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On conflicts

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In this article a mathematical model of conflict situations, based on three binary relations: alliance, conflict and neutrality, is introduced. Axioms for alliance and conflict relations are given and some properties of these relations are investigated.

Further, the strength of an object is introduced. The set of the three relations mentioned above, together with the strength of all objects, is called the situation. Some rules of transformation of situations are introduced and investigated.

Finally, the notion of a capture is defined and the rules of sharing of the capture among objects in a given situation are formulated. Some theorems concerning capture sharing are given.

The approach presented can be used as a starting point for an easy computer simulation of conflict situations.

1. Introduction

This paper is a modified version of ideas introduced in Pawlak (1981).

A formal model of conflict situations, somewhat different from that considered in game thory, is proposed.

Three binary relations: alliance, conflict and neutrality of some set of objects X, are the starting point of our approach. The union of these relations is called the configuration of X. Some axioms for alliance and conflict relations are given, and configurations satisfying these axioms are investigated.

Further, with each object from X a non-negative real number is associated. This is called the strength of the object. The configuration of X together with the strength of all objects in X is called the situation of X. Some rules of transformation of situations are then formulated and studied.

Finally, the non-negative real number called the capture is introduced, and rules of sharing the capture among the objects in the situation are given. Some theorems concerning capture sharing are formulated.

The approach presented can be considered as an extension of ideas presented in Roberts (1976).

2. Configurations

Let X be a finite set. Elements of X will be called *objects*, which can be interpreted as human beings, trading organizations, political groups, governments, etc.

Let ϕ be a function which to each $(x, y) \in D\phi$ associates the number +1, 0 or −1, that is $\phi: X \times X \rightarrow \{+1, 0, -1\}$.

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We assume that the function ϕ satisfies the following conditions:

(1) $\phi(x, x) = +1$,

(2) $\phi(x, y) = \phi(y, x)$,

for every $x, y \in X$.

If $\phi(x, y) = +1$ we say that x and y are *allied* and if $\phi(x, y) = -1$, we say that x and y are in *conflict.* If $\phi(x, y) = 0$ we say that x and y are *neutral.*

The pair $C = (X, \phi)$ will be called the *configuration.*

If $C = (X, \phi)$, then we shall also write X_C, ϕ_C to denote that X and ϕ form the configuration C.

Each configuration $C = (X, \phi)$ defines three disjoint binary relations on X, denoted R_C^+, R_C^0, R_C^-, defined as follows:

$$R_C^+(x, y) \leftrightarrow \phi(x, y) = +1,$$

$$R_C^-(x, y) \leftrightarrow \phi(x, y) = -1,$$

$$R_C^0(x, y) \leftrightarrow \phi(x, y) = 0.$$

We shall call R_C^+, R_C^- and R_C^0 the *alliance, conflict* and *neutrality* relations, respectively.

If $R_C^- = \varnothing$ we shall say that C is a *conflictless* configuration, otherwise C is a *conflict* configuration.

From (1) and (2), it follows that the relation R_C^+ satisfies the following conditions:

A1. $R_C^+(x, x)$,

A2. $R_C^+(x, y) \rightarrow R_C^+(y, x)$.

Moreover, if the condition

A3. $R_C^+(x, y) \,\&\, R_C^+(y, z) \rightarrow R_C^+(x, z)$

is valid, we shall say that R_C^+ is *regular*; otherwise R_C^+ is *nonregular.*

Thus, the regular relation R_C^+ is an equvalence relation on X, and we shall call equivalence classes of R_C^+ *blocks* of the configuration C, or *coalitions* in C.

Obviously, the relation R_C^- satisfies the conditions:

B1. $\sim R_C^-(x, x)$,

B2. $R_C^-(x, y) \rightarrow R_C^-(y, x)$.

If, moreover, the property

B3. $R_C^-(x, y) \,\&\, R_C^-(y, z) \rightarrow R_C^+(x, z)$

is valid, we say that R_C^- is *regular,* otherwise R_C^- is *nonregular.*

From B3, it follows that:

B4. $R_C^-(x, y) \,\&\, R_C^+(y, z) \rightarrow R_C^-(x, z)$,

B5. $R_C^+(x, y) \,\&\, R_C^-(y, z) \rightarrow R_C^-(x, z)$.

If R_C^+ and R_C^- are both regular, then so is the configuration C, otherwise C is nonregular.

