INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES No. 223

THERMOMECHANICS IN SOLIDS

A SYMPOSIUM
HELD AT CISM, UDINE, IN JULY 1974

EDITED BY

W. NOWACKI
UNIVERSITY OF WARSAW

AND

I.N. SNEDDON
UNIVERSITY OF GLASGOW

SPRINGER - VERLAG WIEN - NEW YORK

THERMAL STRESSES IN ANISOTROPIC BODIES

Witold Nowacki

1. Introduction

Thermoelasticity embraces a wide field of phenomena. It contains the theory of heat conduction and the theory of strain and stresses due to the flow of heat, when coupling of temperature and deformation fields occurs.

The coupling between deformation and temperature fields was first postulated by J.M.C. Duhamel [1], the originator of the theory of thermal stresses who introduced the dilatation term in the equation of thermal conductivity. However, this equation was not well grounded in the thermodynamical sense. Later, an attempt at the thermodynamical justification of this equation was untertaken by Voigt [2] and Jeffreys [3]. However, it was only in 1956 that Biot [4] gave the full justification of the thermal conductivity equation based on the thermodynamics of irreversible processes [5]. Biot also presented the basic methods for solving the thermoelasticity equation as well as formulating a variational approach (for isotropic body).

Thermoelasticity describes a broad range of phenomena. It is a generalisation of the classical theories of elasticity and thermal conductivity, and it is now a fully developed scientific discipline.

The theory of thermoelasticity in homogeneous isotropic bodies has been treated in detail in scientific literature, problems of thermoelasticity in anisotropic bodies however have been dealt with, in only very few publications. This fact is due not only to the mathematical difficulties of the problem, but follows from the lack of wide practical applications. More and more frequently, however, engineering structure contain materials of macroscopically anisotropic structure, (plates, discs, shells, thick-walled pipes etc.) the properties of which (both elastic and thermal) are different in different directions.

In the present paper, which is of a survey character, attention is focused on the foundations of the thermodynamical theory, on the differential equations and the general energy and variational theorems. We shall give here also a review of problems solved so far; the reader interested in details will find them in quoted papers.

2. Fundamental assumptions and relations of linear thermoelasticity

Let a body be at temperature T_0 in an undeformed and unstressed state. This initial state will be called the natural state, in which it is assumed that the entropy of the body is zero. Owing to the action of extermal forces. i.s. body and surface forces, and under the influence of internal heat sources and surface heating or cooling, the body will be subjected to deformation and temperature change.

Displacements $\underline{u}$ will occur in the body and the temperature change can be written as $\theta = T - T_0$, where T is the absolute temperature of a point $\underline{x}$ of the body. The temperature change is accompanied by stresses σ_{ij} and strains ϵ_{ij}. The quantities $\underline{u}$, θ, σ_{ij}, ϵ_{ij} are function of position $\underline{x}$ and time t. We assume that the temperature change $\theta = T - T_0$ accompanying the deformation is small and does not result in significant variations of the elastic and thermal coefficients which will be regarded as independent of T. In addition to the assumption $|\theta / T_0| \ll 1$ we shall assume that second powers and products of the components of strain may be neglected in comparison with the strains ϵ_{ij}. Thus, attention is restricted to the linear regime where the strains are related to the displacements by

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,3, \tag{2.1}$$

and the strains satisfy the six compatibility relations

$$\epsilon_{ij,k\ell} + \epsilon_{k\ell,ij} - \epsilon_{j\ell,ik} - \epsilon_{ik,j\ell} = 0, \quad i,j,k,\ell = 1,2,3. \tag{2.2}$$

The main task is now to determine constitutive equations relating the components of the stress tensor σ_{ij} with the components of the strain tensor ϵ_{ij} and of the temperature θ. Thermoelastic disturbances cannot be described in terms of classical thermodynamics and we have to use the relations of the thermodynamics of irreversible processes [5][6].

The constitutive equations are deduced from thermodynamical considerations, taking into account the principle of energy and entropy balance [4 – 6]

$$\frac{d}{dt}\int_V \left(U + \frac{1}{2}\rho v_i v_i\right) dV = \int_V X_i v_i \, dV + \int_A p_i v_i \, dA - \int_A q_i n_i \, dA \tag{2.3}$$

$$\int_V \frac{dS}{dt} dV = -\int_A \frac{q_i n_i}{T} dA + \int_V \Theta \, dV. \tag{2.4}$$

Here U is internal energy, S is the entropy, X_i the components of the body forces, $p_i = \sigma_{ji} n_j$ the components of the stress vector, q_i the components of the vector of heat flux, n_i the components of the normal to the surface A. Further, $v_i = \partial u_i / \partial t$; and the quantity Θ represents the source of entropy, a quantity always positive in a thermodynamically irreversible process.

$$\Theta = -\frac{q_i T_{,i}}{T^2} > 0. \tag{2.5}$$

The terms on the left-hand side of eq. (2.3) represents the rate of increase of the internal and kinetic energies. The first term on the right-hand side is the rate of increase of the work done by the body forces, and the second the rate of increase of the work done by the surface tractions. Finally, the last term of the right-hand side of eq. (2.3) is the entropy acquired by the body by means of thermal conduction. The left-hand side of eq. (2.4) is the rate of increase of the entropy. The first term on the right-hand side of eq. (2.4) represents the exchange of entropy at the surface, and the second term represents the rate of production of entropy due to heat conduction.

Making use of the equations of motion

$$\sigma_{ji,j} + X_i = \rho \ddot{u}_i \,, \quad i,j = 1,2,3, \tag{2.6}$$

and using the divergence theorem to transform the integrals we arrive at the local relations

$$\dot{U} = \sigma_{ij}\, \dot{\epsilon}_{ij} - q_{i,i} \;, \quad \dot{S} = \Theta - \frac{q_{i,i}}{T} + \frac{q_i\, T_{,i}}{T^2} \;. \tag{2.7}$$

Introducing the Helmholtz free energy $F = U - ST$ and eliminating the quantity $q_{i,i}$ from eqs. (2.7) we obtain

$$\begin{aligned} \dot{F} &= \frac{\partial F}{\partial \epsilon_{ij}}\, \dot{\epsilon}_{ij} + \frac{\partial F}{\partial T}\, \dot{T} \\ &= \sigma_{ij}\, \dot{\epsilon}_{ij} - S\dot{T} - T\left(\Theta + \frac{q_i\, T_{,i}}{T^2}\right) . \end{aligned} \tag{2.8}$$

Assuming that the functions Θ , q_i , σ_{ij} do not explicitly depend on the time derivatives of the functions ϵ_{ij} and T we obtain

$$\sigma_{ij} = \frac{\partial F}{\partial \epsilon_{ij}}, \quad S = -\frac{\partial F}{\partial T} \,, \quad \Theta + \frac{q_i\, T_{,i}}{T^2} = 0. \tag{2.9}$$

Let us now expand the function $F(\epsilon_{ij}, T)$ into a infinite series in the neighbourhood of the natural state $(0\ , T_o)$

$$F(\epsilon_{ij}, T) = \frac{1}{2}\, C_{ijk\ell}\, \epsilon_{ij}\, \epsilon_{k\ell} - \beta_{ij}\, \epsilon_{ij}\, \theta + \frac{m}{2}\, \theta^2 + \dots \tag{2.10}$$

From the expansion $F(\epsilon_{ij}, T)$ we retain only the linear and quadratic terms, confining ourselves to linear relations among stresses σ_{ij} strains ϵ_{ij} and temperature change θ.

Let us now ttake advantage of the expressions (2.9)1,2. We obtain

$$\sigma_{ij} = c_{ijk\ell}\ \epsilon_{k\ell} - \beta_{ij}\ \theta,\ \beta_{ij} = \beta_{ji}, \tag{2.11}$$

$$S = \beta_{ij}\ \epsilon_{ij} + m\theta\,. \tag{2.12}$$

In relations (2.11), we identify Hooke's law generalized for thermoelastic problems. There are called the Duhamel – Neumann relations for an anisotropic body. The components $C_{ijk\ell}$, β_{ij} appropriate to the isothermal state play the role of material constants [7]. In the theory of elasticity of an anisotropic body, the following symmetry properties of the tensor $c_{ijk\ell}$ are proved

$$C_{ijk\ell} = C_{jik\ell}\ ,\quad c_{ijk\ell} = C_{ij\ell k}\ ,\quad c_{ij\ell k} = C_{\ell k ij}\ . \tag{2.13}$$

The first symmetry relation is obtained from the symmetry of the stress tensor σ_{ij}, while the second form the symmetry of strain tensor ϵ_{ij}. The third relation is a consequence of the relations

$$\frac{\partial^2 F}{\partial\epsilon_{ij}\ \partial\epsilon_{k\ell}} = \frac{\partial^2 F}{\partial\epsilon_{k\ell}\,\partial\epsilon_{ij}} \qquad \text{or} \qquad \frac{\partial\sigma_{ij}}{\partial\epsilon_{k\ell}} = \frac{\partial\sigma_{k\ell}}{\partial\epsilon_{ij}}\ ,$$

These relations lead to a reduction in the number of mutually independent constants from 81 to 21 for a body with general anisotropy.

