FUZZY LOGIC FOR THE MANAGEMENT OF UNCERTAINTY

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5 Rough sets: A new approach to vagueness

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Abstract. A brief exposition of the concept of a rough set is presented, with an extensive list of literature on its related theories and applications. A rough set is basically meant to represent a vague concept (a vaguely specified set) by two precisely specified sets, called lower and upper approximations, with their difference being a boundary region. Knowledge is basically defined in terms of rough classification whose main underlying concept is an indiscernibility relation. A measure of accuracy (vagueness) is presented. Numerous applications in a wide spectrum of fields are reviewed.

1. INTRODUCTION

The idea of a *rough set* (Pawlak, 1982) has been proposed as a new mathematical tool to deal with *vague concepts*, and seems to be of some importance to AI and cognitive sciences, in particular expert systems, decision support systems, machine learning, machine discovery, inductive reasoning, pattern recognition, and decision tables.

Vagueness is not a clearly understood idea and there are many approaches to it. The rough set approach to vagueness is closely related to the so-called "boundary-line" view, which is credited to Frege (1903) who writes:

The concept must have a sharp boundary. To the concept without a sharp boundary there would correspond an area that had not a sharp boundary-line all around.

Thus, according to Frege, a precise concept must have a sharp boundary, whereas a vague concept is characterized by its boundary-line cases. In other words, if a concept is precise, then for each object it can be decided whether it belongs to the concept or not; for vague concepts this is not the case.

For example, the concept of an odd (even) number is precise, because for each number it can be decided whether it is odd (even) or not. But the concept of a beautiful woman is vague, because for some women it cannot be decided whether they are beautiful or not (there are boundary-line cases).

Cantor's set theory can deal only with precise concepts. There are many approaches to "soften" classical set theory so that vague concepts could be also considered. One of the most successful approaches in this direction is the well-known *fuzzy set theory* of Zadeh.

The basic idea of *rough set theory* consists in replacing vague concepts with a pair of precise concepts (so that classical set theory can be applied). This is called *lower* and *upper approximation.* For example, the lower approximation of the concept of a beautiful woman contains all women that are beautiful with certainty (there is no doubt that they are beautiful), whereas the upper approximation of this concept contains all women that cannot be excluded from being considered beautiful. Clearly the upper and the lower approximations are precise concepts.

With each vague concept a *boundary region* is associated, which consists of all objects that cannot be placed clearly within the concept. For example, all women that cannot be said with certainty to be beautiful belong to the boundary region of the concept of a beautiful woman. The "size" of the boundary region can be used as a *measure of vagueness* of the vague concept. (The greater the boundary region, the more vague is the concept; precise concepts do not have boundary regions at all.) Obviously the boundary region is the difference between the upper and lower approximation of the concept.

Rough set theory is used mainly for data analysis. Among the types of problems that can be solved using rough set theory in data analysis are the following: data reduction (elimination of superfluous data), discovering of data dependencies, data significance, decision (control) algorithms generated from data, approximate classification of data, discovering similarities or differences in data, discovering patterns in data, and discovering cause–effect relationships.

The proposed approach has proved to be very useful in practice and many real-life applications of this concept have been implemented. Some of these are listed below:

- Engineering design (Arciszewski et al. 1986, 1987)
- Generation of cement kiln control algorithm from observation of kiln stoker actions (Mrozek, 1989)
- Approximate (rough) classification of patients after highly selective vagotomy (HSV) for duodenal ulcer (Greenburg, 1987; Pawlak et al., 1986)
- Analysis of peritoneal lavage in acute pancreatitis (Słowiński et al., 1989)
- Analysis of hierarchy of factors of a surgical wound infection (Kandulski et al., 1990)
- Aircraft pilot performance evaluation (Krasowski, 1988)
- Analysis of relationship between structure and activity of drugs (Krysiński, 1990)
- Study of water runoff from a river basin (Reinhard et al., 1989)
- Control of water–air relation on a polder (Reinhard et al., 1989)
- Vibration analysis (Nowak et al., 1990)
- Switching function minimization (Rybnik, 1990)

Machine learning is one of the most important fields of artificial intelligence, and a growing number of researchers are involved in this area. There are a variety of approaches to machine learning; however, at present no commonly accepted theoretical foundations have been developed. It seems that the rough set approach can be used as a theoretical basis for some problems in machine learning. Some ideas concerning the application of rough sets in this area have been published by Grzymała-Busse (1988, 1989), Hadjimichael (1989), Orłowska (1986), Pawlak (1986a, b, 1987), Pawlak et al. (1988), Pettorossi et al. (1987), Raś and Zemankova (1986), Wasilewska (1990a, b), and Wong et al. (1986a, b).

