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PROBLEMS OF THERMOELASTICITY

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1. Introduction

We know from experiment that deformation of a body is associated with a change of heat its content. The time varying loading of a body causes in it not only displacements but also temperature distribution changing in time. Conversely, the heating of a body produces in it deformation and temperature change. The motion of a body is characterized by mutual interaction between deformation and temperature fields. The domain of science dealing with the mutual interaction of these fields is called thermoelasticity.

Owing to the coupling between these fields, the temperature terms appear in the displacement equations of motion, and the deformation terms — in the equation of thermal conductivity.

The coupling between deformation and temperature fields was first postulated by J. M. C. DUHAMEL [1], the originator of the theory of thermal stresses who introduced the dilatation term in the equation of thermal conductivity. However, this equation was not well grounded in the thermodynamical sense. Next, an attempt at thermodynamical justification of this equation was undertaken by W. VOIGT [2] and H. JEFFREYS [3]. However, as recently as in 1956, M. A. BIOT [4] gave the full justification of the thermal conductivity equation on the basis of thermodynamics of irreversible processes [5]. M. A. BIOT also presented the fundamental methods for solving the thermoelasticity equation as well also variational theorem.

Thermoelasticity describes a broad range of phenomena; it is the generalization of the classical theory of elasticity and of the theory of thermal conductivity. Now, the thermoelasticity is a fully formed domain of science. The fundamental relations and differential equations have been formulated. A number of methods for solving thermoelasticity equations and the basic energy and variational theorem have been developed. Some problems concerning the propagation of thermoelastic waves have been solved.

It is known, that research in the field of thermoelasticity was preceded by broad-scale investigations within the framework of what is called the theory of thermal stresses. By this term we mean the investigation of strains and stresses produced by heating a body, with the simplifying assumption that the deformation of an elastic body does not affect the thermal conductivity.

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In this theory, originating from the beginnings of the elasticity theory and recently being intensively developed owing to its growing practical significance, the classical eqation of thermal conductivity, not containing the term associated with the body deformation, has been used.

Elastokinetics has been developed simultaneusly with the theory of thermal stresses. In this case, then has been introduced the simplifying assumption that heat exchange among body parts, caused by the heat conductivity, is so slow that the motion can be regarded of as an adiabatic process.

The domains referred to here now constitute a particular case of the more general theory—namely, of thermoelasticity. The particular theorems and methods of the theory of thermal conductivity and of the classical theory of elasticity are comprised in the general theorems and methods of thermoelasticity.

Note that solutions obtained within the framework of the thermoelasticity differ slightly from solutions of the classical theory of elasticity or the theory of thermal conductivity. The coupling between the deformation and temperature field is weak. But the qualitative differences are fundamental. This, is seen even if, using the examples of elastic waves within the framework of elastokinetics, only undamped waves appear. Thermoelasticity is of fundamental significance in those cases in which the investigation of elastic dissipation is a principal aim. The meaning of thermoelasticity consists principally in recognizing and generalizing the value of this theory.

In the present paper, which is of survey character, attention is focused on foundations of thermodynamical theories, on differential equations of thermoelasticity and more important methods for solving them, and on general energy and variational theorems.

Less attention is devoted to solving concrete problems, and the reader is referred to literature listed at the end of the work. In the function relations and equations, we shall apply the index tensor notation in the Cartesian system of coordinates.

2. Fundamental Assumptions and Relations of Linear Thermoelasticity

In the present section, we shall consider homogeneous anisotropic elastic bodies. For these bodies, we shall derive general relations and extended equations of thermal conductivity, and subsequently, we shall deal with a homogeneous isotropic body which will be the subject of the further sections of the present paper.

Let a body be in the temperature T_0 in an undeformed and unstressed state. This starting state will be called the natural state, assuming that entropy equals zero for this body. Owing to the action of external forces—, i.e., body and surface forces—and under the influence of the heat sources and heating (or cooling) of the body surface, the medium will be subjected to deformation and temperature change. The displacements $\mathbf{u}$ will appear in the body and the temperature change can be written as $\theta = T - T_0$, where T is the absolute temperature of a point $\mathbf{x}$ of the body. The temperature change is accompanied by stresses σ_{ij} and strains ε_{ij}. The quantities $\mathbf{u}, \theta, \varepsilon_{ij}, \sigma_{ij}$ are the functions of position $\mathbf{x}$ and time t.

We assume that the temperature change $\theta = T - T_0$ accompanying deformation is small and increase in the temperature does not result in essential variations of material coefficients either elastic or thermal. These coefficients will regarded as independent of T.

To the introduced assumption $|\theta/T_0| \ll 1$ let us add others concerning small strains. Namely, we assume that second powers and products of the components of strains may be disregarded as quantities small compared with the strains ε_{ij}. Thus, we restrict further considerations to the geometrically linear thermoelasticity. The dependency among strains and displacements is confined to the linear relation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}). \tag{2.1}$$

It is known that strains cannot be arbitrary functions, they must satisfy six relations, called the compatibility relations

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{jl,ik}-\varepsilon_{ik,jl}=0, \quad i,j,k,l=1,2,3. \tag{2.2}$$

The main task becomes that of obtaining state equations relating the components of stress tensor σ_{ij} with the components of strain tensor ε_{ij} and of temperature θ.

Let us note that the mechanical and thermal state of the medium is, at a given time instant, completely described by the distribution of stresses σ_{ij} and temperature θ. We hence conclude that for the isothermal change of state ($T = T_0$), we encounter a process which is elastically and thermodynamically reversible. However, in processes in which temperature changes take place, we observe two interrelated phenomena—namely, the reversible elastic process and the irreversible thermodynamical process. The latter is caused by the spontaneous and thereby irreversible process of carrying the heat by means of thermal conductivity.

Thermoelastic disturbances cannot be described by means of classical thermodynamics, and we have to use the relations of the thermodynamics of irreversible processes [5, 6].

The constitutive relations,—this is, the relations between the state of stress, state of strain and temperature, are deduced from thermodynamical considerations, taking into account the principle of conservation of energy and the entropy balance [4–6]

$$\frac{d}{dt}\int_V \left(U+\frac{1}{2}\varrho v_i v_i\right)dV = \int_V X_i v_i\,dV + \int_A p_i v_i\,dA - \int_A q_i n_i\,dA, \tag{2.3}$$

$$\int_V \frac{dS}{dt}\,dV = -\int_A \frac{q_i n_i}{T}\,dA + \int_V \Theta\,dV. \tag{2.4}$$

Here U is the internal energy, S is the entropy, X_i the components of the body forces, $p_i = \sigma_{ji} n_j$ the components of the stress vector, q_i the components of the vector of heat flux, n_i the components of the normal to the surface A. Further, $v_i = \partial u_i/\partial t$ and the quantity Θ represents the source of entropy—a quantity always positive in a thermodynamically irreversible process.

The terms in the left-hand side of the Eq. (2.3) represent the rate of increase of the internal and kinetic energies. The first term of the right-hand side is the rate of increase of the work of the body forces, and the second the rate of increase of the work of the surface tractions. Finally, the last term of the right-hand side of the Eq. (2.3) is the energy acquired by the body by means of the thermal conductions. The left-hand side of Eq. (2.4) is the rate of increase of the entropy. The first term of the right-hand side of the Eq. (2.4) represents the exchange of entropy with the surroundings and the second term, the rate of production of the entropy due to heat conduction.

Making use of the equations of motion

$$\sigma_{ji,j}+X_i=\varrho\ddot{u}_i, \quad i,j=1,2,3, \tag{2.5}$$

and using the divergence theorem to transform the integrals, we arrive at the local relations

$$\dot{U}=\sigma_{ji}\dot{\varepsilon}_{ji}-q_{i,i}, \quad \dot{S}=\Theta-\frac{q_{i,i}}{T}+\frac{q_iT_{,i}}{T^2}. \tag{2.6}$$

Introducing the Helmholtz free energy $F=U-ST$ and eliminating the quantity $q_{i,i}$ from the Eqs. (2.6), we obtain

$$\dot{F}=\sigma_{ji}\dot{\varepsilon}_{ji}-\dot{T}S-T\left(\Theta+\frac{q_iT_{,i}}{T^2}\right). \tag{2.7}$$

The dot above the function denotes the derivative of this function with respect to time.

Since the free energy is a function of the independent variables ε_{ij}, T, we have:

$$\dot{F}=\frac{\partial F}{\partial\varepsilon_{ji}}\dot{\varepsilon}_{ji}+\frac{\partial F}{\partial T}\dot{T}. \tag{2.8}$$

Assuming that the functions Θ, q_i, σ_{ji} do not explicitly depend on the time derivatives of the functions ε_{ij} and T, we obtains, comparing (2.7) and (2.8):

$$\sigma_{ji}=\frac{\partial F}{\partial\varepsilon_{ji}}, \quad S=-\frac{\partial F}{\partial T}, \quad \Theta+\frac{q_iT_{,i}}{T^2}=0. \tag{2.9}$$

The postulate of the thermodynamics of irreversible processes will be satisfied if $\Theta>0$—i.e., when $-T_{,i}q_i/T^2>0$. This condition is satisfied by the Fourier law of heat conduction [5]

$$-q_i=k_{ij}T_{,j} \quad \text{or} \quad -q_i=k_{ij}\theta_{,ij}, \quad \theta=T-T_0. \tag{2.10}$$

For a homogeneous and isotropic body, the Eq. (2.10) takes the form

$$-q_i=k\theta_{,i} \tag{2.11}$$

Finally, it follows from the Eqs. $(2.6)_2$ and $(2.9)_3$ that

$$T\dot{S}=-q_{i,i}=k\theta_{,jj}. \tag{2.12}$$

Here k is the coefficient of heat conduction. The first two relations (2.9) imply the constitutive relations.