Let us solve the system of eqs. (2.11) for deformations

$$\epsilon_{ij} = s_{ijk\ell}\ \sigma_{k\ell} + \alpha_{ij}\ \theta\,. \tag{2.14}$$

The quantities $s_{ijk\ell}$ (coefficients of elastic susceptibility) then satisfy the symmetry relations

$$s_{ijk\ell} = s_{jik\ell}\ ,\quad s_{ijk\ell} = s_{ij\ell k}\ ,\quad s_{ijk\ell} = s_{k\ell ij}\ . \tag{2.15}$$

Consider now a volume element of the anisotropic body free of stresses on its surface. According to (2.14), we obtain for this element

$$\epsilon^{\circ}_{ij} = \alpha_{ij}\ \theta\,. \tag{2.16}$$

The relation describes the familar physical phenomenon, namely, the proportionality of the element deformation to the increment of temperature θ. The quantities α_{ij} are the coefficients of linear expansion, and it follows from the symmetry of the tensor ϵ_{ij} that α_{ij} is also symmetric. It should be added that the coefficient of volume thermal expression α_{jj} is an invariant.

From relations (2.11) (2.12) and (2.14) we have

$$\left(\frac{\partial\sigma_{ij}}{\partial\epsilon_{k\ell}}\right)_T = C_{ijk\ell} \,, \qquad \left(\frac{\partial\sigma_{ij}}{\partial T}\right) = -\beta_{ij} = \alpha_{k\ell}\; C_{ijk\ell} \,,$$
$$\left(\frac{\partial\epsilon_{ij}}{\partial T}\right)_\epsilon = \alpha_{ij} \,, \qquad \left(\frac{\partial S}{\partial\epsilon_{ij}}\right)_T = \beta_{ij} \,. \tag{2.17}$$

From the thermodynamical considerations we obtain: $m = C_\epsilon / T_o$ and

$$S = \beta_{ij}\,\epsilon_{ij} + \frac{C_\epsilon}{T_o}\,\theta \,, \tag{2.18}$$

where C_ϵ is a specific heat related to unit volume at constant deformation. In the expression for entropy, the first term on the right-hand side is due to the coupling of the deformation and the temperature field, the second term express the entropy caused by the heat flow. The purely elastic term does not appear in this expression.

The postulate of the thermodynamics of irreversible processes will be satisfied if $\Theta > 0$ i.e. when $-q_i T_{,i}/T^2 > 0$. This condition is satisfied by the Fourier law of heat conduction [5]

$$-q_i = k_{ij}\,T_{,j} \quad \text{or} \quad -q_i = k_{ij}\,\theta_{,ij} \,, \quad \theta = T - T_o \,. \tag{2.19}$$

It remains to relate the entropy to the thermal conductivity. Combining the relations (2.19) and

$$T\dot{S} = -\,\mathrm{div}\,\underline{q} = -\,q_{i,i} \tag{2.20}$$

we arrive at the equation

$$T\dot{S} = k_{ij}\,\theta_{,ij} \tag{2.21}$$

From the relations (2.18) and (2.21) we obtain

$$\lambda_{ij}\,T_{,ij} = T\beta_{ij}\,\dot{\epsilon}_{ij} + \frac{C_\epsilon}{T_o}\,T\dot{\theta} \,, \qquad \theta = T - T_o \,. \tag{2.22}$$

We note that the terms on the right-hand side of this equation make it nonlinear. Putting $T = T_o$ on the right-hand side of (2.22) to linearize the equation gives

(2.23) $$\lambda_{ij}\,\theta_{,ij} - C_\epsilon\dot{\theta} - T_o\beta_{ij}\,\dot{\epsilon}_{ij} = 0$$

In this extended equation of thermal conductivity, the term $T_o\beta_{ij}\,\dot{\epsilon}_{ij}$ characterizes the coupling of the deformation and temperature fields. If there are internal sources in the body, we should add to (2.23) the quantity W, which determines the amount of heat produced per unit volume and time

(2.24) $$\lambda_{ij}\,\theta_{,ij} - C_\epsilon\dot{\theta} - T_o\beta_{ij}\,\dot{\epsilon}_{ij} + W = 0.$$

The full set of the differential equations of thermoelasticity comprise the equations of motion and the equations of thermal conductivity. The equations of motion

(2.25) $$\sigma_{ji,j} + X_i = \rho\ddot{u}_i\,(\underline{x},t), \quad \underline{x} \in V, \; t > 0;$$

can be transformed, making use of the Duhamel – Neumann equations

(2.26) $$\sigma_{ij} = C_{ijk\ell}\,\epsilon_{k\ell} - \beta_{ij}\,\theta ,$$

and the strain – displacement relations

(2.27) $$\epsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i})$$

into three equations containing displacements u_i and temperature θ as unknown function

(2.28) $$C_{ijk\ell}\,u_{k,\ell j} + X_i = \rho\ddot{u}_i + \beta_{ij}\,\theta_{,j}\,, \; \underline{x} \in V, \quad t > 0.$$

The above equations and those of thermal conductivity

(2.29) $$\lambda_{ij}\,\theta_{,ij} - C_\epsilon\dot{\theta} - T_o\beta_{ij}\,\dot{\epsilon}_{ij} + W = 0$$

are coupled. Body forces, heat sources, heating and heat flow through the surface result in variations of both displacements and temperature. Boundary conditions of a mechanical type are given in the form of displacements u_i or loadings p_i on the surface A. Thermal conditions can be, in a general way, written in the form

(2.30) $$\alpha\,\frac{\partial\theta}{\partial n} + \beta\theta = f(\underline{x},t), \; \underline{x} \in A, \; t > 0, \qquad \alpha,\beta = \text{const.},$$

determining the heat flow through the surface A. The initial conditions e.q. for $t = 0$, are that the displacement u_i , the velocity of these displacements and temperature are the known functions

$$u_i(\underline{x},0) = f_i(\underline{x}), \quad \dot{u}_i(\underline{x},0) = g(\underline{x}), \quad \theta(\underline{x},0) = h(\underline{x}), \quad \underline{x} \in V, \; t = 0. \tag{2.31}$$

If the variation of body forces, heat sources and heatings is slow, then the inertia terms in the equations of motion can be omitted and the problem can be regarded as quasi-static. The quasi-static equations of thermoelasticity given below are, however, coupled

$$C_{ijk\ell}\, u_{k,\ell j} + X_i = \beta_{ij}\, \theta_{,j} \; , \tag{2.32}$$

$$\lambda_{ij}\, \theta_{,ij} - C_\epsilon \dot{\theta} - T_o \beta_{ij}\, \dot{\epsilon}_{ij} + W = 0 \tag{2.33}$$

Thermoelasticity embraces the following subject previously developed separately: classical elastokinetics and the theories of thermal conduction and thermal stress. We shall determine the differential equations of classical elastokinetics assuming that the motion is adiabatic, i.e. without heat exchange in the body. Since for an adiabatic process $\dot{S} = 0$, eq. (2.18) yields $\dot{\theta} = -T_o / c_\epsilon \beta_{ij}\, \dot{\epsilon}_{ij}$ or, after integrating assuming homogeneous initial conditions:

$$\theta = -\frac{T_o}{c_\epsilon}\, \beta_{ij}\, \epsilon_{ij}. \tag{2.34}$$

This equation replaces the equation of heat conduction. Inserting (2.34) in (2.28), we obtain the displacements equation of classical elastokinetics

$$(c_{ijk\ell})_s\, u_{k\ell j} + X_i = \rho \ddot{u}_i \; , \tag{2.35}$$

where

$$(C_{ijk\ell})_s = (C_{ijk\ell})_T + \frac{(\beta_{ij})_T\, (\beta_{k\ell})_T}{c_\epsilon}\, T_o$$

The quantities $(C_{ijk\ell})_s$ are mechanical constants, measured in adiabatic conditions. The constitutive equations, after substituting (2.34) into (2.11) and (2.14), take the form

$$\sigma_{ij} = (C_{ijk\ell})_s\, \epsilon_{k\ell} \; , \quad \epsilon_{ij} = (s_{ijk\ell})_s\, \sigma_{k\ell} \; , \tag{2.36}$$

where

$$(s_{ijk\ell})_s = (s_{ijk\ell})_T - \frac{\alpha_{ij}\, \alpha_{k\ell}}{C_\sigma}\, T_o \; , \; C_\sigma = C_\epsilon + \alpha_{ij}\, \beta_{ij}\, T_o .$$

In the theory of thermal stresses which considers the influence of surface and internal heating on the deformation and stresses the term $\beta_{ij}\,\dot{\epsilon}_{ij}$ appearing in the thermal conductivity equation is assumed to be negligible. This simplification leads to the following two independent equations

$$C_{ijk\ell}\,u_{k,\ell j} + X_i = \rho\ddot{u}_i + \beta_{ij}\,\theta_{,j} \tag{2.37}$$

$$\lambda_{ij}\,\theta_{,ij} - C_\epsilon\,\dot{\theta} + W = 0 \tag{2.38}$$

The temperature θ is determined from (2.38) i.e. from the classical equation of thermal conductivity. When we know the temperature distribution, we are able to determine the displacements from eq. (2.37).