Each configuration $C = (X, \phi)$ can be depicted by a graph; objects of C are interpreted as vertices of the graph. If $R_C^+(x, y)$, we shall connect vertices x and y by a double line, called a *positive* edge. If $R_C^-(x, y)$, then we shall connect vertices x and y by a single line, called a *negative* edge.

If $C = (X, \phi)$ is a configuration, then the associated graph will be denoted by G_C. In what follows we shall identify configurations and their graphs, and consequently we shall use graph theoretical terminology for configurations (connected configuration, subconfiguration, loop in the configuration, etc.).

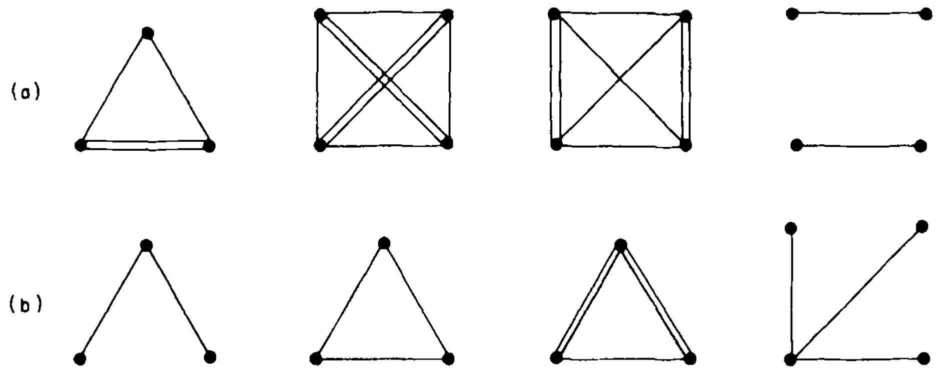

FIG. 1. Examples of (a) regular and (b) nonregular configurations.

Examples of regular and nonregular configurations are shown in Fig. 1.

Let $C = (X, \phi)$ and $C'(X', \phi')$ be configurations. We say that C' is an *extension* of C (or C is a *subconfiguration* of C') if $\phi = \phi'/_{X \times X}$.

Suppose $C = (X, \phi)$ is a nonregular configuration. If $C' = (X', \phi')$ is the last regular extension of C, then C' will be called a *forced extension* of C.

It is obvious that

for every nonregular configuration there exist at most one forced extension of it.

If $C = (X, \phi)$ is a regular configuration, then every extension of C will be called a *free extension of C.*

Examples of forced and free extensions are shown in Fig. 2.

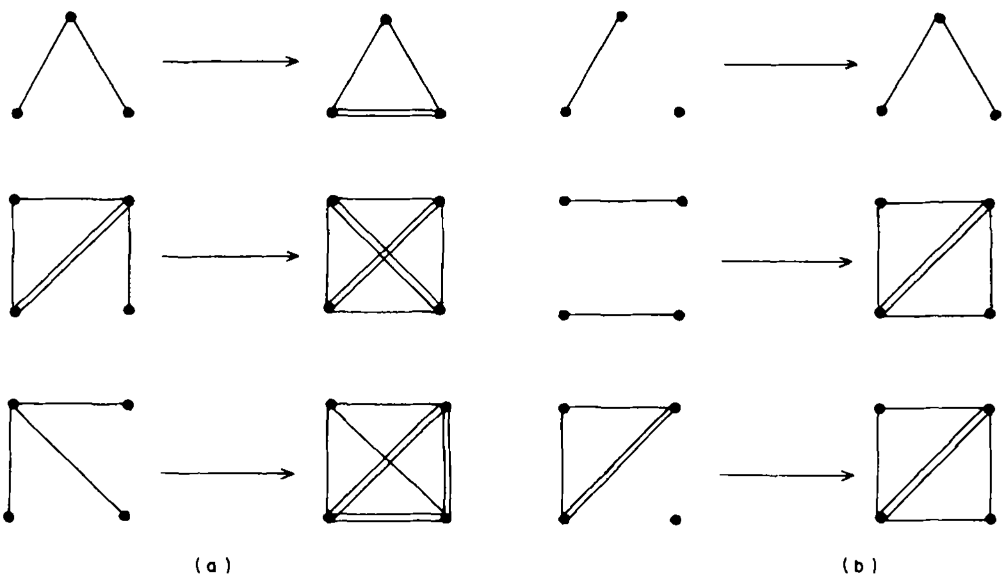

FIG. 2. Examples of (a) forced and (b) free extensions of configurations.