The concept of a rough set has also inspired a variety of logical research: Jian-Ming and Nakamura (1990), Konikowska (1987), Krynicki (1989, 1990a, b), Krynicki and Tuschnick (1990), Nakamura and Jian-Ming (1988), Orłowska (1985a, b, 1989), Pawlak (1987b), Rasiowa (1985, 1986a, b), Rauszer (1985, 1986), Szczerba (1987), Vakarelov (1981, 1989) Wasilewska (1988, 1989), and others. Most of this research has been directed toward creating logical tools to deal with approximate reasoning.

Algebraic properties of rough sets have been studied by Comer (1991), Grzymała-Busse (1986), Iwiński (1987), Nieminen (1988), Novotny and Pawlak (1985–1991), Obtułowicz (1988) and Pomykała and Pomykała (1988).

The rough set concept overlaps in many areas with other mathematical ideas developed to deal with imprecision and vagueness, in particular with fuzzy sets. Fair comparison of rough sets and fuzzy sets can be found in Dubois and Prade (1988). Some remarks on comparison of fuzzy sets and rough sets can be also found in Chanas and Kuchta (1990), and Wygralak (1989). The relation of rough set theory to the Dempster-Shafer evidence theory has been discussed by Grzymała-Busse (1988) and Skowron (1989).

2. PRECISE AND VAGUE KNOWLEDGE

As we mentioned in the introduction, in the proposed approach we replace vague concepts with a pair of precise concepts. In other words, we would like to represent some concepts by means of other concepts. To this end we will need some operations on families of concepts. We must introduce here the idea of *knowledge*, which is simply a family of concepts (as a language in formal linguistics is defined as a set of sentences, or a theory in logic is understood as a set of theorems). Thus, any family of concepts will be called knowledge. If all concepts are precise the corresponding knowledge is precise; otherwise the knowlege is vague. More exactly, let U be a finite set called the *universe of discourse* (in short, the *universe*). Any subset X of U ($X \subseteq U$), will be called *a concept* in U and any family F of concepts in U ($F \subseteq P(u)$, $P(U)$ is the family of all subsets of U), will be referred to as *knowledge* about U. It seems natural to assume that the family F is closed under the set theoretic union, intersection and complement, that is if X and Y are concepts in F, so are $X, \cup Y$, $X \cap Y$, and $-X$.

Suppose we are given knowledge F about U and a concept $Y \subseteq U$. Now we may ask whether Y is precise or vague in F. Of course, if $Y \subseteq F$, then Y is precise in F, otherwise Y is vague in F. How we can approximate the vague concept Y in F? It seems to be justifiable to approximate the concept Y from below and from above as follows:

The *lower approximation* of Y in F, denoted $\underline{F}\,Y$, is the union of all exact concepts X in F that are included in Y.

The *upper approximation* of Y in F, denoted $\bar{F}\,Y$, is the intersection of all exact concepts X in F that include Y.

For practical and mathematical reasons, which will not be discussed here, we will assume a somewhat modified definition of approximation of vague concepts by means of precise concepts. The idea of approximation will be based not on arbitrary families of concepts, but on families of concepts that form classifications (partitions).

The reason we consider classification as a basis for definition of knowledge is that our belief is that knowledge is deep-seated in the classification abilities of human beings and other species. Hence, we assume here that knowledge consists of a family of various classification patterns, of a domain of interest, which provides explicit facts about the reality.

The basic idea underlying classification consists in the fact that objects being in the same equivalence class of the equivalence relation cannot be discerned; therefore we will call these the *indiscernibility classes*. Combining elements of U into indiscernibility classes can be done deliberately or can be due to our lack of knowledge. For example, in order to have the category of the color red we must ignore small differences between various shades of red, otherwise it would be impossible to form the category of the color red. On the other hand, the clustering of objects into categories can be caused by insufficient knowledge. Thus, knowledge about a certain set of objects can be identified with the ability to classify these objects into blocks of the partition induced by the indiscernibility relation. The more knowledge we have about some objects, the more exactly we can classify them. In the next section, we will explain these ideas more precisely.