In the case of stationary heat flow, the production of entropy is compensated by the exchange of entropy with the environment. This exchange is negative and its absolute value is equal to the entropy production in the body. In the equations of thermoelasticity (2.38) (2.29) the derivatives with respect to time disappear and eq. (2.28) becomes

$$C_{ijk\ell}\,u_{k\ell j} + X_i = \beta_{ij}\,\theta_{,j}\ . \tag{2.39}$$

The temperature θ appearing in these equations is a know function, obtained by solving the heat conduction equation in the case of a stationary flow of heat

$$\lambda_{ij}\,\theta_{,ij} + W = 0. \tag{2.40}$$

3. The fundamental theorems for anisotropic bodies.

An important part is played in classical elasticity by variational theorems, which consider either variation of the deformation state or variation of the stress state. In what follows we shall present the thermoelastic (coupled) variational theorem for the deformation state, devised by M.A. Biot [4]. This theorem consist of two parts, the first of which utilizes the familiar d'Alembert principle of the elasticity theory

$$\int_V (X_i - \rho\ddot{u}_i)\,\delta u_i\,dV + \int_A p_i\,\delta u_i\,dA = \int_V \sigma_{ij}\,\delta\epsilon_{ij}\,dV. \tag{3.1}$$

In this equation δu_i are the virtual increments of displacement and $\delta\epsilon_{ij}$ the virtual increments of deformation, each assumed to be arbitrary continuous function, and complying with the conditions contraining the body motion. Supplementing the eq. (3.1) with the constitutive equations (2.11), we obtain

$$\int_V (X_i - \rho \ddot{u}_i)\delta u_i \, dV + \int_A p_i \, \delta u_i \, dA = \delta W - \int_V \beta_{ij} \, \delta\epsilon_{ij} \, \theta dV \,, \qquad (3.2)$$

where

$$W = \frac{1}{2}\int_V C_{ijk\ell}\,\epsilon_{ij}\,\epsilon_{k\ell}\,dV. \qquad (3.3)$$

The second part of the variational theorem uses the laws governing the heat flow so we utilize the expressions interrelating heat flow, temperature and entropy

$$q_i = -\lambda_{ij}\,\theta_{,j}\,,\; T_o\dot{S} \cong -q_{i,i} = \lambda_{ij}\,\theta_{,ij}\,,\; \dot{S}T_o = \beta_{ij}\,\dot{\epsilon}_{ij}\,T_o + C_\epsilon\dot{\theta}. \qquad (3.4)$$

These relations can be written more conveniently by introducing the vector function $\underline{H}$, related to entropy and heat flow by

$$S = -H_{i,i} \quad , \quad q_i = T_o\dot{H}_i \;. \qquad (3.5)$$

Since

$$q_i = T_o\dot{H}_i = -k_{ij}\,\theta_{,j} \qquad (3.6)$$

and solving these relations with respect to $\theta_{,i}$, we obtain

$$\theta_{,i} = -T_o\lambda_{ij}\,\dot{H}_j \;, \qquad (3.7)$$

where λ_{ij} is the matrix inverse to the matrix of the coefficients k_{ij} ; $\lambda_{ij} = [k_{ij}]^{-1}$. Multiplying eqs. (3.7) by δH_i integrating over the volume of the body

$$\int_V (\theta_{,i} + T_o\lambda_{ij}\,\dot{H}_j)\delta H_i \, dV = 0, \qquad (3.8)$$

finally, with use of eq. (3.4)2, we obtain

$$\int_A \theta\delta H_n \, dA + \frac{C_\epsilon}{T_o}\int_V \theta\delta\theta dV + \beta_{ij}\int_V \theta\delta\epsilon_{ij}\,dV + T_o\int_V \lambda_{ij}\,\dot{H}_j\,\delta H_i\,dV = 0. \qquad (3.9)$$

Introducing the notations

$$P = \frac{C_\epsilon}{2T_o}\int_V \theta^2 dV, \qquad \delta D = T_o\int_V \lambda_{ij}\,\dot{H}_j\,\delta H_i\,dV$$

where the functions P and D are called the thermal potential and the dissipation function respectively, and eliminating the integral $\beta_{ij}\int_V \theta\delta\epsilon_{ij}\,dV$ from the equations (3.2) and (3.9), we finally arrive at the required variational principle

$$\delta(W + P + D) = \delta\mathcal{L} - \int_A \theta\delta H_n\, dA, \tag{3.10}$$

where

$$\delta\mathcal{L} = \int_V (X_i - \rho\ddot{u}_i)\delta u_i\, dV + \int_A p_i\,\delta u_i\, dA. \tag{3.11}$$

This is the principle of virtual work for variations of the displacements and the temperature, extended to the problem of thermoelasticity of anisotropic bodies.

a) Consider the particular case when we assume $\theta = -T_o/C_\epsilon\beta_{ij}\,\epsilon_{ij}$, corresponding to the assumption of an adiabatic process.
Then (3.2) transforms into

$$\begin{gathered}\int_V (X_i - \rho\ddot{u}_i)\delta u_i\, dV + \int_A p\delta u_i\, dA = \delta W_s\\ W_s = \int_V (C_{ijk\ell})_s\,\epsilon_{ij}\,\epsilon_{k\ell}\, dV.\end{gathered} \tag{3.12}$$

Equation (3.12) constitutes d'Alembert's principle for classical elastokinetics.

b) In the theory of thermal stresses the coupling between the strain and temperature fields is neglected, by neglecting in eq. (2.23) the term $T_o\beta_{ij}\,\dot{\epsilon}_{ij}$. Thus we arrive at two independent variational equations

$$\int_V (X_i - \rho\ddot{u}_i)\delta u_i\, dV + \int_A p_i\,\delta u_i\, dA + \beta_{ij}\int_V \theta\delta\epsilon_{ij}\, dV = \delta W, \tag{3.13}$$

$$\delta(P + D) + \int_A \theta\delta H_n\, dA = 0 \tag{3.14}$$

Eq. (3.14) constitutes variational theorem for the classical heat conduction problem. In eq. (3.13) the temperature is regarded as known, calculated by means of the classical heat conduction equation.

Let us return to the general variational theorem of thermoelasticity, eq. (3.10) and assume that the variational increments δu_i, $\delta\epsilon_{ij}$, δH_i e.t.c. coincide with the increment occurring when the process passes from a time instant t to t + dt. Then

$$\delta u_i = \frac{\partial u_i}{\partial t}\, dt = v_i\, dt, \quad \delta H_i = \dot{H}_i\, dt = -\frac{k_{ij}}{T_o}\,\theta_{,j}\,, \quad \delta W = \dot{W}dt, \text{ e.t.c.}$$

From (3.10) we obtain

$$\frac{d}{dt}(W+K+P) + X_T = \int_V X_i v_i \, dV + \int_A p_i v_i \, dA + \frac{1}{T_o}\int_V k_{ij}\, \theta\theta_{,i} n_j \, dA,$$

where

$$X_T = \frac{1}{T_o}\int_V k_{ij}\, \theta_{,i}\, \theta_{,j}\, dV, \qquad K = \frac{\rho}{2}\int_V v_i v_i \, dV.$$

Observe that the right-hand side contains the causes producing the motion of the body, i.e. the body forces X_i, surface tractions p_i and the thermal boundary data. Furthermore the integrand $k_{ij}\ \theta_{,i}\,\theta_{,j}$ appearing in the function X_i is a positive definite quadratic function.