If the loop in the graph G_C contains an odd number of negative edges we shall call the loop *improper*; otherwise the loop is *proper.*

One can prove the following property.

If the graph G_C of the configuration $C=(X, \phi)$ contains an improper loop, then $C=(X, \phi)$ is nonregular and there is no regular extension for C. If the graph G_C does not contain an improper loop, then C has a regular forced extension.

Let $B \neq B'$ be two blocks of the configuration $C=(X, \phi)$. If there are $x \in B$ and $y \in B'$ such that $R_C^-(x, y)$, then we shall say that blocks B and B′ are in *weak conflict*; if for every $x \in B$ and $y \in B'$, $R_C^-(x, y)$, we say that blocks B and B′ are in a strong *conflict*: if block B is not in a conflict with any other block in $C=(X, \phi)$ we say that B is neutral in C.

The following theorem is true.

If $C=(X, \phi)$ is a regular configuration, then every maximal connected subconfiguration of C, i.e. maximal connected subgraph of the graph G_C, is a neutral block in C or a pair of blocks in strong conflict.

That is to say that with every regular configuration we can associate besides the graph G_C another graph $\bar{G}_C$, whose vertices are blocks in C, and two blocks B and B′ are connected by an edge in $\bar{G}_C$ if and only if B and B′ are in strong conflict in C.

Examples of graphs G_C and $\bar{G}_C$ for some configurations are shown in Fig. 3.

The above properties of configurations show that if the initial conflict configuration is changed by adding some new alliances and conflicts to it, i.e. by adding some new positive and negative edges to the graph G_C, and if the initial configuration contains

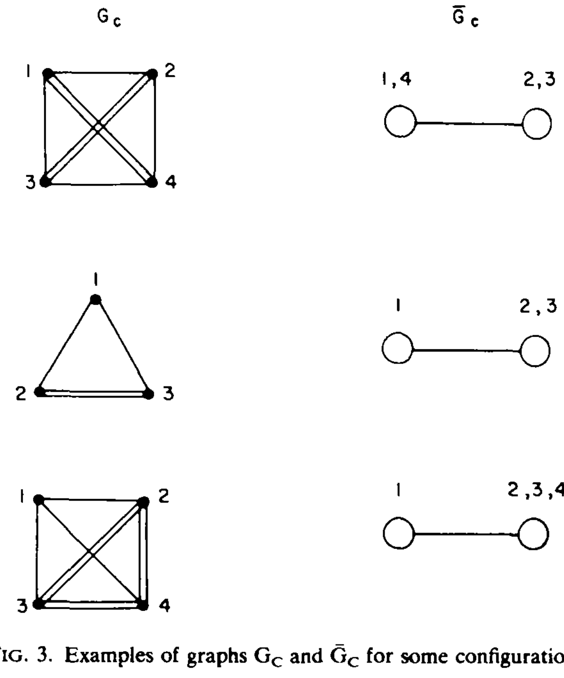

FIG. 3. Examples of graphs G_C and $\bar{G}_C$ for some configurations.

an improper loop, it is impossible to keep the rules A1–A3 and B1–B3, and sooner or later we arrive at contradiction. If the initial configuration does not contain an improper loop and the configuration changes according to the rules A1–A3 and B1–B3, eventually the process ends with the configuration in which there are conflicts only between pairs of blocks, and some neutral blocks may exist in that configuration.

3. The strength of objects

Let $C=(X, \phi)$ be a configuration and let $\mu: X \to R$ be a function which assigns a non-negative real number, called the *strength* of x, to each object $x \in X$.

The strength of $Y \subset X$ is defined as

$$\bar{\mu}(Y)=\sum_{x \in Y} \mu(x).$$

In particular, $\bar{\mu}(X)$ denotes the strength of the whole configuration $C=(X, \phi)$.

The ordered triples $S=(X, \phi, \mu)$ will be called the *situation* of X. If $S=(X, \phi, \mu)$ is a situation, we shall write X_S, ϕ_S and μ_S.