Let us expand the function $F(\varepsilon_{ij},T)$ into an infinite series in the neighbourhood of the natural state $F(0,T_0)$:

$$F(\varepsilon_{ij},T)=F(0,T_0)+\frac{\partial F(0,T_0)}{\partial\varepsilon_{ij}}\varepsilon_{ij}+\frac{\partial F(0,T_0)}{\partial T}(T-T_0)$$
$$+\frac{1}{2}\left[\frac{\partial^2F(0,T_0)}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}\varepsilon_{ij}\varepsilon_{kl}+2\frac{\partial^2F(0,T_0)}{\partial\varepsilon_{ij}\partial T}\varepsilon_{ij}(T-T_0)+\frac{\partial^2F(0,T_0)}{\partial T^2}(T-T_0)^2\right]+\ldots \tag{2.13}$$

From the expansion of $F(\varepsilon_{ij}, T)$, we retain only the linear and quadratic terms, confining ourselves to linear relations among stresses σ_{ij}, strains and temperature change θ.

Taking into account that for $\varepsilon_{ij} = 0$, $T = T_0$, we consider the natural state; it can be assumed that $F(0, T_0) = 0$. The term $\partial F(0, T_0)/\partial T$ will also be equated to zero. Since it results from equating the Eqs. (2.7) and (2.8) that $(\partial F/\partial T)_\varepsilon = -S$, therefore for the natural state, we have

$$\frac{\partial F(0, T_0)}{\partial T} = -S(0, T_0) = 0.$$

Let us now take advantage of the first of the expressions (2.9)

$$(2.14)\quad \sigma_{ij}(\varepsilon_{ij}, T) = \left(\frac{\partial F}{\partial \varepsilon_{ij}}\right)_T = \frac{\partial F(0, T_0)}{\partial \varepsilon_{ij}} + \frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}\varepsilon_{kl} + \frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\partial T}(T - T_0).$$

Thus we have obtained the linear relation for small strains which agrees with the assumption introduced $|\theta/T_0| \ll 1$.

We should put $\partial F(0, T_0)/\partial \varepsilon_{ij} = 0$ in Eq. (2.14) since, for the natural state $\varepsilon_{ij} = 0$ $T = T_0$, we should have $\sigma_{ij} = 0$.

Introducing the denotation

$$\frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = c_{ijkl}, \qquad \frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\partial T} = -\beta_{ij}, \qquad \frac{\partial^2 F(0, T_0)}{\partial T^2} = n,$$

we present the relations (2.13) and (2.14) in the form:

$$(2.15)\qquad F(\varepsilon_{ij}, T) = \frac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \beta_{ij}\varepsilon_{ij}\theta + \frac{n}{2}\theta^2,$$

$$(2.16)\qquad \sigma_{ij} = \frac{1}{2}(c_{ijkl} + c_{klij})\varepsilon_{kl} - \beta_{ij}\theta.$$

Let us note additionally that

$$(2.17)\qquad \left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}\right)_T = c_{ijkl}, \qquad \left(\frac{\partial \sigma_{ij}}{\partial T}\right)_\varepsilon = -\beta_{ij}.$$

In the relations (2.16), we recognize Hooke's law generalized for thermoelastic problems. The Eqs. (2.16) are called the Duhamel-Neumann relations for an anisotropic body. The constants c_{ijkl}, β_{ij}, concerning the isothermal state play the role of material constants [7] The quantities c_{ijkl} are components of the tensor of elastic stiffness.

In the elasticity theory of an anisotropic body, the following symmetry properties of a tensor are proved:

$$(2.18)\qquad c_{ijkl} = c_{jikl}, \qquad c_{ijkl} = c_{ijlk}, \qquad c_{ijkl} = c_{klij}.$$

These relations lead to reduction of the quantity of constants from 81 to 21 of mutually independent constants for a body with general anisotropy.

Let us solve the system of Eqs. (2.16) for deformations

$$(2.19)\qquad \varepsilon_{ij} = s_{ijkl}\sigma_{kl} + \alpha_{ij}\theta.$$

The quantities s_{ijkl} are called the coefficients of elastic susceptibility, Also for these quantities, the following symmetry relations hold

$$s_{ijkl} = s_{jikl}, \quad s_{ijkl} = s_{ijlk}, \quad s_{ijkl} = s_{klij}.$$

Let us now consider a volume element of the anisotropic body free of stresses on its surface. Then, according to (2.19), we obtain for this element:

$$\varepsilon_{ij} = \alpha_{ij}\theta. \tag{2.20}$$

The relation (2.20) describes a familiar physical phenomenon—namely, the proportionality of the deformation of elements to the increment of temperature θ. The quantities α_{ij} are the coefficients of linear thermal expansion. The α_{ij} is a symmetric tensor, as follows from the symmetry of the tensor ε_{ij}. It should added that the coefficient of volume thermal expansion α_{jj} is an invariant.

From the relations (2.16) and (2.19), we obtain the following expressions:

$$\left(\frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}\right)_T = c_{ijkl}, \quad \left(\frac{\partial\sigma_{ij}}{\partial T}\right)_\varepsilon = -\beta_{ij} = -\alpha_{kl}c_{ijkl}, \quad \left(\frac{\partial\varepsilon_{ij}}{\partial T}\right)_\sigma = \alpha_{ij} \tag{2.21}$$

In the further considerations concerning derivation of the extended equation of thermal conductivity, it will be necessary to present the internal energy and entropy as a function of deformation and temperature. We have as starting point the total differentials

$$dU = \sigma_{ij}d\varepsilon_{ij} + T\,dS, \tag{2.22}$$

$$dS = \left(\frac{\partial S}{\partial\varepsilon_{ij}}\right)_T d\varepsilon_{ij} + \left(\frac{\partial S}{\partial T}\right)_\varepsilon dT. \tag{2.23}$$

Inserting (2.23) into (2.22), we obtain

$$dU = \left[T\left(\frac{\partial S}{\partial\varepsilon_{ij}}\right)_T + \sigma_{ij}\right]d\varepsilon_{ij} + T\left(\frac{\partial S}{\partial T}\right)_\varepsilon dT. \tag{2.24}$$

The necessary and sufficient condition for the quantity dU to be a total differential is

$$\frac{\partial}{\partial T}\left[T\left(\frac{\partial S}{\partial\varepsilon_{ij}}\right)_T + \sigma_{ij}\right] = \frac{\partial}{\partial\varepsilon_{ij}}\left[T\left(\frac{\partial S}{\partial T}\right)_\varepsilon\right].$$

From this condition, results the relation

$$\left(\frac{\partial S}{\partial\varepsilon_{ij}}\right)_T + \left(\frac{\partial\sigma_{ij}}{\partial T}\right)_\varepsilon = 0,$$

or taking into account the second term in the group (2.21)

$$\left(\frac{\partial S}{\partial\varepsilon_{ij}}\right)_T = \beta_{ij}. \tag{2.25}$$

On the other hand, we utilize the thermodynamical relation

$$T\left(\frac{\partial S}{\partial T}\right)_\varepsilon = \left(\frac{\partial U}{\partial T}\right)_\varepsilon = c_\varepsilon, \tag{2.26}$$

where c_ε is a specific heat related to unit volume at constant deformation. Substituting (2.25) and (2.26) into (2.23) and (2.24), we obtain:

$$dS = \beta_{ij} d\varepsilon_{ij} + \frac{c_\varepsilon}{T} dT, \tag{2.27}$$

$$dU = \sigma_{ij} d\varepsilon_{ij} + T\beta_{ij} d\varepsilon_{ij} + c_\varepsilon dT. \tag{2.28}$$

Inserting the relations (2.16) into (2:28) and integrating the expressions (2.27) and (2.28), with the assumption that for the natural state ($T = T_0$, $\varepsilon_{ij} = 0$, $\sigma_{ij} = 0$) there is $S = 0$, $U = 0$, we have:

$$S = \beta_{ij}\varepsilon_{ij} + c_\varepsilon \log\left(1 + \frac{\theta}{T_0}\right), \tag{2.29}$$

$$U = \frac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + T_0 \beta_{ij}\varepsilon_{ij} + c_\varepsilon \theta. \tag{2.30}$$

In the expression for entropy, the first term on the right-hand side arises from the coupling of the deformation field with the temperature field, while the second term expresses the entropy caused by the heat flow. The purely elastic term does not appear in this expression. Thus it results that the deformation process is, in the isothermal conditions, reversible, and does not cause an increament in the entropy. In the expression (2.30) for the internal energy, threee terms appear. The first of them is of purely elastic character it representing the strain work, the last-heat content in a unit volume, the second term is a result of mutual interaction between deformation and temperature fields.

Let us return to the expression (2.29). In virtue of the assumption $|\theta/T_0| \ll 1$ introduced, the function ln $(1+\theta/T_0)$ can be expanded into an infinite series and only one term of, the expansion can be taken into account. Thus, we obtain:

$$S = \beta_{ij}\varepsilon_{ij} + \frac{c_\varepsilon}{T_0}\theta. \tag{2.31}$$

For the internal energy $F = U - ST$, we have:

$$F \approx \frac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \beta_{ij}\varepsilon_{ij}\theta - \frac{c_\varepsilon}{2T_0}\theta^2. \tag{2.32}$$

In this way, we have determined the $n = -c_\varepsilon/T_0$, involved in (2.15).