The energy theorem is used in proving the uniqueness of the solution of the fundamental thermoelasticity differential equations. The proof of uniqueness has been given by V. Ionescu Cazimir [8] for the boundary conditions in displacements or loadings on A.

The proof of uniqueness can also be extended to the case when on a part of the boundary A_σ tractions are given, while on the remaining part displacements u_i are known. Similarly, different thermal conditions can be taken on A_σ and A_u . A detailed exposition of this case as well as another one concerning a discontinuity of stresses on the surface A_σ inside the region V + A is given in the paper by V. Ionescu – Cazimir [8].

One on must interesting theorems of thermoelasticity is the reciprocity theorem. The extended reciprocity theorem is thermoelasticity has been formulated by V. Ionescu – Cazimir [9].

Consider two systems of causes and effects. In the first there act the body forces X_i , surface tractions p_i , heat sources W and surface heatings h. These causes produce in the anisotropic body the displacement u_i and the temperature θ. The causes and effects on the second system will be denoted by "primes".

We base on the Duhamel – Neumann relations, to which the Laplace transform has been applied. From these relations we obtain

$$\int_V (\bar{\sigma}_{ij} + \beta_{ij}\,\bar{\theta})\bar{\epsilon}'_{ij}\, dV = \int_V (\bar{\sigma}'_{ij} + \beta_{ij}\,\bar{\theta}')\bar{\epsilon}_{ij}\, dV. \qquad (3.16)$$

With the help of the equation of motion we transfrom eq. (3.16) to the form

$$\int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\, dV + \int_A (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\, dA + \beta_{ij}\int_V (\bar{\epsilon}'_{ij}\,\bar{\theta} - \bar{\epsilon}_{ij}\,\bar{\theta}')\, dV = 0. \qquad (3.17)$$

We assumed here the homogeneous initial conditions. Eq. (3.17) constitutes the first part of the reciprocity theorem. The second part is derived on the basis of the heat conduction equation. We obtain

$$\int_V k_{ij}(\bar{\theta}_{,ij}\ \bar{\theta}' - \bar{\theta}'_{,ij}\ \bar{\theta})\,dV - T_0 p \int_V \beta_{ij}(\bar{\epsilon}_{ij}\ \bar{\theta}' - \bar{\epsilon}'_{ij}\ \bar{\theta})\,dV + \int_V (\bar{W}\bar{\theta}' - \bar{W}'\bar{\theta})\,dV = 0 \qquad (3.18')$$

or

$$-T_0 p \int_V \beta_{ij}(\bar{\epsilon}_{ij}\ \bar{\theta}' - \bar{\epsilon}'_{ij}\ \bar{\theta})\,dV + \int_V (\bar{W}\bar{\theta}' - \bar{W}'\bar{\theta})\,dV + k_{ij}\int_A (\bar{h}'\bar{\theta}_{,i}\ \bar{h}\bar{\theta}'_{,i})n_j\,dA = 0 \qquad (3.18'')$$

We assumed here the homogeneous initial conditions for temperature and the following boundary conditions

$$(3.19')\ \theta(\underline{x},t) = h(\underline{x},t), \quad \theta'(\underline{x},t) = h'(\underline{x},t),\ \underline{x} \in A, \quad t > 0.$$

Eq. (3.18) constitutes the second part of the reciprocity theorem. If we eliminate from eqs. (3.17) and (3.18") the common term, we arrive after inverting the Laplace transform at an equation containing all causes and effects. We have

$$(3.19'')\quad T_0\int_A (p_i \ominus u'_i - p'_i \ominus u_i)\,dA + T_0\int_V (X_i \ominus u'_i - X'_i \ominus u_i)\,dV(\underline{x}) = \int_V (W*\theta' - W'*\theta)\,dV + \int_A (h'*\theta_{,i} - h*\theta'_{,i})\,dA,$$

where

$$p_i \ominus u'_i = \int_0^t p_i(\underline{x}, t-\tau)\,\frac{\partial u_i(\underline{x},\tau)}{\partial\tau}\,d\tau,$$

$$W*\theta' = \int_0^t W(\underline{x}, t-\tau)\theta'(\underline{x},\tau)\,d\tau, \quad \text{e.t.c.}$$

Consider now particular cases of the reciprocity theorem.

a) Assuming that there are no heat sources and no heat exchange between volume elements. We have for this adiabatic process

$$(3.20)\qquad \theta = -\frac{T_0}{C_\epsilon}\beta_{ij}\,\epsilon_{ij}, \qquad \theta' = -\frac{T_0}{C_\epsilon}\beta_{ij}\,\epsilon'_{ij}.$$

We obtain the reciprocity theorem in the form

$$(3.21)\qquad \int_A (p_i * u'_i - p'_i * u_i)\,dA + \int_V (X_i * u'_i - X'_i * u_i)\,dV = 0.$$

b) If we treat the dynamical thermoelasticity problem as uncoupled and apply the approximate theory of thermal stresses (neglecting the term $\beta_{ij}\dot{\epsilon}_{ij}$, in the heat conduction equation!). We obtain

$$\int_A (p_i * u_i' - p_i' * u_i)\,dA + \int_V (X_i * u_i' - X_i' * u_i)\,dV + + \beta_{ij} \int_V (\theta * \epsilon_{ij}' - \theta' * \epsilon_{ij})\,dV = 0 \quad (3.22)$$

$$\int_V (W * \theta' - W' * \theta)\,dV + k_{ij} \int_A (h' * \theta_{,i} - h * \theta'_{,i})\,n_j\,dA = 0. \quad (3.23)$$

c)In the case of the static problem with a stationary temperature field the reciprocity theorem takes the simpler form [10].

$$\int_A (p_i u_i' - p_i' u_i)\,dA + \int_V (X_i u_i' - X_i' u_i)\,dV + \beta_{ij} \int_V (\theta \epsilon_{ij}' - \theta' \epsilon_{ij})\,dV = 0 \quad (3.24)$$

$$\int_V (W \cdot \theta' - W' \theta)\,dV + k_{ij} \int_A (h' \theta_{,i} - h\theta'_{,i})\,n_j\,dA = 0. \quad (3.25)$$

d)Consider finally the particular case of eq. (3.24) when

$$X_i = X_i' = 0, \quad p_i = 0, \quad p_i' = 1 n_i, \quad \sigma_{ji}' = 1\delta_{ij}.$$

We assume that the body is simply-connected, free of body force and its deformation is produced by heating and an action of heat sources. In the primed system we assume the isothermal state and homogeneous extension of the body. Eq. (3.24) is then considerably simplified, namely

$$\int_A n_i u_i\,dA = \beta_{ij} \int_V \theta \epsilon_{ij}'\,dV. \quad (3.26)$$

The first integral represents the volume increment of the body. Therefore [10]

$$\Delta V = \alpha_{ij} \int_V \theta \sigma_{ij}'\,dV$$

or

$$\Delta V = \alpha_{jj} \int_V \theta\,dV. \quad (3.27)$$

The volume increment is equal to the integral of the temperature over the region V, multiplied by the invariant α_{jj}. From (2.14) and (3.27) we obtain

$$\Delta V = \int_V \epsilon_{jj}\,dV = \alpha_{jj} \int_V \theta\,dV + s_{jjk\ell} \int_V \sigma_{k\ell}\,dV \quad (3.28)$$

or

$$s_{jjk\ell} \int_V \sigma_{k\ell}\,dV = 0$$

For the isotropic body, we have [10]

$$\Delta V = 3\alpha_t \int_V \theta \, dV, \qquad \int_V \sigma_{kk} \, dV = 0 \tag{3.29}$$

e). Let the anisotropic body contained within the region V and bounded by the surface A be subjected to heating. Let on the part A_u of the surface A, equal to zero, appear displacements u_i and on the part A_σ of the surface A, equal to zero, appear tractions $\underline{p}$. Moreover let $X_i = 0$.

For the determining the displacement $u_i(\underline{x})$, $\underline{x} \in V$ consider a body of the same shape and with the same boundary conditions.

In this body let $\theta' = 0$ and let a concentrated force $X_i' = \delta(\underline{x} - \underline{\xi})\delta_{ik}$ be acting at the point $\underline{\xi}$ which is, consequently directed along the axis x_k.

This force will cause displacements $u_i' = U_i^{(k)}(\underline{x}, \underline{\xi})$ assuming that the functions $U_i^{(k)}(\underline{x}, \underline{\xi})$ are selected such as to satisfy homogeneous boundary conditions on A_σ and A_u.