The function $\lambda: X \times X \to R$ (R is the set of all non-negative reals) will be called the *strategy* in $S=(X, \phi, \mu)$. The strategy says how the strength of each object in S is distributed against its enemies. We assume that the strategy λ satisfies the following conditions:

(1) if $y=x$ or $R^+(x, y)$ or $R^0(x, y)$, then $\lambda(x, y)=0$,

(2) if $\lambda(x, y) \neq 0$ then $R^-(x, y)$,

(3) $\bar{\lambda}(x) \leq \mu(x)$ for every $x \in X$,

where

$$\bar{\lambda}(x)=\sum_{y \in X} \lambda(x, y)=\sum_{y \in E_x} \lambda(x, y)$$

and E_x denotes the set of all objects being in conflict with x (enemies of x) in $C=(X, \phi, \mu)$, i.e. $E_x=\{y \in X: R_C^-(x, y)\}$.

The strategy λ in S is *balanced* if $\lambda(x, y)=\lambda(y, x)$ for every $(x, y) \in R_C^-$.

If for some $x \in X$, $\bar{\lambda}(x)=\mu(x)$, we say that the strategy λ for x is *total* in S. Let $X^-=\{y \in X: \bigvee_z R_C^-(y, z)\}$, i.e. X^- is the set of all objects being involved in conflicts in S. If $\bar{\lambda}(x)=\mu(x)$ for every $x \in X^-$ we say that it is a *total strategy* in S.

If $\lambda(x, y)=0$ for all $x, y \in X$ we say that λ is a *null strategy* in S.

Let $S=(X, \phi, \mu)$ be a situation such that $\mu(x)>0$ for every $x \in X$, and let λ be a strategy in S. We assume that λ transforms the situation S into a new situation $S_\lambda=(X_\lambda, \phi_\lambda, \mu_\lambda)$ such that

(1) $\mu_\lambda(x)=\mu(x)=\bar{\lambda}x$,

(2) $X_\lambda=\{x \in X: \mu_\lambda(x)>0\}$,

(3) $\phi_\lambda=\phi / X_\lambda \times X_\lambda$.

Of course, if S is a conflictless situation, then every strategy λ in S is a null strategy and $S=S_\lambda$.

Realization of the strategy λ in the situation S reduces the strength of each object being involved in conflict by the strength engaged against its enemies and eliminates all those objects whose strength is reduced to zero.

Let $S=(X, \phi, \mu)$ be a situation. We shall say that the situation $S=(X, \phi, \mu)$ is *balanced* if there exists a strategy λ in S, such that $X_\lambda=X-X^-$, i.e. all objects being involved in conflicts in the situation S are "destroyed", by the realization of the strategy λ.

This balance may be called *balance of fear*, and the corresponding strategy the *strategy of intimidation*.

One can show the following property.

The situation $S = (X, \phi, \mu)$ is balanced if and only if following set of linear equations has a solution (with respect to λ):

$$\bar{\lambda}(x) = \mu(x) \quad \text{for every } x \in X,$$

$$\lambda(x, y) = \lambda(y, x), \quad \text{for every } x, y \in X.$$

Thus, the situation $S = (X, \phi, \mu)$ is balanced if and only if there exists a total strategy in S.

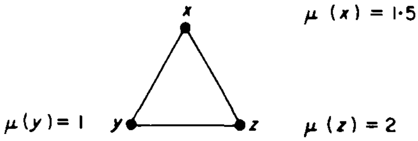

FIG. 4. An example of a balanced situation.

For example, the situation shown in Fig. 4 is balanced because the set of equations

$$\lambda(x, y) + \lambda(x, z) = 1{\cdot}5, \qquad \lambda(z, y) = \lambda(y, z),$$

$$\lambda(z, y) + \lambda(z, x) = 2, \qquad \lambda(x, y) = \lambda(y, x),$$

$$\lambda(y, x) + \lambda(y, z) = 1, \qquad \lambda(x, z) = \lambda(z, x)$$

has the solution

$$\lambda(x, y) = 0{\cdot}25, \qquad \lambda(y, z) = 0{\cdot}75, \qquad \lambda(x, z) = 1{\cdot}25.$$

On the other hand, the situation shown in Fig. 5 is not balanced because the corresponding set of equations has no solution.

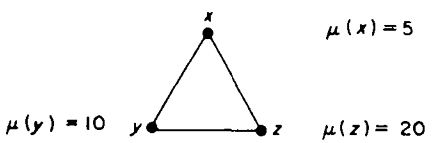

FIG. 5. An example of a situation which is not balanced.

It is obvious that for every situation S there exist a strategy λ in S such that S_λ is conflictless.

The strategy λ such that S_λ is conflictless will be called a *maximal strategy* in S.