3. KNOWLEDGE AND KNOWLEDGE BASE

Suppose we are given a finite set U (the universe) of objects we are interested in, and a family of classification patterns $C = \{C_1, C_2, \ldots, C_m\}$, where each C_i is a disjoint family of concepts in U (i.e., subsets of U).

A pair $K = (U, C)$ will be referred to as a *knowledge base*. Each classification C_i from C will be called an *attribute* in K and each element of C_i will be called a *basic category* of C_i (in U).

For example, if we classify elements of U according to colors, then the basic categories of the attribute color are *red, green, blue*, etc.

Thus, the knowledge base represents a variety of basic classification skills (e.g., according to colors, temperature, etc.) of an "intelligent" agent or group of agents (e.g., organisms or robots).

For mathematical reasons it is often better to use equivalence relations instead of classifications, since these two concepts are mutually exchangeable and relations are easier to deal with. Thus, the knowledge base can defined now as $K = (U, R)$, where $R = \{R_1, R_2, \ldots, R_l\}$ is the family of equivalence relations over U.

Of course, the set theoretical intersection of any family of equivalence relations is also an equivalence relation. Any subset of equivalence relations from our knowledge base also defines a family of categories, which will be called *elementary categories* in the knowledge base. It is obvious that any concept (subset of U) can be expressed in the knowledge base K only if it is the union of some elementary categories in K. Otherwise, the concept cannot be defined in the knowledge base. In other words, elementary categories are fundamental building blocks of our knowledge, or elementary properties of the universe that can be expressed employing the knowledge base.

Evidently, not every concept can be defined in the knowledge base using its elementary categories. This is where approximations come into the picture.

To express approximately an arbitrary concept in the knowledge base we define the lower and upper approximation of any concept in U (subset of U).

The *lower approximation* of $X \subseteq U$ by $\underline{R}$ (i.e., set of categories of R, where R is a relation defined by any subset of R) is the union of equivalence classes of R that are included in X, or formally,

$$\underline{R}X = \bigcup\{Y \in U/R : Y \subseteq X\}$$

where U/R denotes the family of all equivalence classes of R.

The *upper approximation* of $X \subseteq U$ by $\bar{R}$ is the union of all equivalence classes of R that do not have empty intersection with X:

$$\bar{R}X = \bigcup\{Y \in U/R : Y \cap X \neq \emptyset\}$$

It is easily seen that these are special cases of definitions given previously.

The boundary-line region is of course defined as $BN_R(X) = \bar{R}X - \underline{R}X$ and will be called the *R-boundary* of X.

Set $\underline{R}X$ consists of all elements of U that can be *with certainty* classified as elements of X employing knowledge R; set $\bar{R}X$ is the set of all elements of U that can be *possibly* classified as elements of X using knowledge R; *set* $BN_R(X)$ is the set of all elements that cannot be classified either to X or to $-X$ having knowledge R.

Now we are able to give the definition of the rough set:

> Set $X \subseteq U$ is *rough with respect to* R if $\bar{R}X \neq \underline{R}Y$; otherwise set X is *exact with respect to* R.

Thus, a set is rough if it does not have a sharply defined boundary, that is, it cannot be uniquely defined employing available knowledge.

Let us note the difference between imprecision and vagueness that results from our considerations. Imprecision is due to the indiscernibility relation and vagueness is the effect of the borderline region. Thus, imprecision and vagueness are entirely different phenomena.

