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Some Remarks on the Bracket Free Notations

by

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In this paper we discuss some general properties of bracket free notations, first introduced by Łukasiewicz [1].

§ 1. We introduce a preliminary notion of the formula in the tree form. (A finite sequence of the formulae is called also a formula). Let T be the set of all finite non-void sequences of natural numbers. The one-term sequences are identified with their values, and α, β denotes the concatenation of α and β.

Let R be the partial ordering of T defined as follows:

$a_1 a_2 \dots a_k R b_1 b_2 \dots b_n$ iff $k < n$ and for $i \leqslant k$ $a_i = b_i$, where $(a_1 a_2 \dots a_k)$ and $(b_1 b_2 \dots b_n)$ are in T.

The relational system $\langle T, R \rangle$ is called the basic tree. Now let the sequence of symbols F_i and the sequence of numbers k_i and a symbol Λ be given. F_i is called a functor of k_i arguments and Λ — the empty symbol. The symbols with zero arguments are called individual variables. A formula in the tree form is a function Φ defined on T with values F_i or Λ satisfying the following conditions:

1. For almost all $\alpha \in T$ $\Phi(\alpha) = \Lambda$.
2. If $\Phi(\alpha) \neq \Lambda$ and $\beta R \alpha$ then $\Phi(\beta) \neq \Lambda$.
3. If $\Phi(\alpha) = F_i$ then $\Phi(\alpha \cdot n) \neq \Lambda$ if and only if $n < k_i$.
4. If $\Phi(n) \neq \Lambda$ and $m < n$, then $\Phi(m) \neq \Lambda$.

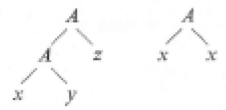
The set of formulae in the tree form is called the language L in the tree form.

Let R_1 be an extension of partial ordering R to the linear ordering. The relation N_{R_1} (bracket free notation for the language in the tree form) is a function, defined on the language whose values are finite sequences of symbols F_i defined as follows:

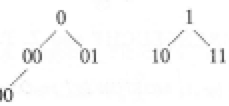
If Φ is in L and $\alpha_1, \alpha_2, \dots, \alpha_k$ is a sequence of all $\alpha \in T$ such that $\Phi(\alpha) \neq \Lambda$, and $\alpha_i R_1 \alpha_j$ for $i < j$ then

$$N_{R_1}(\Phi) = \Phi(\alpha_1), \Phi(\alpha_2), \dots, \Phi(\alpha_k).$$

§ 2. A. If R_1 is the lexicographic ordering of T, then N_{R_1} is the Łukasiewicz notation. For example, let Φ be



All vertices for which $\Phi(a) = \Lambda$ are omitted in the diagram and, of course, the corresponding part of the basic tree looks as follows



Then $N_{R_1}(\Phi) = AAxyzAxx$, because R_1 gives the following ordering of vertices shown in the diagram

$$0,\ 00,\ 000,\ 001,\ 01,\ 1,\ 10,\ 11.$$

B. Let R_2 be the ordering defined as follows:

$a_1 a_2 \dots a_k R_2 b_1 b_2 \dots b_n$ if $k < n$ or $k = n$ and $a_1 \dots a_k Rb_1 \dots b_k$.

(This means that $a_1 \dots a_k$ precedes $b_1 \dots b_n$ in the lexicographic ordering).

For the formula Φ given in the example A

$$N_{R_2}(\Phi) = AAAzxxxy$$

because R_2 gives the following ordering of vertices

$$0,\ 1,\ 00,\ 01,\ 10,\ 11,\ 000,\ 001.$$

§ 3. Let R_0 be an extension of the partial ordering T to a linear ordering. Let us denote by L_{R_0} the range of N_{R_0} (This is the set of well formed formulae in the notation N_{R_0}).

THEOREM 1. $L_{R_0} = L_{R_1}$ *(the set of wff does not depend on chosen notation and is the same as in Łukasiewicz notation*)).*

THEOREM 2. *The function N_{R_0} maps in a one-to-one way the set L onto L_{R_0} (Each formula in the tree form T has a unique representation in L_{R_0} and each formula in L_{R_0} corresponds to exactly one formula in L).*

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REFERENCES

[1] J. Łukasiewicz, *Elementy logiki matematycznej* [in Polish], [*Elements of mathematical logic*], Warszawa, 1958.

*) Speaking informally different "languages" L_{R_0} has the same correct sentences but different grammars.