Making use of formula (3.24) we obtain

$$- \int_V \delta(\underline{x} - \underline{\xi})\delta_{ik} u_i(\underline{x}) dV(\underline{x}) + \beta_{ij} \int_V \theta(\underline{x}) \epsilon_{ij}^{(k)}(\underline{x}, \underline{\xi}) dV(\underline{x}) = 0. \tag{3.30}$$

As a result we obtain the following formula

$$u_k(\underline{\xi}) = \frac{1}{2} \beta_{ij} \int_V \theta(\underline{x}) \left[\frac{\partial U_i^{(k)}(\underline{x}, \underline{\xi})}{\partial x_j} + \frac{\partial U_j^{(k)}(\underline{x}, \underline{\xi})}{\partial x_i} \right] dV(\underline{x}). \tag{3.31}$$

The formula (3.31) may be treated as a generalization of Maysel's formula [11]. For the isotropic body, we obtain

$$u_k(\underline{\xi}) = \gamma \int_V \theta(\underline{x}) \frac{\partial U_j^{(k)}(\underline{x}, \underline{\xi})}{\partial x_j} dV(\underline{x}). \tag{3.32}$$

Here $U_{j,j}^{(k)}(\underline{x}, \underline{\xi})$ should be treated as a dilatation caused at the point $\underline{\xi}$ by a concentrated force X_i applied at the point $\underline{x}$.

5. Three-dimensional problems in anisotropic thermoelasticity

Let us consider the non-coupled quasi-static problem of thermoelasticity. We write the equation of heat conduction

$$\lambda_{ij} \theta_{,ij} - c_\varepsilon \dot{\theta} = - W, \tag{4.1}$$

and the displacement equations

$$C_{ijk\ell}\ u_{k,\ell j} = \beta_{ij}\ \theta_{,j} \tag{4.2}$$

in the following form

$$L_{ij}\ u_j = -W\delta_{4i} \qquad i = 1,2,3,4, \quad u_4 = \theta. \tag{4.3}$$

The L_{ij} are certain linear differential operators of first and second order. The operators $L_{ij} = L_{ji}$ $(i,j = 1,2,3)$ are associated with he displacements equation. We have further

$$L_{i4} = -\beta_{ij}\partial_j\,, \quad L_{4i} = 0, \quad L_{44} = \lambda_{ij}\partial_i\partial_j - c_\epsilon\partial_t\,, \quad i = 1,2,3.$$

Let us express the function u_i by means of four functions X_i $(i = 1,2,3,4)$ as follows [12]:

$$u_1 = \begin{vmatrix} X_1 & L_{12} & L_{13} & L_{14} \\ X_2 & L_{22} & L_{23} & L_{24} \\ X_3 & L_{32} & L_{33} & L_{34} \\ X_4 & 0 & 0 & L_{44} \end{vmatrix}, \quad u_2 = \begin{vmatrix} L_{11} & X_1 & L_{13} & L_{14} \\ L_{21} & X_2 & L_{23} & L_{24} \\ L_{31} & X_3 & L_{33} & L_{34} \\ 0 & X_4 & 0 & L_{44} \end{vmatrix}, \text{ e.t.c.} \tag{4.4}$$

The functions X_i should satisfy the equations

$$\begin{vmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ 0 & 0 & 0 & L_{44} \end{vmatrix} X_i = -W\delta_{i4}\,, \tag{4.5'}$$

or

$$\| L_{ij} \| X_i = -W\delta_{i4}\,, \quad i,j = 1,2,3,4. \tag{4.5''}$$

The functions X_i can be regarded as Galerkin's functions, generalized to the case of anisotropic thermoelasticity. The Galerkin method has been used to solving some thermoelastic problem in a simply anisotropic body, the transversely isotropic body [13].

The system of coordinates will be assumed in such a way that the three planes coincide with those of elastic symmetry. Denote by E, ν Young's modulus and Poisson's ratio in the direction x_1, x_2 and by E', ν' the same quantities for the direction x_3. Let α. λ denote the coefficients of thermal expansion and heat transfer, respectively in the direction x_1, x_2 and α', λ' the same quantities in the direction x_3. The constitutive equations have the form (*)

$$(4.6)\qquad \begin{aligned} \sigma_{11} &= c_{11}\epsilon_{11} + c_{12}\epsilon_{22} + c_{13}\epsilon_{33} - \beta\theta, \quad \sigma_{23} = 2c_{44}\epsilon_{23}, \\ \sigma_{22} &= c_{12}\epsilon_{11} + c_{11}\epsilon_{22} + c_{13}\epsilon_{33} - \beta\theta, \quad \sigma_{31} = 2c_{44}\epsilon_{31}, \\ \sigma_{33} &= c_{13}\epsilon_{11} + c_{13}\epsilon_{22} + c_{33}\epsilon_{33} - \beta'\theta, \quad \sigma_{12} = 2c_{66}\epsilon_{12}, \end{aligned}$$

where the material constants $c_{11}, \ldots, c_{44}$ are expressed throught the four quantities E, E', ν, ν'; and β, β' through the quantities E, E', α, α', ν, ν'. Introducing (4.6) into the equilibrium equations, we obtain the system of equations (4.5''). The quantities L_{ij} have the form

$$(4.7)\qquad \left\{ \begin{aligned} L_{11} &= c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2, \\ L_{22} &= c_{66}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2, \\ L_{33} &= c_{44}(\partial_1^2 + \partial_2^2) + c_{33}\partial_3^2, \\ L_{23} &= L_{32} = (c_{13} + c_{44})\partial_2\partial_3, \\ L_{31} &= L_{13} = (c_{13} + c_{44})\partial_1\partial_3, \\ L_{12} &= L_{21} = (c_{12} + c_{66})\partial_1\partial_2, \\ L_{14} &= -\beta\partial_1, \quad L_{24} = -\beta\partial_2, \quad L_{34} = -\beta'\partial_3, \\ L_{41} &= L_{42} = L_{43} = 0, \quad L_{44} = \lambda(\partial_1^2 + \partial_2^2) + \lambda'\partial_3^2 - C_e\partial_t. \end{aligned} \right.$$

Introducing (4.7) into (4.5') and performing the operations indicated in (4.5) we can represent that equation in the form

$$(4.8)\qquad \begin{aligned} &\lambda' c_{33} c_{44}^2 (\mu_1^2\nabla_1^2 + \partial_3^2)(\mu_3^2\nabla_1^2 + \partial_3^2) \times \\ &\times (\mu_5^2\nabla_1^2 + \partial_3^2 - \sigma^2\partial_t)(\mu_7^2\nabla_1^2 + \partial_3^2)X_i = -W\delta_{4i} \end{aligned} \qquad i = 1,2,3,4,$$

(*) The linear elasticities are components of a positive definite fourth-order tensor. Necessary and sufficient conditions for the satisfaction of the latter requirement are

$$c_{11} > 0,\ c_{11} > c_{12},\ c_{11}^2 > c_{12}^2,\ c_{44} > 0, \quad c_{33}(c_{11} + c_{12}) > 2c_{13}^2$$

where

$$\mu_{1,3}^2 = \begin{cases} \epsilon^2\left(\rho \pm \sqrt{\rho^2 - 1}\right) & \text{for } \rho > 1 , \\ \epsilon^2 & \text{for } \rho = 1 , \\ \epsilon^2\left(\sqrt{\frac{1+\rho}{2}} \pm i\sqrt{\frac{1-\rho}{2}}\right)^2 & \text{for } \rho < 1 , \end{cases}$$

$$\mu_5^2 = \frac{\lambda}{\lambda'} , \quad \mu_7^2 = \frac{c_{66}}{c_{44}} , \quad \epsilon^4 = \frac{c_{11}}{c_{33}} , \quad \sigma^2 = \frac{C_\epsilon}{\lambda'} , \tag{4.9}$$

$$\rho = \frac{c_{11}\, c_{33} - 2c_{13}\, c_{44} - c_{13}^2}{2c_{44}\,(c_{11}\, c_{33})^{1/2}} , \quad \nabla_1^2 = \partial_1^2 + \partial_2^2 .$$

For the solution of solved problems, two function χ_3 and χ_4 were sufficient. Since in the expressions for u_i , the operator $\mu_7^2 \nabla_1^2 + \partial_3^2$ will appear, therefore we assume that

$$\begin{cases} \varphi = c_{44}\,(\mu_7^2 \nabla_1^2 + \partial_3^2)(\mu_3^2 \nabla_1^2 + \partial_3^2 - \sigma^2 \partial_t)\chi_3 , \\ \psi = c_{44}\,(\mu_7^2 \nabla_1^2 + \partial_3^2)\chi_4 . \end{cases} \tag{4.10}$$

The functions ψ and φ satisfies the equations

$$\lambda' c_{33}\, c_{44}\,(\mu_1^2 \nabla_1^2 + \partial_3^2)(\mu_3^2 \nabla_1^2 + \partial_3^2)(\mu_5^2 \nabla_1^2 + \partial_3^2 - \sigma^2 \partial_t)\psi = - W , \tag{4.11}$$

$$(\mu_1^2 \nabla_1^2 + \partial_3^2)(\mu_3^2 \nabla_1^2 + \partial_3^2)\varphi = 0. \tag{4.12}$$

It is easy to see that the function φ is the Galerkin function generalized to the case of transverse isotropy [14 – 15].