It remains to interrelate the entropy with the thermal conductivity. In a solid body, the heat transfer is realized through the thermal conductivity meant as a transfer of heat from points with higher to those with lower temperature. The equation of thermal conductivity is derived from the principle of energy conservation expressed in the form of entropy flow. This law, constituting the local formulation of the second principle of thermodynamics, can be written in the form:

$$T\dot{S} = -\operatorname{div}\mathbf{q}, \quad \dot{S} = -\frac{1}{T} q_{i,i}. \tag{2.33}$$

Combining the relations (2.33) and (2.10), and differentiating (2.31) with respect to time, we arrive at the equations:

$$T\dot{S} = \lambda_{ij} T_{,ij}, \tag{2.34}$$

$$T\dot{S} = T\beta_{ij} \frac{d\varepsilon_{ij}}{dt} + \frac{c_\varepsilon}{T_0} T \frac{d\theta}{dt}. \tag{2.35}$$

Comparing these equations yields the equation of thermal conductivity:

$$\lambda_{ij} T_{,ij} = T\beta_{ij} \dot{\varepsilon}_{ij} + \frac{c_\varepsilon}{T_0} T\dot{\theta}, \quad \theta = T - T_0. \tag{2.36}$$

Let us note that this is a nonlinear equation on account of its right-hand side. Putting $T = T_0$ in the right-hand side of (2.36), we linearize this equation. Finally, we obtain:

$$\lambda_{ij} \theta_{,ij} - c_\varepsilon \dot{\theta} - T_0 \beta_{ij} \dot{\varepsilon}_{ij} = 0. \tag{2.37}$$

In this extended equation of thermal conductivity, the term $T_0 \beta_{ij} \dot{\varepsilon}_{ij}$ appears characterizing the coupling of deformation field with temperature field. If sources of heat act in the body, we should add to (2.34) the quantity which determines the amount of heat produced in a unit of volume and time:

$$T\dot{S} = \lambda_{ij} T_{,ij} + W$$

to Eq. (2.37), in the case of the appearence of a heat source in the body, is extended to the form:

$$\lambda_{ij} \theta_{,ij} - c_\varepsilon \dot{\theta} - T_0 \beta_{ij} \dot{\varepsilon}_{ij} = -W. \tag{2.38}$$

On the basis of the Duhamel-Neumann relations derived for an anisotropic body, we shall easily proved to isotropic body, applying the following relation:

$$\begin{aligned} c_{ijkl} &= \mu [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}] + \lambda \delta_{ij} \delta_{kl}, \\ s_{ijkl} &= \mu' [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}] + \lambda' \delta_{ij} \delta_{kl}, \\ \beta_{ij} &= \gamma \delta_{ij}, \quad \alpha_{ij} = \alpha_t \delta_{ij}. \end{aligned} \tag{2.39}$$

Here, μ, λ are Lamé's constants for an isothermal state, and

$$\gamma = (3\lambda + 2\mu)\alpha_t, \quad \mu' = \frac{1}{4\mu}, \quad \lambda' = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}.$$

The quantity α_t is the coefficient of linear thermal expansion. In this way, the relations (2.16) and (2.19) transform in to the Duhamel-Neumann relations for an isotropic body:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma\theta)\delta_{ij}, \tag{2.40}$$

$$\varepsilon_{ij} = \alpha_t \theta \delta_{ij} + 2\mu' \sigma_{ij} + \lambda' \delta_{ij} \sigma_{kk}. \tag{2.41}$$

For an isotropic body we have: $\lambda_{ij} = \lambda_0 \delta_{ij}$.

Thus, the equation of thermal conductivity (2.42) assumes the form [4]:

$$\lambda_0 \theta_{,jj} - c_\varepsilon \dot{\theta} - T_0 \gamma \dot{\varepsilon}_{kk} = -W,$$

or

$$\theta_{,jj} - \frac{1}{\kappa} \dot{\theta} - \eta \dot{\varepsilon}_{kk} = -\frac{Q}{\kappa}, \tag{2.42}$$

where

$$\kappa = \frac{\lambda_0}{c_\varepsilon}, \quad \eta = \frac{\gamma T_0}{\lambda_0}, \quad Q = \frac{W}{\lambda_0}$$

Let us give, further, the expressions of U, F for an anisotropic body. We obtain:

(2.43)
$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + \frac{1}{2}e\gamma(\theta+2T_0)+c_\varepsilon\theta,$$
$$F = \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}e^2 - \gamma e\theta - \frac{c_\varepsilon}{2T_0}\theta^2,$$
$$S = \gamma e + c_\varepsilon\frac{\theta}{T_0}, \quad e = \varepsilon_{kk}.$$

The equation of state and the equations of thermal conductivity derived in this section should be joined with the equations of motion of a solid deformable body. In this way, we shall obtain a full set of thermoelasticity equations.

Note fact the coupling of temperature and deformation fields vanishes when external forces or heating the body is stationary. In this case, the time derivatives disappear in the equation of thermal conductivity, and the Eq. (2.42) transforms into Poisson's equation.

3. Differential Equations of Thermoelasticity and Methods for Solving them

The full set of differential equations of thermoelasticity is composed of the equations of motion and the equations of thermal conductivity. The equations of motion:

(3.1)
$$\sigma_{ij,j} + X_i = \varrho\ddot{u}_i(\mathbf{x}, t), \quad \mathbf{x}\in V, \quad t>0,$$

can be transformed, making use of the Duhamel-Neumann equations,

(3.2)
$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma\theta)\delta_{ij}, \quad \mathbf{x}\in V+\Sigma, \quad t>0,$$

and of the relations among displacements and deformations

(3.3)
$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \quad \mathbf{x}\in V+\Sigma, \quad t>0,$$

into the three equations containing displacements u_i and temperature θ as unknown functions

(3.4)
$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + X_i = \varrho\ddot{u}_i + \gamma\theta_{,i}, \quad \mathbf{x}\in V, \quad t>0.$$

The above equations and those of thermal conductivity

(3.5)
$$\theta_{,jj} - \frac{1}{\kappa}\dot{\theta} - \eta\dot{u}_{k,k} = -\frac{Q}{\kappa}, \quad \mathbf{x}\in V, \quad t>0,$$

are coupled. Body forces, heat sources, heating and heat flow through the surface enveloping the region, and initial conditions are the causes of arising both displacements and the temperature accompanying them in a body.

Boundary conditions of a mechanical type are given in the form of either given dis-

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placements u_i or loadings $p_i = \sigma_{ji} n_j$ on the surface Σ. Thermal conditions can, in a general way be, written in the form

$$\alpha \frac{\partial \theta}{\partial n} + \beta\theta = f(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma, \quad t > 0, \quad \alpha, \beta = \text{const}, \tag{3.6}$$

determining the heat exchange through the surface Σ. If $\beta = \infty$, then the temperature θ on the boundary is equal to zero; if $\alpha = \infty$, then we have the case of the surface Σ thermally isolated. The initial conditions suggest that at an initial time instant—e.g, for $t = 0$—displacement u_i, the velocity of these displacements, and temperature are the known functions

$$u_i(\mathbf{x}, t)_{t=0} = f_i(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, t)_{t=0} = g_i(\mathbf{x}), \quad \theta(\mathbf{x}, t)_{t=0} = h(\mathbf{x}). \tag{3.7}$$

The system of Eqs. (3.4) and (3.5) is greatly complicated and there is an obvious tendency to lead this system to a system of simpler equations—viz., wave equations. The necessary simplification is obtained by decomposition of the displacement vector and the vector of body forces into a potential part and a solenoidal part. Substituting then, into the Eq. (3.4) and (3.5) the formulae:

$$u_i = \Phi_{,i} + \epsilon_{ijk}\psi_{k,j}, \quad X_i = \varrho(\vartheta_{,i} + \epsilon_{ijk}\chi_{k,j}), \tag{3.8}$$

where Φ and ϑ are the scalar functions, whereas ψ_i and χ_i are vector functions, we lead the thermoelasticity equations to the following system of equations [8]:

$$\square_1^2 \Phi - m\theta = -\frac{1}{c_1^2}\vartheta, \tag{3.9}$$

$$\square_2^2 \psi_i = -\frac{1}{c_2^2}\chi_i, \tag{3.10}$$

$$D\theta - \eta\nabla^2\dot{\Phi} = -\frac{Q}{\kappa}, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}, \quad m = \frac{\gamma}{c_1^2\varrho}. \tag{3.11}$$

The following denotations are introduced here:

$$\square_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\partial_t^2, \quad \alpha = 1,2; \quad D = \nabla^2 - \frac{1}{\kappa}\partial_t, \quad \partial_t = \frac{\partial}{\partial t}.$$

The Eqs. (3.9) and (3.11) are coupled in a direct manner. Elimination of the function θ leads to the equation of a longitudinal wave:

$$(\square_1^2 D - \eta m \partial_t \nabla^2)\Phi = -\frac{mQ}{\kappa} - \frac{1}{c_1^2} D\vartheta. \tag{3.12}$$

The Eqs. (3.10) describes a transverse wave. Let us note that the functions Φ and ψ_i are joined mutually through boundary conditions which will be expressed, in every case, by displacements u_i, by derivatives of these functions, and by temperature θ.

Eliminating the function Φ from the Eqs (3.9) and (3.11), we obtain the equation:

$$(\square_1^2 D - \eta m \partial_t \nabla^2)\theta = -\frac{m}{\kappa}\square_1^2 Q - \frac{1}{c_1^2}\eta\partial_t\nabla^2\vartheta. \tag{3.13}$$

We see that the Eqs (3.12) and (3.13) have the same form. The structure of these formulae, which will be discussed later, indicates that we are considering a wave damped and subjected to disperssion. In an infinite thermoelastic space, the longitudinal and transverse waves propagate independently of each other. Let us assume that heat sources Q and body forces $X_i = \varrho\vartheta_{,i}$ are the source of motion. Under the assumption that $\chi_i = 0$, and that initial conditions connected with the Eq. (3.10) are equal to zero, we have $\psi_i \equiv 0$ in the whole space.

In the infinite thermoelastic space, there will arise only longitudinal waves of dilatation character.

Taking into account (3.2) and (3.8), we have:

$$u_i = \Phi_{,i} \quad \varepsilon_{ij} = \Phi_{,ij}, \quad \varepsilon_{kk} = \nabla^2\Phi,$$

and

$$\sigma_{ij} = 2\mu(\Phi_{,ij} - \delta_{ij}\Phi_{,kk}) + \varrho\delta_{ij}(\ddot{\Phi} - \vartheta).$$

If in the infinite space the body forces $X_i = \varrho\in_{ijk}\psi_{k,j}$ act, whereas $Q = 0$, $\vartheta = 0$ and $\Phi(\mathbf{x}, 0) = 0$, $\dot{\Phi}(\mathbf{x}, 0) = 0$, then only the functions ψ_i are different from zero, but $\Phi \equiv 0$, $\theta \equiv 0$ in the whole region. Only the transverse waves are propagated and their velocity is $c_2 = (\mu/\varrho)^{1/2}$. These waves are not accompanied by heat production.
Let us observe that for transverse waves we have:

$$u_i = \in_{ijk}\psi_{k,j}, \quad u_{k,k} = 0, \quad \theta = 0, \quad \sigma_{ij} = 2\mu\varepsilon_{ij} = \mu(u_{i,j} + u_{j,i}).$$

In a bounded body, there appear simultaneously, in principle, two kinds of waves. The solution for the Eqs. (3.10) and (3.12) will be constructed of two parts—viz., of the particular integrals of these equations Φ^0, ψ_i^0 and of the general integrals of the homogeneous equations:

$$(\Box_1^2 D - \eta m\partial_t\nabla^2)\Phi' = 0, \quad \Box_2^2\psi_i' = 0,$$

where the functions Φ' and ψ_i' should be so chosen as to satisfy all boundary conditions.