It is easy to show that approximations have the following properties:

(1) $\underline{R}X \subseteq \bar{R}X$

(2) $\underline{R}\emptyset = \bar{R}\emptyset = \emptyset$; $\underline{R}U = \bar{R}U = U$

(3) $\bar{R}(X \cup Y) = \bar{R}X \cup \bar{R}Y$

(4) $\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y$

(5) $X \subseteq Y$ implies $\underline{R}X \subseteq \underline{R}Y$

(6) $X \subseteq Y$ implies $\bar{R}X \subseteq \bar{R}Y$

(7) $\underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}Y$

(8) $\bar{R}(X \cap Y) \subseteq \bar{R}X \cap \bar{R}Y$

(9) $\underline{R}(-X) = -\bar{R}X$

(10) $\bar{R}(-X) = -\underline{R}X$

(11) $\underline{R}\underline{R}X = \bar{R}\underline{R}X = \underline{R}X$

(12) $\bar{R}\bar{R}X = \underline{R}\bar{R}X = \bar{R}X$

I would like to stress properties (7), (8), (9) and (10), but the detailed discussion is left to the interested reader.

It is interesting to note that the lower and the upper approximations are respectively interior and closure operations in a topology generated by the equivalence relation R. In other words, vagueness is strictly related to granulation of knowledge, which induces topological structure in the knowledge base.

For practical applications we need a numerical characterization of vagueness, which will be defined as

$$\alpha_R(X) = \text{card } \underline{R}X/\text{card } \overline{R}X$$

where $X \neq \emptyset$; this is called the *accuracy measure*.

The accuracy measure $\alpha_R(X)$ is intended to capture the degree of completeness of our knowledge about the set X.

Obviously, $0 \leqslant \alpha_R(X) \leqslant 1$, for every R and $X \subseteq U$; if $\alpha_R(X) = 1$ the R-boundary region of X is empty and the set X is definable in knowledge R; if $\alpha_R(X) < 1$ the set X has some nonempty R-boundary region and consequently is undefinable in knowledge R.

4. KNOWLEDGE REPRESENTATION

The assumed model of knowledge, as a family of equivalence relations, is very well suited to prove some mathematical properties of the concepts introduced. However, the definition has some disadvantages when considering algorithmic properties of knowledge and the method of processing knowledge. To avoid this drawback we need a special representation of the set of equivalence relations so that all necessary algorithms can be easily derived. Therefore, for algorithmic reasoning knowledge bases will be represented in tabular form, sometimes called *information system*, or *attribute-value system*. We will refer to it as *knowledge representation system*.

Knowledge representation system is a finite table with rows labeled with elements from U, and columns labeled with elements from a set A, called the set of attributes. With each attribute α from A a finite set of values V_a is associated, and is referred to as *domain of* α.

To each object x and attribute α there corresponds an entry in the table, which is a value of attribute α associated with object X. For example, if the object were an *apple* and the attribute *color*, then the corresponding entry in the table could be *red*.

In Table 5.1, set $\{1, 2, 3, 4, 5, 6, 7\}$ is the set of objects, $\{a, b, c, d, e\}$ is the set of attributes, and the domain of each attribute is the set $\{0, 1, 2\}$.

Table 5.1.

U	*a*	*b*	*c*	*d*	*e*
1	1	0	0	1	1
2	1	0	0	0	1
3	0	0	0	0	0
4	1	1	0	1	0
5	1	1	0	2	2
6	2	2	0	2	2
7	2	2	2	2	2

It is easily seen that each attribute in the table defines an equivalence relation, such that two objects x, y belong to the same equivalence class if they have the same attribute values. Thus, such a table can be considered as representation of a knowledge base with the family of equivalence relations defined by the set of attributes. Each subset of objects (concept) can be now described in terms of attributes and their values. If the concept is exact it can be described uniquely; otherwise, the concept can be described approximately, by its lower and upper approximations.

Moreover, we can now easily define a variety of other concepts needed to analyze knowledge represented by the table. We are mostly interested in discovering various relations between attributes, for instance, exact or approximate dependency of attributes (cause–effect relations), redundancy of attributes, significance of attributes, etc. The proposed approach has also given rise to new efficient methods of decision rule generation from data.

The rough set theory has proved to be a very effective tool for data analysis. Several systems based on the ideas discussed in this paper were implemented on personal computers (IBM PC) and work stations (SUN) in Poland and elsewhere, and have found many real-life, nontrivial applications.

It is worthwhile to observe that the rough sets philosophy is close to statistical data analysis and perhaps can be viewed as "deterministic statistics." Comparison of statistical and rough set methods can be found in Krusińska et al. (1990).

Keywords: rough set, indiscernibility, approximation, vagueness knowledge representation, learning, classification

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