13. Appendix B.

Using dimensionless variables:

$$\xi = \frac{r}{R}, \quad \textbf{(B.1a)} \qquad \tau = \int_0^t \frac{D\,dt'}{R^2(t')}, \quad \textbf{(B.1b)} \qquad C = \frac{c}{c_0}, \quad \textbf{(B.1c)}$$

where **R** is the radius of an elongated filament:

$$R = R_0 \cdot \exp(-\alpha \cdot t/2) \ , \qquad \textbf{(B.2)}$$

one can rewrite equations (3.37) into simpler form:

$$\frac{\partial C}{\partial \tau} = \frac{1}{\xi} \cdot \frac{\partial}{\partial \xi}\left(\xi \cdot \frac{\partial C}{\partial \xi}\right) , \qquad \textbf{(B.3a)}$$

$$\left.\frac{\partial C}{\partial \xi}\right|_{\xi=0} = 0 \ , \quad \textbf{(B.3b)} \qquad C(\xi \to +\infty, \tau) = 0 \ , \quad \textbf{(B.3c)}$$

$$C(\xi,0) = \begin{cases} f(R_0 \cdot \xi) & \xi \le 1 \\ 0 & 1 > \xi \end{cases} , \qquad \textbf{(B.3d)}$$

Hankel's transformation of the order zero

$$\bar{C}(\sigma,\tau) = \mathcal{H}\{C(\xi,\tau)\} = \int_0^\infty C(\xi,\tau) \cdot J_0(\xi \cdot \sigma) \cdot \xi \, d\xi \ , \qquad \textbf{(B.4)}$$

when applied to equations (B.3) yields:

$$\frac{d\bar{C}(\sigma,\tau)}{d\tau} + \sigma^2 \cdot \bar{C}(\sigma,\tau) = 0 \ . \qquad \textbf{(B.5)}$$

Symbol $\mathbf{J_0}$ in definition (B.4) represents the first kind Bessel function of the order zero. Solution of equation (B.5) has the following form:

$$\bar{C}(\sigma,\tau) = \bar{C}(\sigma,0) \cdot \exp(-\tau\,\sigma^2) \ . \qquad \textbf{(B.6)}$$

The inverse Hankel's transform is given by:

$$C(\xi,\tau) = \mathcal{H}^{-1}\{\bar{C}(\sigma,\tau)\} = \int_0^\infty \bar{C}(\sigma,\tau) \cdot J_0(\xi \cdot \sigma) \cdot \sigma \, d\sigma \ . \qquad \textbf{(B.7)}$$

To find the inverse transform of function (B.6) one can use of the Perseval theorem [89]:

$$\int_0^\infty P(\sigma) \cdot Q(\sigma) \cdot \sigma \, d\sigma = \int_0^\infty p(\eta) \cdot q(\eta) \cdot \eta \, d\eta \ , \qquad \textbf{(B.8)}$$

where $\mathbf{P}(\sigma)=\mathcal{H}\{\mathbf{p}(\eta)\}$ and $\mathbf{Q}(\sigma)=\mathcal{H}\{\mathbf{q}(\eta)\}$. The theorem (B.8) allows to write

$$\begin{aligned} C(\xi,\tau) &= \int_0^\infty \bar{C}(\sigma,0)\cdot e^{-\tau\sigma^2}\cdot J_0(\xi\cdot\sigma)\cdot\sigma\,d\sigma = \\ &= \int_0^\infty C(\eta,0)\cdot\mathcal{H}^{-1}\left\{e^{-\tau\sigma^2}\cdot J_0(\xi\cdot\sigma)\right\}\cdot\eta\,d\eta\ . \end{aligned} \quad \textbf{(B.9)}$$

Thanks to the integral formula valid for $\gamma>0$ [90]:

$$\int_0^\infty e^{-\gamma\sigma^2}\cdot J_0(a\cdot\sigma)\cdot J_0(b\cdot\sigma)\cdot\sigma\,d\sigma = \frac{1}{2\cdot\gamma}\cdot\exp\left(-\frac{a^2+b^2}{4\cdot\gamma}\right)\cdot I_0\left(\frac{a\cdot b}{2\cdot\gamma}\right)\ , \quad \textbf{(B.10)}$$

where I_0 is the modified Bessel function of the order zero:

$$I_0(z) = J_0\left(z\cdot e^{-i\pi/2}\right) \quad \textbf{(B.11)}$$

the inverse transform of function $\exp(-\sigma^2\tau)\cdot J_0(\xi\cdot\sigma)$ reads:

$$\begin{aligned} \mathcal{H}^{-1}\left\{e^{-\tau\sigma^2}\cdot J_0(\xi\cdot\sigma)\right\} &= \int_0^\infty e^{-\tau\sigma^2}\cdot J_0(\xi\cdot\sigma)\cdot J_0(\eta\cdot\sigma)\,\sigma\,d\sigma = \\ &= \frac{1}{2\cdot\tau}\cdot\exp\left(-\frac{\xi^2+\eta^2}{4\cdot\tau}\right)\cdot I_0\left(\frac{\xi\cdot\eta}{2\cdot\tau}\right)\ . \end{aligned} \quad \textbf{(B.12)}$$

Finally, combining equations (B.3c), (B.9) and (B.12) results in:

$$C(\xi,\tau) = \frac{1}{2\cdot\tau}\cdot\int_0^1 f(R_0\cdot\eta)\cdot\exp\left(-\frac{\xi^2+\eta^2}{4\cdot\tau}\right)\cdot I_0\left(\frac{\xi\cdot\eta}{2\cdot\tau}\right)\cdot\eta\,d\eta\ . \quad \textbf{(B.13)}$$

Returning to the original variables one receives:

$$c(r,t) = \frac{\alpha}{2\cdot D\cdot(e^{\alpha t}-1)}\cdot\int_0^{R_0}\exp\left(-\frac{\alpha}{4\cdot D}\cdot\frac{r^2\cdot e^{\alpha t}+u^2}{e^{\alpha t}-1}\right)\cdot I_0\left(\frac{\alpha}{2\cdot D}\cdot\frac{u\cdot r\cdot e^{\alpha t/2}}{e^{\alpha t}-1}\right)\cdot f(u)\cdot u\,du\ . \quad \textbf{(B.14)}$$