The next method used for solving the differential equations of thermoelasticity is that of disjoining the equations which consists in leading the Eq. (3.4) and (3.5) to a system of four equations decoupled. Only one unknown function appears in each equation. Presumably, this method was first used by Hilbert [9] for the differential equations of optics. A certain variant of it in the operator form developed by G. Moisil [10] was applied for the quasi-static equations of thermoelasticity by V. Ionescu-Cazimir [12] S. Kaliski [11] has disjoined the dynamical equations of thermoelasticity in a different way. This result was repeated, in a different manner, by J. S. Postrigatz [13] and D. Rüdiger [14].

We shall present only the final results, of this method omitting details. We introduce one scalar function ψ and one vector function φ_i, and by means of them we express displacement and temperature as follows

$$u_i = (\Omega\delta_{ij} - \Gamma\partial_i\partial_j)\varphi_j + \gamma_0\partial_i\psi, \tag{3.14}$$

$$\theta = \eta\partial_t\partial_j\Box_2^2\varphi_j + (1 + a)\Box_1^2\psi, \tag{3.15}$$

where

$$\Omega = (1+a)\,\Box_1^2 D - \gamma_0 \eta \partial_t \nabla^2, \qquad \Gamma = aD - \gamma_0 \eta \partial_t,$$
$$a = (\lambda+\mu)/\mu, \qquad \gamma_0 = \gamma/\mu.$$

Substituting u_i and θ into the Eqs. (3.4) and (3.5) we obtain four already decoupled equations for the functions φ_i and ψ

$$\Box_2^2(\Box_1^2 D - \eta m \partial_t \nabla^2)\varphi_i + \frac{X_i}{\varrho c_1^2} = 0, \tag{3.16}$$

$$(\Box_1^2 D - \eta m \partial_t \nabla^2)\psi + \frac{Q\mu}{\kappa c_1^2 \varrho} = 0. \tag{3.17}$$

To these equations, we should add boundary and initial conditions. In the boundary conditions, there appear, of course, the functions φ_i and ψ. The simplicity of the differential Eqs. (3.16) and (3.17) is, however, ransomed with the complicated form of boundary conditions. Therefore, the Eqs. (3.16) and (3.17) will be applied, first of all, in a infinite space, where boundary conditions in the strict sense do not exist and are replaced by the requirement of zero values of displacements and temperature in infinity. This postulate is fulfilled if distribution of body forces and heat sources is restricted to a finite region.

There is an interesting way of solving the differential Eqs. (3.4) and (3.5) into a system of three differential equations for displacements u_i. We shall present it briefly in reference to an infinite space with the assumption of homogeneous initial conditions. We write the conductivity equation in such a form that the term containing dilatation velocity is on the right-hand side of the equation

$$\theta_{,jj} - \frac{1}{\kappa}\dot{\theta} = \eta \dot{u}_{j,j}. \tag{3.18}$$

Regarding the function $\eta \dot{u}_{j,j}$ as a heat source, we can give the solution of the Eq. (3.18), using the Green's function valid for the classical equation of thermal conductivity:

$$G_{,jj} - \frac{1}{\kappa}\dot{G} = -\frac{1}{\kappa}\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t),$$
$$G(\mathbf{x},\boldsymbol{\xi},t) = \frac{1}{8(\pi\kappa t)^{3/2}} \exp\left(\frac{-\varrho^2}{4\kappa t}\right). \tag{3.19}$$

Inserting the solution of the Eq. (3.18)

$$\theta(\mathbf{x},t) = -\eta\kappa \int_0^t d\tau \int_V G(\boldsymbol{\xi},\mathbf{x},t-\tau)\frac{\partial}{\partial\tau}\operatorname{div}\mathbf{u}(\boldsymbol{\xi},\tau)\,dV(\boldsymbol{\xi}), \qquad \varrho^2 = (\xi_i - x_i)(\xi_i - x_i).$$

into the displacement Eqs. (3.4), we arrive at the following differential-integral equation:

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \varrho\ddot{\mathbf{u}} = -\eta\kappa\gamma \operatorname{grad}\int_0^t d\tau \int_V G(\boldsymbol{\xi},\mathbf{x},t-\tau)\frac{\partial}{\partial\tau}\operatorname{div}\mathbf{u}(\boldsymbol{\xi},\tau)\,dV)(\boldsymbol{\xi}). \tag{3.20}$$

If the displacement vector is decomposed according to the formula (3.8), then the Eq. (3.20) disintegrates into the system of equations:

$$\square_1^2 \Phi + \frac{\gamma\eta}{c_1^2 \varrho} \int_0^t d\tau \int_V G(\boldsymbol{\xi}, \mathbf{x}, t-\tau) \frac{\partial}{\partial \tau} \nabla^2 \Phi(\boldsymbol{\xi}, \tau)\, dV(\boldsymbol{\xi}) = 0, \tag{3.21}$$

$$\square_1^2 \boldsymbol{\psi} = 0. \tag{3.22}$$

The differential-integral Eq. (3.21) is equivalent to the Eqs. (3.9) and (3.11).

In certain cases, especially when boundary conditions are given in terms of stresses, it is useful to utilize the equations analogous to Beltrami-Michell equations. These equations for uncoupled problems have been derived by J. Ignaczak [16], for coupled problems by E. Soós [17]. Another method of solutions in terms of stresses in reference to a plane state of deformation was given by W. Nowacki [18].

If the variability of body forces, heat sources, surface loadings and heatings is slow, then the inertial terms in the equations of motion can be deleted and the problem can be regarded as quasi-static. The quasi-static equations of thermoelasticity

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + X_i = \gamma \theta_{,i}, \tag{3.23}$$

$$\theta_{,jj} - \frac{1}{\kappa} \dot{\theta} - \eta \dot{u}_{j,j} = -\frac{Q}{\kappa}, \tag{3.24}$$

continue to be coupled. A solution for this system of equations is particularly simple for an unbounded thermoelastic medium in which act heat sources Q and body forces of the potential type $X_i = \varrho \vartheta_{,i}$.

By introducing the thermoelastic potential of displacement Φ, we obtain, from (3.23) and (3.24), the disjoined system of equations [15]:

$$\nabla^2 \theta - \frac{1}{\kappa_1} \dot{\theta} = -\frac{Q}{\kappa} - \frac{\eta}{c_1^2} \dot{\vartheta}, \quad \nabla^2 \Phi = m\theta - \frac{\vartheta}{c_1^2}, \qquad \kappa_1 = \frac{\kappa}{1+\varepsilon}, \quad \varepsilon = \eta m \kappa. \tag{3.25}$$

The temperature θ is determined here from a parabolic differential equation whose structure is similar to the classical equation of thermal conductivity.

For disjoining the system of Eqs. (3.23) and (3.24) we can also apply the manner presented previously (Eqs. (3.14) to (3.17)) which consists in disregarding the inertial terms appearing there. The mode given by M. A. Biot [4] is also interesting.

By introducing the expression for entropy

$$S = \gamma \varepsilon_{kk} + \frac{c_\varepsilon}{T_0} \theta, \tag{3.26}$$

into the Eqs. (3.23) and (3.24) with the assumption $Q = 0$, $X_i = 0$ we obtain the system of equations:

$$\mu u_{i,jj} + (\lambda+\mu+\delta) u_{j,ji} = \gamma\beta S_{,i}, \tag{3.27}$$

$$S_{,jj} - \frac{1}{\kappa_2} \dot{S} = 0, \qquad \delta = \gamma^2 \beta, \quad \beta = \frac{T_0}{c_\varepsilon}, \quad \kappa_2 = \kappa \frac{\lambda+2\mu}{\lambda+2\mu+\beta^2}. \tag{3.28}$$

These equations are disjoined and the entropy fulfills the parabolic equations. The solution of the Eqs. (3.27) can be written in the form of Papkowich-Boussinesq potentials

$$u_i = -(\psi_{0,i}+x_j\psi_j)_{,i}+B\psi_i, \quad B = 2\frac{\lambda+2\mu+\delta}{\lambda+\mu+\delta}, \tag{3.29}$$

with the assumption that the vector function ψ_i is harmonic. To determine the functions φ_i, ψ_0, we have at our disposal the following equations:

$$\nabla^2\psi_i = 0, \quad \nabla^2\psi_0' = 0 \quad \left(\nabla^2-\frac{1}{\kappa_2}\partial_t\right)\psi_0'' = 0, \tag{3.30}$$

where $\psi_0 = \psi_0'+\psi_0''$.

After determining the functions ψ_0, ψ_i and taking into account boundary and initial conditions, we shall obtain the displacements from the formula (3.29).

As was indicated at the outset, the thermoelasticity comprises complete division of directions hitherto developed so far separately: classical elastokinetics assuming that the motion operates in adiabatic conditions—i.e., without heat exchange between particular parts of body. Since, for an adiabatic process, we have $\dot{S} = 0$, we obtain from the formula (3.26) $\dot{\theta} = -\eta\kappa\dot{\varepsilon}_{kk}$ or after integrating and assuming homogeneous initial conditions:

$$\theta = -\eta_T\kappa\varepsilon_{kk}. \tag{3.31}$$

This equation replaces that of heat conduction.

Inserting (3.31) into (3.4), we obtain the displacement equation of classical elastokinetics

$$\mu_s u_{i,jj}+(\lambda_s+\mu_s)u_{j,ji}+X_i = \varrho\ddot{u}_i, \tag{3.32}$$

where $\lambda_s = \lambda_T+\gamma_T\eta_T\kappa$, $\mu_T = \mu_s$.

