14. Appendix C.

Let $\mathbf{T_r^*(x)}$ denote the **r**-th degree, modified Chebyshev polynomial, defined by

$$T_r^*(x) = \cos(r\cdot\theta)\ , \quad \cos\theta = 2\cdot x - 1\ , \quad 0 \le x \le 1\ . \tag{C.1}$$

Any function integrable with the weight $\mathbf{(x\text{-}x^2)^{-1/2}}$ in the interval [0,1] can be expanded into

$$f(x) = \frac{a_0}{2}\cdot T_0^*(x) + \sum_{r=1}^{\infty} a_r\cdot T_r^*(x)\ , \tag{C.2}$$

where

$$a_r = \frac{2}{\pi}\cdot\int_0^1 f(x)\cdot T_r^*(x)\cdot\frac{dx}{\sqrt{x-x^2}}\ . \tag{C.3}$$

The error arising from taking only **N** terms of this expansion decreases more rapidly than any power of **1/N** as **N→∞** [92]. Multiplying the finite series by $\mathbf{x^n}$ yields:

$$x^n\cdot\left(\frac{a_0}{2}\cdot T_0^*(x) + \sum_{r=1}^{N} a_r\cdot T_r^*(x)\right) = \frac{b_0}{2}\cdot T_0^*(x) + \sum_{r=1}^{N+n} b_r\cdot T_r^*(x)\ , \tag{C.4}$$

where

$$b_r = 2^{-2\cdot n}\cdot\sum_{s=r-n}^{s=r+n}\binom{2\cdot n}{r+n-s}\cdot a_s \quad \text{if} \quad r>n \tag{C.5}$$

or

$$b_r = c_r\cdot 2^{-2\cdot n}\cdot\left[\sum_{s=0}^{s=r+n}\binom{2\cdot n}{r+n-s}\cdot a_s + \sum_{s=1}^{s=n-r}\binom{2\cdot n}{r+n+s}\cdot a_s\right]$$
$$\text{if} \quad r\le n \quad \text{and} \quad c_r = \begin{cases} 0.5 & r=0 \\ 1 & r>0\ . \end{cases} \tag{C.6}$$

Differentiation of the finite series (C.2) gives:

$$\left[\frac{a_0}{2}\cdot T_0^*(x) + \sum_{r=1}^{N} a_r\cdot T_r^*(x)\right]^{(n)} = \left[\frac{a_0^{(n)}}{2}\cdot T_0^*(x) + \sum_{r=1}^{N-n} a_r^{(n)}\cdot T_r^*(x)\right] \tag{C.7}$$

and for the first four derivatives one receives:

$$a_r^{(1)} = 4\cdot\sum_{p=r+1}^{N} p\cdot a_p\ , \quad p + r \equiv 1\,(\mathrm{mod}\ 2)\ , \tag{C.8}$$

$$a_r^{(2)} = 4 \cdot \sum_{p=r+2}^{N} (p-r) \cdot p \cdot (p+r) \cdot a_p \ , \quad p \equiv r (\mathrm{mod}\ 2) \ , \tag{C.9}$$

$$a_r^{(3)} = 2 \cdot \sum_{p=r+3}^{N} (p-r-1) \cdot (p-r+1) \cdot p \cdot (p+r-1) \cdot (p+r+1) \cdot a_p \ , \quad p+r \equiv 1 (\mathrm{mod}\ 2) \ , \tag{C.10}$$

$$a_r^{(4)} = \frac{2}{3} \cdot \sum_{p=r+4}^{N} (p-r-2) \cdot (p-r) \cdot (p-r+2) \cdot p \cdot (p+r) \cdot (p+r) \cdot (p+r+2) \cdot a_p \ , \quad p \equiv r (\mathrm{mod}\ 2) \ , \tag{C.11}$$

where $\mathbf{i} \equiv \mathbf{j}$ means that $\mathbf{i} - \mathbf{j}$ is divisible by 2.

The above properties make the Chebyshev series particularly useful in solving the ordinary differential equation of type

$$\sum_{i=0}^{K} \sum_{j=0}^{L_i} \alpha_{i,j} \cdot x^j \cdot \frac{d^{(i)} y(x)}{dx^{(i)}} = 0 \tag{C.12}$$

with the boundary conditions:

$$\sum_{j=0}^{K-1} \beta_{i,j} \cdot \left. \frac{d^{(j)} y}{dx^{(j)}} \right|_{x=x_i} = 0 \ , \quad i = 0,1,\ldots,K-1 \ . \tag{C.13}$$

When the finite approximation

$$y(x) = \frac{a_0}{2} \cdot T_0^*(x) + \sum_{r=1}^{N} a_r \cdot T_r^*(x) \tag{C.14}$$

is to be applied, the simplest method to solve (C.12) and (C.13) is Lanczo's tau method [91]. Introduction of (C.14) into (C.12) gives

$$\sum_{r=0}^{Q} \left(\sum_{n=0}^{N} \gamma_{r,n} \cdot a_n \right) \cdot T_r^*(x) = 0 \ , \quad Q = \max\{N + L_i - i\} \quad for \quad i = 0,1,\ldots,K \ , \tag{C.15}$$

whereas elimination of $\mathbf{y(x)}$ between (C.13) and (C.14) results in $\mathbf{K}$ equations:

$$\sum_{r=0}^{N} \left(\sum_{n=0}^{N} \delta_{i,r,n} \cdot a_n \right) \cdot T_r^*(x_i) = 0 \ , \quad i = 0,1,\ldots,K-1 \ . \tag{C.16}$$