In the case of axially symmetric problems, it will be more convenient to use displacements and stresses expressed in cylindrical coordinates. Introducing the notation $\nabla_r^2 = \partial^2/\partial r^2 + 1/r\, \partial/\partial r$ we obtain the differential equations for the function φ and ψ, substituting in the place of ∇_1^2 in the Eqs. (4.10) (4.11) the operator ∇_r^2 .

The displacements and the temperature are given by the equations

$$u_r = \beta c_{44} \frac{\partial}{\partial r}(\nabla_r^2 + a\partial_z^2)\psi - c_{44}\, s\, \partial_r \partial_z \varphi , \quad u_\varphi = 0 ,$$

$$W = \beta c_{44} \frac{\partial}{\partial z}(b\nabla_r^2 + c\partial_z^2)\psi + c_{44}\,(t\nabla_r^2 + \partial_z^2)\varphi , \tag{4.13}$$

$$\theta = c_{33}\, c_{44}\,(\mu_1^2 \nabla_r^2 + \partial_z^2)(\mu_3^2 \nabla_r^2 + \partial_z^2)\psi , \quad r = (x_1^2 + x_2^2)^{1/2}, \; z = x_3 ,$$

where

$$a = \eta - \varkappa(1+\gamma\eta), \quad b = \mu_1^2\mu_3^2\,\varkappa\eta - \gamma\eta - 1, \quad \gamma = \frac{a_{13}}{a_{33}}$$

$$c = \varkappa = \beta'/_{\beta} \quad , \quad s = 1+\gamma\eta \ , \quad t = \mu_1^3\mu_3^2\eta, \quad \eta = \frac{\mu_1^2 + \mu_3^2 + 2\gamma}{\mu_1^2\mu_3^2 - \gamma^2} = \frac{a_{33}}{a_{44}} \ ,$$

The assumption of the functions φ and ψ suffices for the determination of thermal stresses in a simple system: an infinite space, a semi-space and an elastic layer. On the other-hand, for the solution of the state of stress in thick circular and rectangular plates, we should, to satisfy all the boundary conditions, take for the solution, besides the function φ and ψ, also the functions χ_1 and χ_2. With the help of the two functions φ and ψ a few problems have been solved.

The problem was solved in closed form derived for thermal stresses produced by heat sources in an infinite elastic space and an elastic semi-space, for various static and thermal boundary conditions. Similar solutions were given in the case of an action of a nucleus of thermoelastic strain. The case of stationary heating of an elastic semi space and a layer was also examined. Further it was proved that the stresses the vector of which is perpendicular to the plane bounding the semi-space do not vanish, which was the case in the problem of Sternberg and Mac Dowell [16]. Finally, solutions were given for a few quasi-static problems concerning an action of an instantaneous heat source in an elastic space and a semi-space.

In the case of an infinite elastic solid, to although it is possible to obtain a formal solution of the system of equations (4.1) and (4.2) by applying the quadruple Fourier integral transforms, it has not been possible so far to obtain the solution in a form suitable for calculations of a three-dimensional problem of either general anisotropy or orthogonal anisotropy (orthotropy) [17]. Only in the case of a transversally isotropic body we obtain the results in closed form.

We return to the displacement equations (4.2). A solution of equations (4.2) can be represented in the form of a sum, the first component $\bar{u}_i$ satisfying the non homogeneous system of equations (4.2), while the second component $\bar{\bar{u}}_i$ satisfies the homogeneous system

$$L_{ij}\,\bar{\bar{u}}_j = 0, \qquad i = 1,2,3. \tag{4.14}$$

$\bar{\bar{u}}_i$ can be represented in terms of the three functions $\eta_i\,(i = 1,2,3,)$ these functions satisfy the homogeneous equation

$$\| L_{ij} \|\, \eta_j = 0, \quad i,j = 1,2,3. \tag{4.15}$$

The particular solution $\bar{u}_i$ for the isotropic body can be devised by the method presented by Goodier [18], who introduced the so-called thermoelastic displacement potential ϕ according to the relation $\bar{u}_i = \phi_{,i}$.

In the case of transversely isotropic body Borş [19] introduced three thermoelastic displacement potentials ϕ_i (i= 1,2,3) according to the relations

$$\bar{u}_1 = \partial_1(\phi_1 + \phi_2) - \partial_2\phi_3, \quad \bar{u}_2 = \partial_2(\phi_1 + \phi_2) + \partial_1\phi_3, \quad \bar{u}_3 = \partial_3(k_1\phi_1 + k_2\phi_2). \tag{4.16}$$

Introducing (4.16) into the system of equations (4.2), we obtain the following three differential equations

$$(\nabla_1^2 + a_1^2\partial_3^2)\phi_1 = A_1\theta, \quad (\nabla_1^2 + a_2^2\partial_3^2)\phi_2 = A_2\theta, \quad (\nabla_1^2 + \hat{\beta}^2\partial_3^2)\phi_3 = 0, \quad \hat{\beta}^2 = \frac{c_{44}}{c_{66}}, \quad \nabla_1^2 = \partial_1^2 + \partial_2^2, \tag{4.17}$$

where

$$A_1 = \frac{[c_{13} + c_{44}(1 + k_1)]\beta - c_{11}\beta'}{c_{11}c_{44}(k_1 - k_2)}$$

$$A_2 = \frac{[c_{13} + c_{44}(1 + k_2)]\beta - c_{11}\beta'}{c_{11}c_{44}(k_1 - k_2)} .$$

We obtain the roots k_1, k_2 from the equation

$$c_{44}(c_{13} + c_{44})k^2 + [(c_{13} + c_{44})^2 + c_{44}^2 - c_{11}c_{33}]k + (c_{13} + c_{44})c_{44} = 0. \tag{4.18}$$

The quantities a_1^2, a_2^2 are given by the formulae

$$a_\alpha^2 = \frac{c_{33}k_\alpha}{c_{13} + c_{44}(1 + k_\alpha)} = \frac{c_{44} + (c_{13} + c_{44})k_\alpha}{c_{11}}, \quad \alpha = 1,2. \tag{4.19}$$

For a bounded body the functions ϕ_i (i=1,2,3) satisfy a part of the boundary conditions. Therefore the incomplete solution $\bar{u}_i$ must be completed by a solution $\bar{\bar{u}}_i$ of the system of homogeneous equations (4.14). The functions $\bar{\bar{u}}_i$ must so be chosen that the final solution $u_i = \bar{u}_i + \bar{\bar{u}}_i$ satisfies all the boundary conditions of the problem. The system of equations (4.14) can be solved means of the methods, indicated by Elliot [20], Hu-Hai-Chang [15] and Nowacki [10].

We now give the formule for the displacement $\bar{u}_i$ for the axi-symmetrical case (21). Introducing two displacement potentials $\phi_\alpha (\alpha = 1,2)$, we obtain

$$(4.20) \qquad u_r = \frac{\partial}{\partial r}(\phi_1 + \phi_2), \quad u_z = \frac{\partial}{\partial z}(k_1\phi_1 + k_2\phi_2).$$

Introducing the above relations into the equations (4.2), written for transversally isotropic body in cylindrical coordinate system, we arrive at two equations

$$(4.21) \qquad (\nabla_r^2 + a_1^2\partial_z^2)\phi_1 = B_1\theta, \qquad (\nabla_r^2 + a_2^2\partial_z^2)\phi_2 = B_2\theta,$$

where

$$B_1 = \frac{\beta[(c_{13} + c_{44}) + c_{44}k_2] - \beta' c_{11}}{c_{11}\, c_{44}\, (k_2 - k_1)},$$

$$B_2 = -\frac{\beta[(c_{13} + c_{44}) + c_{44}k_1] - \beta' c_{11}}{c_{11}\, c_{44}\, (k_2 - k_1)}, \quad \nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

The quantities k_1, k_2 and μ_α^2 ($\alpha = 1,2$) can be calculated by means of the formulae (4.18) (4.19). A number of problems on axisymmetric thermal stresses in a semi-space of transversal isotropy were solved by Artar Singh [22] who employed two displacement functions.