The quantities λ_s, μ_s are the Lamé constants measured in adiabatic conditions. The equations of state take after substituting (3.31) into (3.2), the form:

$$\sigma_{ij} = 2\mu_s\varepsilon_{ij}+\lambda_s\delta_{ij}\varepsilon_{kk}. \tag{3.33}$$

4. The Dynamic Problems of the Theory of Thermal Stresses

In the theory of thermal stresses in which the influence of body surface heating and the action of heat sources on deformation and the stress state of a body is considered the influence of the term appearing in the thermal conductivity equation on the body deformation is assumed to be very small and in practice negligible. This simplification leads to a system of two equations independent one of the other

$$\mu_T u_{i,jj}+(\lambda_T+\mu_T)u_{j,ji} = \varrho\ddot{u}_i+\gamma_T\theta_{,i}, \tag{4.1}$$

$$\theta_{,jj}-\frac{1}{\kappa}\dot{\theta} = -\frac{Q}{\kappa}. \tag{4.2}$$

The temperature θ is determined from (4.2)—i.e. from the classical equation of thermal conductivity. Knowing the temperature distribution, we are able to determine the displacements from the Eqs (4.1).

Decomposing the displacement vector into its potential and solenoidal parts

$$u_i = \Phi_{,i} + \epsilon_{ijk}\psi_{k,j} \tag{4.3}$$

we arrive at the system of wave equations:

$$\Box_1^2 \Phi = m\theta, \quad \Box_2^2 \psi_i = 0, \quad i = 1,2,3 \tag{4.4}$$

Eliminating temperature from the Eqs (4.2) and $(4.4)_1$, we obtain

$$\Box_1^2 D\Phi = -\frac{m}{\kappa} Q, \quad \Box_2^2 \psi_i = 0, \tag{4.5}$$

where

$$D = \nabla^2 - \frac{1}{\kappa}\partial_t .$$

The methods for solving the system of Eqs. (4.1) to (4.2) have been elaborated in detail [19–22].

The first of these equations refers to the longitudinal wave and the second to the transverse wave. It is evident that in an infinite space under the action of temperature (heat sources), only longitudinal waves arise. In a finite region, both types of waves occur, since the functions Φ and ψ_i, are connected by means of the boundary conditions.

There exist also other methods of solving the system of Eqs. (4.1). Representing the solution of the Eqs. (4.1) in the form

$$u_i = u_i' + u_i'' \tag{4.6}$$

where

$$u_i' = \Phi_{,i}, \quad u_i'' = \frac{\lambda+2\mu}{\mu}\Box_1^2 \varphi_j \delta_{ij} - \frac{\lambda+\mu}{\mu}\partial_i\partial_j\varphi_j, \tag{4.7}$$

we have to solve the system of equations:

$$\Box_1^2 \Phi = m\theta, \quad \Box_1^2 \Box_2^2 \varphi_i = 0, \quad i = 1,2,3. \tag{4.8}$$

Here, $u_i' = \Phi_{,i}$ is the particular solution of the non-homogeneous system of equations (4.1) and $\boldsymbol{\varphi}$ is the solution of the homogeneous system:

$$\mu\nabla^2 u_i'' + (\lambda+\mu) u_{k,ki}'' = \varrho \ddot{u}_i'' . \tag{4.9}$$

The vector $\boldsymbol{\varphi}$ is the familiar function of M. IACOVACHE [23].

It is also possible to determine the thermal stresses on the basis of the differential equations in stresses. They can be deduced by means of an appropriate transformation of the displacement equations [24].

Here we have the system of equations:

$$\Box_1^2 \sigma_{ij} + \frac{2(\lambda+\mu)}{3\lambda+2\mu}\sigma_{kk,ij} + \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\frac{\lambda\delta_{ij}}{3\lambda+2\mu}\ddot{\sigma}_{kk} + \tag{4.10}$$

$$+2\mu\alpha_t\left(\theta_{,ij} + \frac{3\lambda+2\mu}{\lambda+2\mu}\theta_{,kk}\delta_{ij}\right) - \frac{5\lambda+4\mu}{\lambda+2\mu}\alpha_t\varrho\delta_{ij}\ddot{\theta} = 0,$$

$$i,j,k = 1,2,3,$$

which should be completed by the boundary and initial conditions.

As in the case of stationary problems, we can use the concept of the nucleus of thermoelastic strain in order to determine the dynamic thermal stresses. By this concept, we understand the displacement field $u_i(\mathbf{x}, \boldsymbol{\xi}, t)$ produced by the temperature θ^* in the form of the Dirac delta function $\theta^* = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)$.

Denoting by $[S^*(\mathbf{x}, \boldsymbol{\xi}, t)]$ the solution of this problem, the solution $[S(\mathbf{x}, t)]$ referring to the temperature field $\theta(\mathbf{x}, t)$ is given by the formula:

$$(4.11)\qquad [S(\mathbf{x},t)] = \int_0^t d\tau \int_V [S^*(\boldsymbol{\xi},\mathbf{x},t-\tau)]\,\theta(\boldsymbol{\xi},\tau)\,dV(\boldsymbol{\xi})\,.$$

This method of solution is particularly convenient when the temperature field is discontinuous as a function of both position and time, so that it does not satisfy the heat conduction equation (4.2). J. Ignaczak and W. Piechocki [25, 26] obtained by this method many interesting results.

5. Stationary and Quasi-static Problems

In the case of steady flow of heat, the production of entropy is compensated by the exchange of entropy with environment. This exchange is negative and its absolute value is equal to entropy production in a body. In the equations of thermoelasticity (3.4) and (3.5) the derivatives with respect to time disappear. The Eq. (3.4) becomes

$$(5.1)\qquad \mu u_{i,jj}+(\lambda+\mu)u_{k,ki} = \gamma\theta_{,i}, \quad i,k = 1,2,3\,.$$

The temperature θ appearing in these equations is a known function, obtained by solving the heat conduction equation in the case of a stationary flow of heat:

$$(5.2)\qquad \nabla^2\theta = -\frac{Q}{\kappa},$$

completed by the appropriate boundary condition.

Assume that the boundary conditions for the Eqs. (5.1) are homogeneous. Set

$$(5.3)\qquad u_i(\mathbf{x}) = 0, \quad \mathbf{x}\in\Sigma_u, \quad p_i = \sigma_{ji}(\mathbf{x})\,n_j(\mathbf{x}) = 0, \quad \mathbf{x}\in\Sigma_u+\Sigma_\sigma\,.$$

The solution of the system (5.1) can be represented in the form

$$(5.4)\qquad u_i = u_i'+u_i'',$$

where u_i' is the particular solution of the non-homogeneous system of Eqs. (5.1), and u_i'' is the general solution of the homogeneous system (5.1). The particular solution can easily be found by introducing the potential of thermoelastic deformation related to the displacement u_i' by means of [27, 28]

$$(5.5)\qquad u_i' = \Phi_{,i}\,.$$

Substituting (5.4) and (5.5) into the system of Eqs. (5.1) we arrive at the system of equations,

$$(5.6)\qquad \nabla^2\Phi = m\theta, \quad m = \frac{\gamma}{\lambda+2\mu}$$

$$(5.7)\qquad \mu\nabla^2 u_i''+(\lambda+\mu)\,u_{k,ki}'' = 0\,.$$

The problem has been reduced to the solution of the Poisson equation and the system of the displacement equations of the theory of elasticity.

In an infinite elastic space, the solution of the Poisson equation (5.6) is the final solution:

$$\Phi(\xi) = -\frac{m}{4\pi}\int_V \frac{\theta(\mathbf{x})\,dV(\mathbf{x})}{R(\mathbf{x},\boldsymbol{\xi})}\,. \tag{5.8}$$

Here $R(\mathbf{x},\boldsymbol{\xi})$ is the distance between the points $\mathbf{x}$ and $\boldsymbol{\xi}$. The displacements, deformations and stresses are expressed in terms of the functions Φ by the relations:

$$u_i = \Phi_{,i}, \quad \varepsilon_{ij} = \Phi_{,ij}, \quad \sigma_{ij} = 2\mu(\Phi_{,ij} - \delta_{ij}\Phi_{,kk})\,. \tag{5.9}$$

In a bounded body, the function Φ can satisfy at most a part of the conditions, and therefore an additional solution is always necessary. The solution of the Eqs. (5.1) for abounded body can be represented in various forms. If we apply to the solution of Eqs. (5.7) the Papkovitch-Neuber functions, the displacement is assumed in the form:

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{grad}(\varphi + \mathbf{R}\cdot\boldsymbol{\psi}) - 4(1-\nu)\boldsymbol{\psi}\,, \tag{5.10}$$

where the functions φ, ψ_i satisfy the harmonic equation and the function Φ the Eq. (5.6).

If we introduce the Galerkin functions, then

$$u_i = \Phi_{,i} + \frac{\lambda+2\mu}{\mu}\nabla^2\chi_j\delta_{ij} - \frac{\partial+\mu}{\mu}\partial_i\partial_j\chi_j\,, \tag{5.11}$$

where the function Φ satisfies the Eq. (5.6) and the functions χ_i the biharmonic equations:

$$\nabla^2\nabla^2\chi_i = 0\,, \quad i = 1,2,3\,. \tag{5.12}$$

In problems possessing axial symmetry with respect to the z-axis, it is convenient to use the Love functions. In the system of cylindrical coordinates (r, ϑ, z), we express the displacement as follows:

$$u_r = \frac{\partial\Phi}{\partial r} - \frac{\partial^2\chi}{\partial r\,\partial z}\,, \quad u_z = \frac{\partial\Phi}{\partial z} + 2(1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\,; \tag{5.13}$$

here, the function Φ satisfies the Eq. (5.6) and the Love function the biharmonic equation:

$$\nabla^2\nabla^2\chi(r,z) = 0\,. \tag{5.14}$$

To determine the thermal stresses in bodies of simple shape (elastic space, semi-space, elastic layer, etc.), the method of integral transforms has been successfully applied to the Eqs. (3.1), [29, 30]. One more method of solving problems of the theory of thermal stresses is worth mentioning here. It consists in determination of the Green function as the solution of the equation:

$$\nabla^2\Phi^*(\mathbf{x},\boldsymbol{\xi}) = m\delta(\mathbf{x}-\boldsymbol{\xi})\,. \tag{5.15}$$

Here, $\delta(\mathbf{x}-\boldsymbol{\xi})$ is the Dirac delta function. The solution Φ^* of the Eq. (5.15) does not satisfy all boundary conditions and therefore, in the case of a bounded body, an additional solution $u'' = U^*(\mathbf{x}, \boldsymbol{\xi})$ is needed, which satisfies the system of homogeneous equations:

$$\mu\nabla^2 U_i^* + (\lambda+\mu)\,U_{k,ki}^* = 0\,. \tag{5.16}$$

Knowing the function $u_i^* = \Phi_{,i}^* + U_i^*(\mathbf{x}, \boldsymbol{\xi})$, we are in a position to determine the displacements $u_i(\mathbf{x})$ produced by the action of the temperature field $\theta(\mathbf{x})$. They are determined from the formula

$$u_i(\boldsymbol{\xi}) = \int_V \theta(\mathbf{x}) u_i^*(\mathbf{x}, \boldsymbol{\xi}) dV(\mathbf{x}). \tag{5.17}$$

This procedure is of considerable importance in the case of a discontinuous temperature field, when the function θ does not satisfy the heat conduction equation. Discontinuous temperature fields are encountered in certain specific cases,—for instance, when a part of the body is heated to a constant temperature $\theta^{(e)}$ and a part to the temperature $\theta^{(i)}$. A discontinuous temperature field is also obtained in the case of a body with different thermal properties and uniform elastic properties, heated to a constant temperature θ_0.

