}$; AT(A,X,11111) $= \{Y_2, Y_3, Y_4\}$;
AT(A,X,01010) $= \{Y_2\}$; AT(A,X,00111) $= \{Y_1\}$;
AT(A,X,00000) $= \{Y_1, Y_3\}$; AT(A,X,11100) $= \{Y_3\}$;

Thus, we have T(A,X) as in Table 12.2 specifying the functions $f(\mathrm{A},\mathrm{X},u)$ for $u \in$ INF(A).

Let us denote by [A,X] the extension of the data table corresponding to A by the columns corresponding to the characteristic functions of sets from X.

One can show that the tabel T(A,X) can be constructed from [A,X] in the number of steps of order $0(n^2(m + k))$, where $n = |U|$, $m = |A|$, and $k = |\mathrm{X}|$.

Table 12.2

u	Y_1	Y_2	Y_3	Y_4
11000	1/3	1/3		1/3
00101	1/4	1/4	1/4	1/4
10101		1/2		1/2
11111		1/4	1/2	1/4
01010		1		
00111	1			
00000	2/3		1/3	
11100		1		

Let us observe that by a slight modification of the construction of the table $T(\mathbf{A},\mathbf{X})$ one can obtain a table for computing the belief and plausibility functions of the information systems (Skowron, 1991; Skowron and Grzymała-Busse, 1991). This modification can be realized by adding to $T(\mathbf{A},\mathbf{X})$ one additional column in which in the position corresponding to u the cardinality of $u_{\mathbf{A}}$ is stored.

After such a modification one can easily compute the A-basic probability assignment $m_{\mathbf{A}}(\theta)$ for any nonempty set θ of atoms. It is sufficient, in fact, first to find all rows with nonempty entries corresponding exactly to elements of θ, second, to compute the sum s of all numbers appearing in the last column of these rows, and third to put $m_{\mathbf{A}}(\theta) = s/|U|$.

Now we are ready to present a simple method for computing the *rm*-function values.

We assume that the family $\{f(\mathbf{A},\mathbf{X},u)\}_{u\in \mathrm{INF}(\mathbf{A})}$ is represented by its data table $T(\mathbf{A},\mathbf{X})$ in the way described before. We also assume that the information system **A** is represented in the standard way by its data table. The data table of a given information system **A** is extended by one additional column containing for any $x \in U$ a pointer to the row labeled by $Inf_{\mathbf{A}}(x)$ in the table $T(\mathbf{A},\mathbf{X})$. A set X of objects is represented by marking all columns in the table $T(\mathbf{A},\mathbf{X})$ corresponding to atoms included in X.

ROUGH MEMBERSHIP FUNCTION PROCEDURE:

INPUT: representations of **X**, **A**, $\{f(\mathbf{A},\mathbf{X},u)\}_{u\in \mathrm{INF}(\mathbf{A})}$ and $X \in \mathbf{B}(\mathbf{X})$ in the form described above.

OUTPUT: $\mu_X^{\mathbf{A}}$.

1. For any $x \in U$ perform the following steps:
   1.1. For a given x find in the table $T(\mathbf{A}, \mathbf{X})$ the row corresponding to $u = Inf_{\mathbf{A}}(x)$;
   1.2. Compute $\mu_{X}^{\mathbf{A}}(x) = \sum f(\mathbf{A}, \mathbf{X}, u)(Y)$, where the above sum is taken for all $\mathbf{Y}$ such that, first, the entry in $T(\mathbf{A}, \mathbf{X})$ corresponding to the column labeled Y and the row labeled u is nonempty, and second, Y corresponds to a marked column in $T(\mathbf{A}, \mathbf{X})$.

The correctness of this method follows from Proposition 2 (part (ix)) and from the construction of the table $T(\mathbf{A}, \mathbf{X})$. One can see that the sum in Step 1.2 is taken for all $Y \in \mathbf{Y} \cap AT(\mathbf{A}, \mathbf{X}, u)$, where $\mathbf{Y}$ is a set of atoms such that $X = \bigcup \mathbf{Y}$.

The number of steps to realize Step 2 is of order $0(n^2)$ (at most n additions for each u), where $n = |U|$.

Example 2. (continuation of Example 1). Let $X = X_1 \cup X_2$. We have $X = X_1 \cap X_2 \cup X_1 \cap -X_2 \cup -X_1 \cap X_2 = Y_1 \cup Y_2 \cup Y_3$. Hence $\mathbf{Y} = \{Y_1, Y_2, Y_3\}$.

Let $x = 7$. Then $Inf_{\mathbf{A}}(7) = 10101$, $\mathbf{Y} \cap AT((\mathbf{A}, \mathbf{X}, 10101) = \{Y_2\}$, and $\mu_{X}^{\mathbf{A}}(7) = f(\mathbf{A}, \mathbf{X}, 10101)(Y_2) = 1/2$.

Let $x = 6$. Then $Inf_{\mathbf{A}}(6) = 11111$, $\mathbf{Y} \cap AT((\mathbf{A}, \mathbf{X}, 11111) = \{Y_2, Y_3\}$, and $\mu_{X}^{\mathbf{A}}(6) = f(\mathbf{A}, \mathbf{X}, 11111)(Y_2) + f(\mathbf{A}, \mathbf{X}, 11111)(Y_3) = 1/4 + 1/2 = 3/4$.

7. CONCLUSIONS

We introduced the rough membership functions (*rm*-functions) as a new tool for reasoning with uncertainty. The definition of those functions is based on the observation that objects are classified by means of partial information that is available. That definition allows us to overcome some problems that may be encountered if we use other approaches (like the ones mentioned in Section 5). We have investigated the properties of the *rm*-functions, and in particular, we have shown that the *rm*-functions are computable in an algorithmic way so that their values can be derived without the help of an expert.

We would also like to point out one important topic for further research based on the results presented here. Our *rm*-functions are defined relative to information systems. We will look for a calculus with rules based on properties of *rm*-functions and also on belief and plausibility functions for information systems. One important problem to be studied is the definition of strategies that can allow us to reconstruct those rules when the information systems are modified by environment. In some sense we would like to embed a nonmonotonic reasoning on our *rm*-functions approach as well as the belief and plausibility functions related to the information systems (Shafer, 1976; Skowron, 1991; Skowron and Grzymała-Busse, 1991).

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