A different methods of solving the problem of stationary and quasi-static thermal stresses in bodies of transversal isotropy have been developed by Grindei [23] and Sharma [24]. Sharma investigated thermal stresses due to the heating of the plane bounding an elastic semi-space; he deduced the solution by introducing two stress functions, which satisfy a differential equation of second order.

5. Plane problems of anisotropic thermoelasticity

Two-dimensional problems have been dealt with fairly extensively. Thus W.H. Pell [27] examined the problem of simultaneous bending and compression of an anisotropic plate, produced by a stationary field varying linearly with the thickness of the plate; in particular he investigated in detail the circular plate.

Mossakowski [22] applied the complex variable method to derive a number of solutions for the action of a heat source in a semi-infinite disc of isogonal anisotropy. It is convenient here to introduce a stress function analogous to Airy's function in isotropic discs [28]. A method of solution for orthotropic discs making use of the type of function of Airy and Marguerré was presented for static and dynamics problems by P.P. Teodorescu [29].

For the orthotropic plate we denote by E_1 and E_2 the Young's moduli in the direction of x_1 and x_2 axis, respectively, by ν–Poisson's ratio and by G the shear modulus. Finally α_1 and α_2 denote the coefficients of thermal expansion and λ_1, λ_2 coefficients of thermal conductivity in the directions of x_i and x_2 axes, respectively.

The heat equation for an orthotropic plate has the form

$$(\lambda_1 \partial_1^2 + \lambda_2 \partial_2^2 - C_e \partial_t)\theta = - W. \tag{5.1}$$

The relations between stress, strain and temperature in the plane state of stress are

$$\begin{aligned} \epsilon_{11} &= a_{11}\,\sigma_{11} + a_{12}\,\sigma_{12} + \alpha_1\theta , \\ \epsilon_{22} &= a_{21}\,\sigma_{11} + a_{22}\,\sigma_{22} + \alpha_2\theta , \\ \epsilon_{12} &= a_{66}\,\sigma_{12} , \quad a_{11} = \frac{1}{E_1} , \quad a_{22} = \frac{1}{E_2} \quad a_{12} = a_{21} = -\frac{\nu}{E_1} , \quad a_{66} = \frac{1}{2\theta} . \end{aligned} \tag{5.2}$$

Substituting the strains into the compatibility relation

$$\partial_1^2\epsilon_{22} + \partial_2^2\epsilon_{11} = 2\partial_1\partial_2\epsilon_{12} , \tag{5.3}$$

and expressing the stresses by means of the Airy function

$$\sigma_{\alpha\beta} = - \partial_\alpha\partial_\beta F + \delta_{\alpha\beta}\nabla^2 F, \quad \alpha,\beta = 1,2 \tag{5.4}$$

we obtain the differential equation

$$\begin{aligned} \varkappa^4\partial_1^4 F + 2\eta\,\varkappa^2\partial_1^2\partial_2^2 F + \partial_2^4 F &= - E_1(\alpha_1\partial_2^2 + \alpha_2\partial_1^2)\theta , \\ \varkappa^4 = \frac{E_1}{E_2}, \quad 2\varkappa^2\eta &= E_1\left(\frac{1}{G} - \frac{2\nu}{E_1}\right) . \end{aligned} \tag{5.5}$$

Let us write the solution of eq. (5.5) as the sum of two components $\bar{F}$ and $\bar{\bar{F}}$ where $\bar{\bar{F}}$ is the particular integral of eq. (5.5) and $\bar{\bar{F}}$ satisfies the homogeneous quasi-biharmonic equation

$$\varkappa^4\partial_1^4\bar{\bar{F}} + 2\eta\varkappa^2\,\partial_1^2\partial_2^2\bar{\bar{F}} + \partial_2^4\bar{\bar{F}} = 0, \tag{5.6}$$

and the boundary conditions. The described procedure is particularly convenient in the case of boundary conditions expressed in stresses. It is also easy to extend to the case of orthotropic discs the "plate analogy" devised by Dubas [30] and Trommel [31].

If we solve the equations (5.2) with respect to the stresses and introduce into the equilibrium equations, we obtain with (5.1) a system of three equations

$$L_{ij}\, u_j = -W\delta_{i3}\,, \quad i = 1,2,3, \quad u_3 = \theta, \tag{5.7}$$

where

$$\begin{aligned} &L_{11} = c_{11}\,\partial_1^2 + c_{66}\,\partial_2^2\,, \quad L_{22} = c_{66}\,\partial_1^2 + c_{22}\,\partial_2^2\,, \\ &L_{12} = L_{21} = (c_{12} + c_{66})\,\partial_1\partial_2\,, \quad L_{13} = -\beta_1\partial_1\,, \quad L_{23} = -\beta_2\partial_2\,, \\ &L_{33} = \lambda_1\partial_1^2 + \lambda_2\partial_2^2 - c_\epsilon\partial_t\,, \quad L_{31} = L_{32} = 0. \end{aligned} \tag{5.8}$$

Let us express the functions u_α ($\alpha = 1,2$), $u_3 = \theta$ by means of three displacement functions χ_i ($i = 1,2,3$) as follows

$$u_1 = \begin{vmatrix} \chi_1 & L_{12} & L_{13} \\ \chi_2 & L_{22} & L_{23} \\ \chi_3 & 0 & L_{33} \end{vmatrix}, \quad u_2 = \begin{vmatrix} L_{11} & \chi_1 & L_{13} \\ L_{21} & \chi_2 & L_{23} \\ 0 & \chi_3 & L_{33} \end{vmatrix}, \quad u_3 = \begin{vmatrix} L_{11} & L_{12} & \chi_1 \\ L_{21} & L_{22} & \chi_2 \\ 0 & 0 & \chi_3 \end{vmatrix}. \tag{5.9}$$

The functions χ_i should satisfy the equations

$$\begin{vmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ 0 & 0 & L_{33} \end{vmatrix} \chi_i = -W\delta_{i3}\,, \quad i = 1,2,3, \tag{5.10}$$

or

$$c_{22}\, c_{66}\, (\lambda_1\partial_1^2 + \lambda_2\partial_2^2 - c_\epsilon\,\partial_t)(\mu_1^2\partial_1^2 + \partial_2^2)(\mu_2^2\partial_1^2 + \partial_2^2)\chi_i = -W\delta_{i3}\,, \tag{5.11}$$

where

$$\mu_{1,2}^2 = \varkappa^2 \begin{cases} \sigma \pm \sqrt{\sigma^2 - 1} & \text{for } \sigma > 1, \\ \sigma & \text{for } \sigma = 1, \\ \left(\sqrt{\dfrac{1+\sigma}{2}} \pm i\sqrt{\dfrac{1-\sigma}{2}}\right)^2 & \text{for } \sigma < 1. \end{cases}$$

The functions χ_1, χ_2 satisfy the homogeneous and the function χ_3 the non-homogeneous differential equation. The solution procedure is a follows. From the eq. (5.11) we determine for i = 3 the particular integral χ_3. By means of the functions χ_1, χ_2 we satisfy the given boundary conditions.

The plane problem can also be solved by means of two termoelastic displacement potentials [33] ϕ_α (α = 1,2).

The displacement equations for a medium of an arbitrary curvilinear orthotropy were derived and examined in the paper of Nowiński, Olszak and Urbanowski [34]; these authors solved three examples, the first of which concerns a non-uniformly heated thick-walled cylinder of cylindrical orthotropy, the second deals with an analogous problem for a disc, and the third as that of the non-uniform heating of a shperical shell of spherical orthotropy.

Interesting investigations on states free of stresses in anisotropic bodies was carried out by Olszak [35]. He proved that for bodies which deform freely and possess rectlinear anisotropy only linear distribution of temperature result in no stresses. However, in the case of bodies of curvilinear anisotropy the compatibility equations constitute much stronger limitation than for bodies with rectilinear anisotropy. For instance, for bodies of spherical orthotropy only a constant distribution of temperature produces no stresses, while for bodies of spherical orthotropy any non-vanishing field result in a state of stress.