The functions Φ^*, u_i^* have been determined for bodies of simple shapes, such as elastic semi-space, sphere, infinite cylinder, layer, etc. [31, 32, 33].

Another method for the determination of the thermal stresses consists in making use of the Beltrami-Michell equations, in which in accordance with the body forces analogy, the body forces are replaced by the quantity $-\gamma\theta_{,i}$. Thus, we obtain the system of equations:

$$\nabla^2 \sigma_{ij} + \frac{2(\lambda+\mu)}{3\lambda+2\mu} \sigma_{kk,ij} + 2\mu\alpha_t \left(\theta_{,ij} + \frac{3\lambda+2\mu}{\lambda+2\mu} \delta_{ij} \theta_{,kk}\right) = 0, \quad i,j,k = 1,2,3. \tag{5.18}$$

Except for a certain few cases, these equations for spatial problems have not been widely applied.

On the other hand, the above method has been successfully used in two-dimensional problems, and in problems of the plane state of stress and strain. Introducing the Airy stress function related to the stresses by means of the expressions:

$$\sigma_{\alpha\beta} = -\partial_\alpha \partial_\beta F + \delta_{\alpha\beta} \nabla_1^2 F, \quad \nabla_1^2 = \partial_1^2 + \partial_2^2, \quad \alpha, \beta = 1,2, \tag{5.19}$$

we obtain after the elimination of the temperature, the equation:

$$\nabla_1^2 \nabla_1^2 F = \beta Q, \tag{5.20}$$

where for the plane state of stress $\beta = E\alpha_t$, while for the plane state of strain $\beta = E\alpha_t/(1-\nu)$. The Eq. (5.20) should be completed by the boundary conditions for a boundary free of tractions:

$$F = 0, \quad \partial F/\partial n = 0. \tag{5.21}$$

It follows from the Eq. (5.20) and the boundary conditions (5.21) that for a simply connected region and the plane state of stress we have $F \equiv 0$—i.e., the body deforms without stresses. In the plane state of strain, besides the stresses $\sigma_{\alpha\beta}$ given by the formulae (5.19), there appear the stresses:

$$\sigma_{33} = \nu \nabla_1^2 F - 2\mu m\theta. \tag{5.22}$$

Consequently, if there are no heat sources in a simply connected infinite cylinder (plane state of strain), then $F \equiv 0$ and the only non-vanishing stress is $\sigma_{33} = -2\mu m\theta$ (MUSKEL-

ISHVILI [35]). In the case of the presence of heat sources and a multiply connected body, the solution of the Eq. (5.20) can be deduced by using the complex variable functions (MUSKELISHVILI [36], GATEWOOD [37]). There is an interesting observation by DUBAS [38] and TREMMEL [39] concerning the analogy between the Eq. (5.20) and the equation of deflection of a thin plate fixed on the boundary. It is known that this deflection is described by the equation:

$$\nabla_1^2 \nabla_1^2 w = \frac{p}{N}, \tag{5.23}$$

and the boundary conditions

$$w = 0, \quad \frac{\partial w}{\partial n} = 0. \tag{5.24}$$

This „plate analogy" makes it possible to make use of the numerous solutions in the theory of plates to determine the thermal stresses in discs.

Every non-stationary problem of the theory of thermal stresses is a dynamic problem, but in the case of a slow variation of the temperature in time, the influence of the inertia forces is negligible and the problem may be regarded as quasi-static, the inertia terms in the equations of motion being disregarded. Thus, in the quasi-static case we are faced (assuming that $\eta = 0$) with the system of equations:

$$\mu \nabla^2 u_i + (\lambda + \mu) u_{k,ki} = \gamma \theta_{,i} \tag{5.25}$$

$$\left(\nabla^2 - \frac{1}{\kappa} \partial_t\right) \theta = -\frac{Q(\mathbf{x}, t)}{\kappa}. \tag{5.26}$$

From the Eq. (5.26) completed by the appropriate boundary and initial conditions we, find the temperature θ as a function of the variables $\mathbf{x}$, t, and as a known function it is introduced into the Eq. (5.25). The solution of (5.25) yields the displacements $u_i(\mathbf{x}, t)$, the time however, which appears in the displacements is regarded as a parameter. The method of solving the Eqs (5.25) is the same as for the stationary case. The function $\Phi(\mathbf{x}, t)$, however, can be determined in a different way, found by J. N. GOODIER [28]:

$$\Phi = m\kappa \int_0^t \theta \, dt + \Phi_0 + t\Phi_1,$$

here Φ_1 is a harmonic function and $\Phi_0 = \Phi(\mathbf{x}, 0)$ is the displacement potential corresponding to the initial temperature $\theta_0 = \theta(\mathbf{x}, 0)$. The function Φ_0 should satisfy the equation:

$$\nabla^2 \Phi_0(\mathbf{x}) = m\theta(\mathbf{x}).$$

The stationary and quasistatic problems of the theory of thermal stresses have already been treated in numerous scientific papers. Important methods of solution and important papers have been removed in various monographs—for instand E. MELAN and H. PARKUS *Wärmespannungen infolge stationäre Temperaturfelder* [19], H. PARKUS *Instationäre Wärmespannungen* [20], B. A. BOLEY and J. H. WEINER *Theory of thermal stresses* [21], W. NOWACKI *Thermoelasticity* [22]; we shall therefore not deal here with particular problems.

6. Variational Theorems of Thermoelasticity

We know how important a part is played by the variational theorems in the elasticity theory with variation of deformation state or stress state. They not only make it possible to derive the differential equations describing the bending of plates, shells, discs, membranes, etc., bul also to construct approximate solutions. In what follows we shall present the variational theorem with the variation of deformation state for thermoelasticity. This method was devised by M. A. Biot [4]. The theorem will consist of two parts; the first utilizes the d'Alambert principle familiar in elasticity theory:

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\,dV = \int_V (X_i-\varrho\ddot{u}_i)\delta u_i\,dV + \int_\Sigma p_i\,\delta u_i\,d\Sigma. \tag{6.1}$$

In this equation, δu_i are the virtual increments of displacements, $\delta\varepsilon_{ij}$ the virtual increments of deformations. We assume that δu_i and $\delta\varepsilon_{ij}$ are arbitrary continuous functions independent of time and complying with the conditions constraining the body motion.

The d'Alambert principle is valid irrespectively of the body material—i.e. for all forms of dependency of stress state on deformation state. Supplementing (6.1) with the equation of state, and introducing the quantity

$$\mathcal{W}_\varepsilon = \int_V \left(\mu\varepsilon_{ij}\,\varepsilon_{ij}+\frac{\lambda}{2}\,\varepsilon_{kk}\,\varepsilon_{nn}\right)dV, \tag{6.2}$$

where the integrand is a quadratic form positive definite, we obtain from (6.1) the following equation:

$$\delta\,\mathcal{W}_\varepsilon = \int_V (X_i-\varrho\ddot{u}_i)\delta u_i\,dV + \int_\Sigma p_i\delta u_i\,d\Sigma+\gamma\int_V \theta\,\delta e\,dV, \qquad e=\varepsilon_{kk}. \tag{6.3}$$

The second part of the variational theorem should make use advantage of the laws governing heat flow. Therefore, we shall utilize the expressions interrelating heat flow, temperature and entropy:

$$q_i = -\lambda_0\,\theta_{,i}, \qquad -q_{i,i} = \dot{S}T_0 = \gamma\dot{\varepsilon}_{kk}\,T_0+c_\varepsilon\dot{\theta}. \tag{6.4}$$

These relations can be written in a form more convenient for further studies by introducing the vector function H interrelated with entropy and flow in the following way:

$$S=-H_{i,i}, \qquad q_i = T_0\,\dot{H}_i \tag{6.5}$$

Combining (6.4) and (6.5) we obtain:

$$T_0\,\dot{H}_i = -\lambda_0\,\theta_{,i}, \qquad -T_0\,\dot{H}_{i,i} = c_\varepsilon\dot{\theta}+T_0\,\gamma\dot{\varepsilon}_{kk}. \tag{6.6}$$

Let us multiply the first of the Eqs (6.6) by the virtual increment δH_i, and integrate over the body region:

$$\int_V \left(\theta_{,i}+\frac{T_0}{\lambda_0}\,\dot{H}_i\right)\delta H_i\,dV = 0. \tag{6.7}$$

Through transforming this integral and taking into account the second of the relations (6.6), we obtain the equation:

$$\frac{c_\varepsilon}{T_0}\int_V \theta\,\delta\theta\,dV+\frac{T_0}{\lambda_0}\int_V \dot{H}_i\,\delta H_i\,dV+\int_\Sigma \theta n_i\,\delta H_i\,d\Sigma+\gamma\int_V \theta\,\delta e\,dV = 0, \tag{6.8}$$

in which is involved the term $\int_V \theta\, \delta e\, dV$ identical with that appearing in (6.3). Eliminating this term from the Eqs. (6.3) and (6.8), we obtain the final form of the variational theorem:

$$(6.9)\qquad \delta(\mathcal{W}_\varepsilon + P + D) = \int_V (X_i - \varrho \ddot{u}_i)\, \delta u_i\, dV + \int_\Sigma p_i\, \delta u_i\, d\Sigma - \int_V \theta n_i\, \delta H_i\, d\Sigma.$$

We have introduced here the following denotations:

$$(6.10)\qquad P = \frac{c_\varepsilon}{2T_0} \int_V \theta^2\, dV, \qquad D = \frac{T_0}{2\lambda_0} \int_V (\dot{H}_i)^2\, dV.$$

The function P is called the thermal potential, D—the dissipation function. Let us consider, moreover, the particular cases. If we assume $\theta = -\eta_T\, \kappa \varepsilon_{kk}$ in the Eq. (6.3), which corresponds to assuming the adiabatic process, then (6.3) transforms into

$$(6.11)\qquad \delta \mathcal{W}_s = \int_V (X_i - \varrho \ddot{u}_i)\, \delta u_i\, dV + \int_\Sigma p_i\, \delta u_i\, d\Sigma,$$

where

$$(6.12)\qquad \mathcal{W}_s = \int_V \left(\mu_s \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda_s}{2}\, \varepsilon_{kk} \varepsilon_{nn} \right) dV,$$

and μ_s, λ_s are the Lamé adiabatic constants. The Eq. (6.11) constitutes the d'Alambert principle for classical elastokinetics.