We say that object $x \in X$ is *strong* in the situation S iff for every strategy λ in S $\mu_\lambda(x) > 0$; otherwise the object $x \in X$ is *weak* in S.

The following properties are valid.

(1) Object $x \in X$ is strong in S iff $\mu(x) > \bar{\mu}(E_x)$.

(2) If x and y are strong in S, then x and y must not be in a conflict in S, i.e. $\sim R^-(x, y)$.
(3) The strategy of intimidation exists in S iff all objects in S are weak.

Let us remark that if S is conflictless situation, then every object in S is strong.

4. How do conflicts arise?

Let $S = (X, \phi, \mu)$ be a situation and let q be a certain non-negative real number, called the *capture.*

We assume that in each situation $S = (X, \phi, \mu)$ the capture q is shared among the objects of X, i.e. increases the strength of each object (possibly by zero), according to some rules.

Thus, if $S = (X, \phi, \mu)$ is a situation, λ is a strategy in S and q is a capture, we assume that the strategy λ and the capture q transforms the situation $S = (X, \phi, \mu)$ into a new situation $S^q = (X^q, \phi^q, \mu^q)$.

If $S = (X, \phi, \mu)$ is a conflictless situation, λ is a strategy (let us note that in conflictless situation every strategy in S is a null strategy) and q is a capture, we assume that

(1) $X^q_\lambda = X$
(2) $\phi^q_\lambda = \phi$
(3) $\mu^q_\lambda(x) = \mu(x) + [\mu_x/\bar{\mu}(X)]q$,

i.e. we assume that the capture in conflictless situation is shared in proportion to the strength of each object.

If $S = (X, \phi, \mu)$ is a conflict situation, then we assume that the capture is shared only among winners according to some prefixed rules (for example, in proportion to the strength of each winner) to the strength of all winners. (Other principles are possible, but we shall not discuss them here.)

Let $S = (X, \phi, \mu)$ be a conflict situation, λ a maximal strategy in S and q the capture. Then

(4) $X^q_\lambda = X_\lambda$
(5) $\phi^q_\lambda = \phi / X^q_\lambda \times X^q_\lambda$

$$(6)\quad \mu^q_\lambda(x) = \begin{cases} \mu(x), & \text{if } x \in X - X^-, \\ \mu_\lambda(x) + \dfrac{\mu_\lambda(x)}{\bar{\mu}_\lambda(X^+_\lambda)} q & \text{if } x \in X^+_\lambda, \end{cases}$$

where

$$X^+_\lambda = X_\lambda \cap X^-$$

is the set of *winners* in the situation $S = (X, \phi, \mu)$ and the strategy λ.

We say that the situation $S = (X, \phi, \mu)$, is *better* for $x \in X$ with respect to the capture q than the situation $S' = (X, \phi', \mu)$ if for every maximal strategy λ' in S' there exists a maximal strategy λ in S such that

$$\mu^q_\lambda(x) > \mu^q_{\lambda'}(x).$$

Let $S = (X, \phi, \mu)$ be a situation. We shall say that the situation S is *stable* if for every situation $S' = (X, \phi', \mu)$ and every capture q the situation S' is not better than the situation S for every $x \in X$; otherwise the situation S is *unstable.*

The following theorem is true.

A situation $S=(X, \phi, \mu)$ is stable if and only if card $(X)=2$, $\mu(x_1)=\mu(x_2)$ and $(x_1, x_2) \in R^+$ where $X=\{x_1, x_2\}$.

One can show by simple computation the following (sad) theorem.

For every unstable conflictless situation $S=(X, \phi, \mu)$ there exists a conflict situation $S'=(X, \phi', \mu)$ such that S' is better than S for every $x \in X_\lambda^+$, with respect to every $q > \bar{\mu}(X)$.

Proof is by simple computation.

EXAMPLE

Let $S=(X, \phi, \mu)$ be a conflictless situation with $X=\{x_1, x_2, x_3\}$ and $\mu(x_1)=1$, $\mu(x_2)=1$ and $\mu(x_3)=1$, and let the capture $q=6$.

If the capture is shared among objects without "war" the new strength of objects will be $\mu'(x_1)=3$, $\mu'(x_2)=3$, $\mu'(x_3)=3$.

If two objects, say x_1 and x_2 make a coalition against object x_3, then $\mu''(x_1)=3{\cdot}5$ and $\mu''(x_2)=3{\cdot}5$.

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