Only a few dynamic problems have been solved so far in the literature of the subject. Above all the work by T. Chadwik and L.T.C. Seet [36] deserves notice. The main aim of their work are considerations on the propagation of elastic waves in a transversely isotropic heat-conduction material. The authors are concerned with the system of equations

$$\begin{aligned} &\frac{1}{2}(c_{11}-c_{12})\nabla_1^2 u_\alpha + \frac{1}{2}(c_{11}+c_{12})u_{\beta,\alpha\beta} + c_{44}u_{\alpha,33} + \\ &+ (c_{13}+c_{44})u_{3,\alpha 3} - \rho\ddot{u}_\alpha = \beta_o\theta_{,\alpha} , \\ &c_{44}\nabla_1^2 u_3 + c_{33}u_{3,33} + (c_{13}+c_{44})u_{\beta,3\beta} - \rho\ddot{u}_3 = \beta_o'\theta_{,3} \\ &c_\epsilon\dot{\theta} + T_o(\beta_o\dot{u}_{\alpha,\alpha} + \beta_o'\dot{u}_{3,3}) = \lambda\nabla_1^2\theta + \lambda'\theta_{,33} . \quad \alpha,\beta = 1,2 \end{aligned} \tag{5.13}$$

This system may be partialy divided through the introduction of three new scalar funcitons ϕ, χ, ψ, related to the displacements in the following manner

$$u_1 = \phi_{,1} + \chi_{,2} , \quad u_2 = \phi_{,2} - \chi_{,1} , \quad u_3 = \psi_{,3} . \tag{5.14}$$

Setting the above to (5.13), we obtain

$$
\begin{aligned}
& c_{11}\nabla_1^2\phi + c_{44}\phi_{,33} + (c_{13}+c_{44})\psi_{,33} - \rho\ddot{\phi} = \beta_o\theta , \\
(5.15) \qquad & (c_{13}+c_{44})\nabla_1^2\phi + c_{44}\nabla_1^2\psi + c_{33}\psi_{,33} - \rho\ddot{\psi} = \beta'_o\theta , \\
& C_e\dot{\theta} + T_0(\beta_o\nabla_1^2\dot{\phi} + \beta'_o\dot{\psi}_{,33}) = \lambda\nabla_1^2\theta + \lambda'\theta_{,33} ,
\end{aligned}
$$

as well as

$$
(5.16) \qquad \frac{1}{2}(c_{11} - c_{12})\nabla_1^2\chi + c_{44}\chi_{,33} - \rho\ddot{\chi} = 0.
$$

Owing to substitution of (5.1), it was possible to isolate the SH-wave generated by the function χ. It is obvious that only the waves characterized by potentials ϕ and ψ are coupled with the temperature field.

It is clear here that a transversely isotropic elastic material can transit three body waves in each direction, a quasi-longitudinal wave, a quasi-transverse wave, and a purely transverse wave. The quasi-longitudinal and quasi-transverse waves respectively are modified; both these waves suffer dispersion and attenuation.

In the paper discussed in detail – and exemplified also – the numerical results related to the propagation of plane harmonic waves in a single crystal of zinc. A further expansion of the topic is the paper of the above quoted authors concerning the second-order thermoelasticity theory for isotropic and transversely isotropic materials [37].

REFERENCES

[1] DUHAMEL J.M.C. : Seconde mémoire sur les phénomènes thermomécaniques. J. de l'Ecole Polytechnique 15 (1839), 1 – 15.

[2] VOIGT W. : Lehrbuch der Kristallphysik, Teubner, 1910.

[3] JEFFREYS H. : The thermodynamics of an elastic solid. Proc. Cambr.Phil. Soc., 26 (1930).

[4] BIOT M.A. : Thermoelasticity and irreversible thermodynamics. J. Appl.Phys., 27 (1956).

[5] de GROOT S.R. : Thermodynamics of irreversible processes. Amsterdam 1952.

[6] BOLEY B.A. and WEINER J.H. : Theory of thermal stresses. John Viley, New York, 1960.

[7] NEY J.F. : Physical properties of crystals. Oxford, Clarendon Press, 1957.

[8] IONESCU–CAZIMIR V. : Problem of linear Thermoelasticity. Uniqueness theorems (I) (II), Bull.Acad. Polon.Sci., Série Sci Techn., 12, 12 (1964).

[9] IONESCU–CAZIMIR V. : Problem of linear coupled thermoelasticity, (I) (II). Bull.AcAcad. Polon.Sci., Série Sci. Techn. 9, 12 (1964) and 9, 12 (1964).

[10]NOWACKI W. : Thermal stresses in anisotropic bodies (I). (in Polish), Arch. Mech. Stos. 3, 6 (1954).

[11]MEYSEL V.M. : Temperature problems of the theory of elasticity (in Russian), Kiew, 1951.

[12]MOSIL Gr.C. : Matricile asociate sistemelor de ecuatii cu derivate partiale. Introducere in stidiul cercetazilor lui I.N. Lopatinski. Edit. Acadaemiei R.P.R. Bucuresti, (1950).

[13] MOSSAKOWSKA Z. and NOWACKI W. : Thermal stresses in transversely isotropic bodies Arch. Mech. Stos. 4, 10, (1958).

[14] NOWACKI W. : The determining of stresses and deformation in transversely isotropic elastic bodies. (in Polish). Arch. Mech. Stos. 5, 4 (1953).

[15] HU HAI–CZANG. : On the threedimensional problems of the theory of elasticity of a transversely isotropic body. Acta Sci. Sinica, 2, 2 (1953).

[16] STERNBERG E. and MAC DOWELL E.L. : On the steady-state thermoelastic problem for the half-space. Quart. Appl. Math. (1957).

[17] CARRIER C.P. : The thermal stress and body force problem of infinite orthotropic solid. Quart. Appl. Math., (1944).

[18] GOODIER J.N. : On the integration of the thermoelastic equations. Phil. Mag. VI, 23, (1937), p. 1017.

[19] BORS C.I. : Sur le problème à trois dimensions de la thermoelasticité des corps transversalement isotropes. Bull. de l'Acad. Polon. Sci. Ser. Sci. Tech., 11, 5 (1963), 177-181.

[20] ELLIOT A.H. : The three-dimensional stress distribution in hexagonal aelotropic crystals. Proc. Cambridge Phil. Soc. 44 (1948), 621-630.

[21] BORS C.I. : Tensions axiallement symétriques, dans les corps transversalement isotropes. An. stiint. Univ. Iasi, 8, 1, (1962), 119-126.

[22] SINGH Avtar : Axisymmetrical thermal stresses in transversely isotropic bodies. Arch. Mech. Stos. 12, 3, (1960), 287-304.

[23] GRINDEI I : Tensiuni termice in medii elastice transversal izotrope. An. stiint. Univ. Iasi, 14, 1, (1968), 169-176.

[24] SHARMA B. : Thermal stresses in transversely isotropic semi-infinite elastic solids. J. Appl. Mech. 1958.

[25] IESAN D. : Tensiuni termice in bare ortho ropic. An. stiint. Univ. Iasi, 12, 2 (1967).

[26] IESAN D. : Tensions thermiques dans des barres élastiques non-homogènes. An. stiint. Univ. Iasi, 14, 1 (1968).

[27] PELL W.H. : Thermal deflection of anisotropic thin plates. Quart. Appl. Mech., (1946.).

[28] MOSSAKOWSKI J. : The state of stress and displacemnt in a thin anisotropic plate to a concentrated source of heat. Arch. Mech. Stos. 9, 5 (1957), p. 595.

[29] TEODORESCU P.P. : Asupra problemei plane a elasticitatii unor corpuri anizotrope. VIII. Influenta variatiei de temperatura. Com. Acad. R.P.R. 8, 11 (1958), p. 1119-1126.

[30] TREMMEL F. : Uber die Anwendung der Plattentheorie zur Bestimmung der Wärmespannungsfelder. Osterr. Ing. Arch. 1957.

[31] DUBAS P. : Calcul numérique des plaques et des parois minces, Zürich, 1955.

[32] NOWACKI W. : Thermal stresses in orthotropic plates. Bull. de l'Acaad. Polon. Sci., Ser Sci. Techn. 7, 1 (1959), 1-6.

[33] BORŞ C.I. : Tensioni termice la corpurile orthotrope in cazul problemelor plane. St. si cerc. stiint. Filiala Iasi Acad. R.P.R. 14, 1 (1963), 187-192.

[34] NOWINSKI J., OLSZAK W. and URBANOWSKI W. : On thermoelastic problems in the case of a body of an arbitrary type of curvilinear orthotropy (in polish), Arch. Mech. Stos. 7, 2, (1955), 247-265.

[35] OLSZAK W. : Autocontraints des milieux anisotropes. Bull. Acad. Polon. Sci. Lettr. Cl. Ser. Math. (1950).

[36] CHADWICK P. and SEET L.T.C. : Wave propagation in a transversely isotropic heat-conducting elastic material. Mathematika 17, (1970) p. 255.

[37] CHADWICK P. and SEET L.T.C. : Second-order thermoelasticity theory for isotropic and transversely isotropic materials. Trends in Elasticity and Thermoelasticity. Wolters-Nordhoff Publ., Groningen, 1971.