Let us now return to the general variational theorem of thermoelasticity (6.9) and assume that the virtual increments δu_i, $\delta \varepsilon_{ij}$, δH_i, etc. coincide with the increments really occurring when the process passes from a time instant t to $t+dt$. Then

$$(6.13)\qquad \delta u_i = \frac{\partial u_i}{\partial t}\, dt = v_i\, dt, \qquad \delta H_i = \frac{\partial H_i}{\partial t}\, dt = \dot{H}_i\, dt, \qquad \delta \mathcal{W}_\varepsilon = \dot{\mathcal{W}}_\varepsilon\, dt,$$

and so forth.

Putting (6.13) into (6.9), we obtain:

$$(6.14)\qquad \frac{d}{dt}(K + \mathcal{W}_\varepsilon + P) + \chi_T = \int_V X_i v_i\, dV + \int_\Sigma p_i v_i\, d\Sigma + \frac{\lambda_0}{T_0} \int_\Sigma \theta \theta_{,n}\, d\Sigma,$$

where $K = \frac{\varrho}{2} \int_V v_i v_i\, dV$ is the kinetic energy, χ_T—dissipation function, where

$$\chi_T = \lambda_0 T_0 \int_V \left(\frac{\theta_{,i}}{T_0} \right)^2 dV = \lambda_0 T_0 \int_V \left(\frac{q_i}{\lambda_0 T_0} \right)^2 dV.$$

The Eq. (6.14) is called the basic energy theorem of thermoelasticity. This theorem can be utilized to determine the uniqueness of solutions for the thermoelasticity Eqs. [21, 40].

Proceeding in a manner similar to the elasticity theory, we assume that the thermoelasticity equations are satisfied by two groups of functions u_i', θ' and u_i'', θ''. Constructing the difference between these functions $\hat{u}_i = u_i' - u_i''$, $\hat{\theta} = \theta' - \theta''$, and inserting it into the

Eqs. (6.4) and (6.5), we see that these equations are homogeneous and satisfy homogeneous boundery and initial conditions.

To the functions $\hat{u}_i$, $\hat{\theta}$ there then corresponds a thermodynamical body the interior of which is free of heat sources and body forces, which is not loaded on its surface, and is in conditions of zero temperature $\hat{\theta}$. The formula (6.14) will answer the question as to whether or not the displacements $\hat{u}_i$ and temperature $\hat{\theta}$ will appear in the interior of the body. Eq. (6.14) takes the form:

$$(6.15)\quad \frac{d}{dt}\int_V \left(\frac{\varrho}{2}\hat{v}_i\hat{v}_i+\mu\hat{\varepsilon}_{ij}\hat{\varepsilon}_{ij}+\frac{\lambda}{2}\hat{\varepsilon}_{kk}\hat{\varepsilon}_{nn}+\frac{\gamma}{2\eta\kappa}\hat{\theta}^2\right)dV = -\frac{\lambda_0}{T_0}\int_V (\hat{\theta}_{,i})^2\,dV \leqslant 0.$$

The integral appearing on the left-hand side of the equation is equal to zero at initial time instant since the functions $\hat{u}_i$, $\hat{\theta}$, $\hat{v}_i$, $\hat{\varepsilon}_{ij}$ satisfy the homogeneous initial conditions. On the other hand, the inequality derived indicates that the left-hand side of the equation either decreases assuming negative values, or is equal to zero. Since the expression under the integral sign is a sum of the second powers, and the integrand is equal to zero for $t = 0$, hence only the second of the alternatives referred to is possible. As a result, we obtain $\hat{v}_i = 0$, $\hat{\varepsilon}_{ij} = 0$, $\hat{\theta} = 0$ for $t \geqslant 0$. Since the stresses $\hat{\sigma}_{ij}$ are linearly related to the quantities $\hat{\varepsilon}_{ij}$, $\hat{\theta}$, then also $\hat{\sigma}_{ij} = 0$ for $t \geqslant 0$.

In consequence, we obtain:

$$(6.16)\qquad u_i' = u_i'', \quad \theta' = \theta'', \quad \sigma_{ij}' = \sigma_{ij}' \quad \text{for} \quad t \geqslant 0.$$

Then, there exists only one solution for the thermoelasticity equations.

In the theory of thermal stresses, we disregard the mutual interaction of the deformation and temperature fields which is expressed by deleting the term $\gamma\dot{\varepsilon}_{kk}T_0$ in the second of the Eqs. (6.4). Disregarding this term leads to a modified Eq. (6.8). We obtain:

$$(6.17)\qquad \delta(P+D)+\int_\Sigma \theta n_i\delta H_i\,d\Sigma = 0.$$

The Eq. (6.17) expresses the variational theorem for the classical uncoupled problem of thermal conductivity. In the theory of thermal stress, we have at our disposal two equations-viz., the Eqs. (6.17) and (6.3) in which the function θ is regarded of as the known function.

The Eq. (6.3) may be written as follows:

$$(6.18)\qquad \int_V (X_i-\varrho\ddot{u}_i-\gamma\theta_{,i})\,\delta u_i\,dV+\int_\Sigma (p_i+n_i\gamma\theta)\,\delta u_i\,d\Sigma = \delta\mathcal{W}_\varepsilon.$$

Comparing this expression with the virtual work performed by the forces X_i^*, p_i^* on the displacements δu_i in the body of the same shape and volume, and assuming that the process is isothermal, we have (for $\theta^* = 0$):

$$(6.19)\qquad \int_V (X_i^*-\varrho\ddot{u}_i)\,\delta u_i\,dV+\int_\Sigma p_i^*\,\delta u_i\,d\Sigma = \delta\mathcal{W}_\varepsilon.$$

We have assumed here that the external forces X_i^*, p_i^* are chosen in such a way that the displacement field u_i is identical with that produced by the action of the forces X_i, p_i and the temperature field θ. Comparing the Eqs (6.3) and (6.19) we obtain:

$$X_i^* = X_i - \gamma\theta_{,i}, \quad \mathbf{x}\in V; \quad p_i^* = p_i + n_i\gamma\theta, \quad \mathbf{x}\in\Sigma. \tag{6.20}$$

This is known as the analogy of body forces [21]. The relations (6.20) make it possible to reduce the problem of the theory of thermal stresses to the problems of the theory of elasticity.

Making use of the analogy of the body forces, we are in a position to state certain variational theorems for the theory of thermal stresses. Thus, the theorem of minimum of potential energy generalized to the theory of thermal stresses has the form:

$$\delta\Pi = 0, \quad \Pi = \textit{Minimum}, \tag{6.21}$$

where

$$\Pi = \mathcal{W}_\varepsilon - \int_V X_i u_i\, dV - \int_{\Sigma_\sigma} p_i u_i\, d\Sigma - \gamma \int_V \theta\varepsilon_{kk}\, dV.$$

Here Σ_σ is the part of the surface Σ bounding the body, on which the tractions p_i are known.

The theorem of minimum complementary work in the theory of thermal stresses has the form:

$$\delta\Gamma = 0 \quad \Gamma = \textit{Minimum}, \tag{6.22}$$

where

$$\Gamma = \mathcal{W}_\sigma - \int_V X_i u_i\, dV - \int_{\Sigma_u} p_i u_i\, dA + \alpha_t \int_V \theta\sigma_{kk}\, dV.$$

Here,

$$\mathcal{W}_\sigma = \int_V \left(\mu'\sigma_{ij}\sigma_{ij} + \frac{\lambda'}{2}\sigma_{kk}\sigma_{nn}\right) dV, \quad \mu' = \frac{1}{4\mu}$$

$$\lambda' = -\frac{\lambda}{2\mu(3\lambda+2\mu)},$$

Σ_u denotes the part of the surface Σ on which the displacements are prescribed.

Let us now return to the variational principle (6.3). If we assume that the virtual increments δu_i, $\delta\varepsilon_{ij}$ are identical with the real increments occurring in passing from the instant t to the instant $t+dt$, and bearing in mind that $\delta u_i = \dot{u}_i\, dt = v_i\, dt$, $\delta\varepsilon_{ij} = \dot{\varepsilon}_{ij}\, dt$, $\delta\mathcal{W}_\varepsilon = \dot{\mathcal{W}}_\varepsilon\, dt$, we have:

$$\frac{d}{dt}(\mathcal{W}_\varepsilon + K) = \int_V (X_i - \gamma\theta_{,i})\, v_i\, dV + \int_\Sigma (p_i + n_i\gamma\theta)\, v_i\, d\Sigma, \quad v_i = \dot{u}_i. \tag{6.23}$$

The Eq. (6.22) constitutes the fundamental energy theorem of the theory of thermal stresses. From this theorem we can deduce the uniqueness of the solution of the differential equations of the theory of thermal stresses for a simply connected body, and, moreover the generalization of the Kirchhoff theorem of the elasticity theory [21].