Constants $\gamma_{r,n}$ and $\delta_{i,r,n}$ appearing in expressions (C.15) and (C.16) are linear functions of constants $\alpha_{i,j}$ and $\beta_{i,j}$, and can be determined from relations (C.5), (C.6) and (C.8)÷(C.11) provided that $\mathbf{K} \leq \mathbf{4}$. Equations (C.15) should be satisfied for each $\mathbf{x}$ from 0 to 1, which results in $\mathbf{Q+1}$ equations of type:

$$\sum_{n=0}^{N} \gamma_{r,n} \cdot a_n = 0 \quad \text{where} \quad 0 \le r \le Q \ . \tag{C.17}$$

The system of equations (C.16) and (C.17) is overdetermined since one has **N+1** coefficients $\mathbf{a}_n$ and **Q+K+1** equations. The idea of the tau method is to leave **Q-N+K** equations (C.17) unsatisfied [91,92] and determine the high frequency (i.e. high **r**) behavior of the solution from **K** boundary conditions (C.16).

In the case of the system composed of two Orr-Sommerfeld equations (4.26) and eight boundary conditions (4.24) and (4.25) the constants $\alpha_{i,j}$ and $\beta_{i,j}$ are linear functions of the wave speed. Hence, the application of the tau method yields general eigenvalue problem (4.28). Such a problem can be solved by the LZ-algorithm [93], which is a generalization of well known Rutishauser's LR-method. The LZ-algorithm is based on three observations:

1. If $\overline{\overline{\mathbf{L}}}$ and $\overline{\overline{\mathbf{M}}}$ are nonsingular matrices, the eigenvalue problems:

$$\overline{\overline{A}} \cdot \overline{x} = \lambda \cdot \overline{\overline{B}} \cdot \overline{x} \ , \tag{C.18}$$

$$\overline{\overline{L}} \cdot \overline{\overline{A}} \cdot \overline{\overline{M}} \cdot \overline{y} = \lambda \cdot \overline{\overline{L}} \cdot \overline{\overline{B}} \cdot \overline{\overline{M}} \cdot \overline{y} \tag{C.19}$$

have the same eigenvalues, λ, and their eigenvectors are related by

$$\overline{x} = \overline{\overline{M}} \cdot \overline{y} \ , \tag{C.20}$$

2. If $\overline{\overline{\mathbf{A}}}$ is a triangular matrix with diagonal elements, $\mathbf{a}_{i,i}$, and $\overline{\overline{\mathbf{B}}}$ is a triangular matrix with diagonal elements, $\mathbf{b}_{i,i}$, then the ratio $\mathbf{a}_{i,i}/\mathbf{b}_{i,i}$ is an eigenvalue of the generalized problem (C.18) if $\mathbf{b}_{i,i} \neq 0$,

3. There exist matrices $\overline{\overline{\mathbf{L}}}$ and $\overline{\overline{\mathbf{M}}}$ such that $\overline{\overline{\mathbf{L}}} \cdot \overline{\overline{\mathbf{A}}} \cdot \overline{\overline{\mathbf{M}}}$ and $\overline{\overline{\mathbf{L}}} \cdot \overline{\overline{\mathbf{B}}} \cdot \overline{\overline{\mathbf{M}}}$ are upper triangular and $\overline{\overline{\mathbf{L}}}$ and $\overline{\overline{\mathbf{M}}}$ are the products of lower triangular and permutation matrices.

The aim of the LZ-algorithm is to construct iteratively $\overline{\overline{\mathbf{L}}}$ and $\overline{\overline{\mathbf{M}}}$ matrices by means of stabilized elementary transformations. The superiority of this algorithm over similar ones is expressed by the fact that it can be successfully applied even if both matrices $\overline{\overline{\mathbf{A}}}$ and $\overline{\overline{\mathbf{B}}}$ are singular; in system (4.28) matrix $\overline{\overline{\mathbf{B}}}$ is singular.

The accuracy of the proposed here numerical procedure consisting of Lanczo's tau method and the LZ-algorithm depends on the number of terms in Chebyshev series (4.27). Considering data presented in table C-I one can see that setting **k=l=20** gives accuracy up to 5 significant digits, whereas **k=l=26** gives accuracy up to 8 significant digits. Stabilty curves presented in chapter 4 were obtained for **k=l=20**.

Table C.I. Accuracy of computations; ρ_1/ρ_2=1.026, μ_1/μ_2=2.36e-3, $\alpha \cdot$a=0.7, a/R=0.10512, Re_w=29.83.

number of terms k=l	$Im(c \cdot a \cdot \rho_1/\mu_1)$
10	0.65928180
12	0.66034212
14	0.65938258
16	0.65917069
18	0.65912998
20	0.65912260
22	0.65912128
24	0.65912104
26	0.65912100