The principle of virtual work (6.2) makes it possible to derive the Hamilton principle for the theory of thermal stresses:

$$\delta \int_0^t (U-K)\, dt = \int_0^t \delta L\, dt, \tag{6.24}$$

where

$$U = \mathcal{W}_\varepsilon - \gamma \int_V \theta \varepsilon_{kk}\, dV, \quad \delta L = \int_V X_i \delta u_i\, dV + \int_\Sigma p_i \delta u_i\, d\Sigma.$$

In the case of conservative forces $\delta L = -\delta \mathcal{V} = \dfrac{\partial \mathcal{V}}{\partial u_i} \partial u_i$ we obtain from (6.24):

$$\delta \int_0^t (U - \mathcal{V} - K)\, dt = 0.$$

7. The Reciprocity Theorem

One of the most interesting theorems of the thermoelasticity theory is E. Betti's theorem on reciprocity. Not only does the symmetry fundamental solutions (of Green's function) follows from this theorem, but it also provides a basis for developing further methods for integrating the differential equations of the thermoelasticity theory.

The extended theorem on reciprocity concerning the thermoelasticity problems has been fully formulated by V. Ionescu-Cazimir [41]. The elements of this theorem can, although expressed in a less general form, be found in works by M. Biot [42].

We shall present the reciprocity theorem in its main outlines emphasizing its numerous applications.

Let two systems of forces act in an isotropic body. We assume that inside the body V, the heat sources and body forces operate, and on the body surface the loadings p_i and temperature $\theta = \vartheta$ are given. We denote these causes in the abbreviation $I = \{X_i, p_i, Q, \vartheta\}$, and the consequences ensuning from them—by the symbol $C = \{u_i, \theta\}$. The second system of causes and consequences is denoted by $I' = \{X_i', p_i', Q', \vartheta'\}$, $C' = \{u_i', \theta'\}$. The initial conditions are assumed to be homogeneous. Starting from motion equations, thermal conductivity equations, and Duhamel-Neumann relations written for both systems, adding those systems in an appropriate manner and integrating over the region V, we obtain two equations of reciprocity for the transforms of functions involved in the two systems:

$$\int_V (\overline{X}_i \bar{u}_i' - \overline{X}_i' \bar{u}_i)\, dV + \int_\Sigma (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\, d\Sigma + \gamma \int_V (\bar{\theta}\bar{e}' - \bar{\theta}'\bar{e})\, dV = 0, \tag{7.1}$$

$$\int_V (\bar{Q}'\bar{\theta} - \bar{Q}\bar{\theta}')\, dV + \eta\kappa p \int_V (\bar{\theta}'\bar{e} - \bar{\theta}\bar{e}')\, dV + \kappa \int_\Sigma (\bar{\vartheta}\bar{\theta}'_{,n} - \bar{\vartheta}'\bar{\theta}_{,n})\, d\Sigma = 0, \tag{7.2}$$

where

$$\bar{u}_i(\mathbf{x}, p) = \int_0^\infty u_i(\mathbf{x}, t)\, e^{-pt}\, dt,$$

on so forth.

The first of these equations arises from employing the equations of motion and of state, with application of the Green's transformation. Eliminating from these equations the common terms, we arrive at:

$$(7.3)\qquad \eta\kappa p\Big[\int_V (\bar{X}_i\bar{u}_i'-\bar{X}_i'\bar{u}_i)\,dV+\int_\Sigma (\bar{p}_i\bar{u}_i'-\bar{p}_i'\bar{u}_i)\,d\Sigma\Big]$$
$$=\kappa\gamma\int_\Sigma (\bar{\vartheta}'\bar{\theta}_{,n}-\bar{\vartheta}\bar{\theta}'_{,n})\,d\Sigma+\gamma\int_V (\bar{Q}\bar{\theta}'-\bar{Q}'\bar{\theta})\,dV.$$

The Eq. (7.3) should be subjested to the Laplace inverse transformation. After utilizing the theorem on convolution, we have

$$(7.4)\qquad \eta\kappa\Big\{\int_V dV(\mathbf{x})\int_0^t\Big[X_i(\mathbf{x},t-\tau)\frac{\partial u_i'(\mathbf{x},\tau)}{\partial\tau}-X_i'(\mathbf{x},t-\tau)\frac{\partial u_i(\mathbf{x},\tau)}{\partial\tau}\Big]d\tau$$
$$+\int_\Sigma d\Sigma(\mathbf{x})\int_0^t\Big[p_i(\mathbf{x},t-\tau)\frac{\partial u_i'(\mathbf{x},\tau)}{\partial\tau}-p_i'(\mathbf{x},t-\tau)\frac{\partial u_i(\mathbf{x},\tau)}{\partial\tau}\Big]d\tau\Big\}$$
$$=\gamma\int_V dV(\mathbf{x})\int_0^t[Q(\mathbf{x},t-\tau)\theta'(\mathbf{x},\tau)-Q'(\mathbf{x},t-\tau)\theta(\mathbf{x},\tau)]\,d\tau$$
$$+\gamma\kappa\int_\Sigma d\Sigma(\mathbf{x})\int_0^t[\vartheta'(\mathbf{x},t-\tau)\theta_{,n}(\mathbf{x},\tau)-\vartheta(\mathbf{x},t-\tau)\theta'_{,n}(\mathbf{x},\tau)]\,d\tau.$$

The Eq. (7.4) is valid for both dynamical and quasi-static problems. But in both cases, the functions u_i, θ and u_i', θ' have different meaning. We have assumed in our considerations that on the surface Σ, the loading p_i and the temperature $\theta=\vartheta$ are given. It is seen from the structure of the Eq. (7.4) that we can assume that on Σ, displacements and heat flow are proportional to the gradient of temperature $\theta_{,n}=\vartheta_{,n}$.

The Eqs. (7.4) are also satisfied for mixed boundary conditions. The Eq. (7.4) assumes a particularly simple form for an unbounded body, because, in this case, the surface integrals vanish.

If we encounter vibration harmonically varying in time $X_i(\mathbf{x},t)=X_i^*(\mathbf{x})\,e^{i\omega t}$, $p_i(\mathbf{x},t)=$ $=p_i^*(\mathbf{x})\,e^{i\omega t}$, and so forth, then the equation of reciprocity takes the form:

$$(7.4)\qquad \eta\kappa i\omega\Big[\int_V (X_i^*u_i'^*-X_i'^*u_i^*)\,dV+\int_\Sigma (p_i^*u_i'^*-p_i'^*u_i^*)\,d\Sigma\Big]$$
$$=\kappa\gamma\int_\Sigma (\vartheta'^*\theta^*_{,n}-\vartheta^*\theta'^*_{,n})\,d\Sigma+\gamma\int_V (Q^*\theta'^*-Q'^*\theta^*)\,dV.$$

We shall derive from the Eq. (7.4) a number of interesting conclusions. Let us assume that at the point $\boldsymbol{\xi}$ of the region V, the instantaneous force $X_i=\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{ij}$ acts, and is directed along the x_j -axis. If we assume that the boundary conditions are homogeneous, the relation (7.4) gives:

$$\frac{\partial u_j'(\boldsymbol{\xi},\boldsymbol{\xi}',t)}{\partial t}=\frac{\partial u_k(\boldsymbol{\xi}',\boldsymbol{\xi},t)}{\partial t}.$$

For the heat source $Q = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)$ and the source $Q' = \delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t)$, we have:

$$\theta'(\boldsymbol{\xi},\boldsymbol{\xi}',t) = \theta(\boldsymbol{\xi}',\boldsymbol{\xi},t).$$

If the concentrated and instantaneous force $X_i = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{ij}$ is applied at the point $\boldsymbol{\xi}$, and the heat source $Q' = \delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t)$ at the point $\boldsymbol{\xi}'$, then the following relation is obtained from the Eq. (7.4)

$$\theta(\boldsymbol{\xi}',\boldsymbol{\xi},t) = -\frac{\eta\kappa}{\gamma}\,\frac{\partial u'_j(\boldsymbol{\xi},\boldsymbol{\xi}',t)}{\partial t}.$$

Let the heat source $Q = \delta(x_1)\,\delta(x_2)\,\delta(x_3-vt)$ move with a constant velocity v in the direction of the x_3 -axis. Assuming that in the system of causes with „primes" $Q' = \delta\,(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t)$, we obtain from (7.4):

$$\theta(\xi_1,\xi_2,\xi_3,t) = \int_0^t \theta'(0,0,v\tau;\,\xi_1,\xi_2,\xi_3;t-\tau)\,d\tau.$$

The above formula enables determination of the temperature caused by the moving heat source, making use of the expression for temperature caused by the action of an instantaneous but not moving heat source.

From the Eqs. (7.1), (7.2) or (7.3), we can obtain particular forms of the reciprocity theorem which concern the classical elastokinetics and the thermal stresses theory.

If we assume that deformation takes place in adiabatic conditions, then we should put $\theta = -\eta_T\,\kappa\varepsilon_{kk}$, $\theta' = -\eta_T\,\kappa\varepsilon'_{kk}$ in (7.1). Then, the following equation remains:

$$\int_V (\overline{X}_i\bar{u}'_i-\overline{X}'_i\bar{u}_i)\,dV+\int_\Sigma (\bar{p}_i\bar{u}'_i-\bar{p}'_i\bar{u}_i)\,d\Sigma = 0. \tag{7.5}$$

The Eq. (7.2) disappears, since in elastokinetics we assume that heat sources do not exist in a body and the body surface is thermally isolated.

In the theory of thermal stresses, we disregard the dilatation term in the thermal conductivity equation. This omission is formally equivalent to putting $\eta = 0$ in the Eq. (7.2).

Thus, we obtain the equations:

$$\int_V (\overline{X}_i\bar{u}'_i-\overline{X}'_i\bar{u}_i)\,dV+\int_\Sigma (\bar{p}_i\bar{u}'_i-\bar{p}'_i\bar{u}_i)\,d\Sigma+\gamma\int_V (\bar{\theta}\bar{e}'-\bar{\theta}'\bar{e})\,dV = 0, \tag{7.6}$$

$$\int_V (\bar{Q}\bar{\theta}'-\bar{Q}'\bar{\theta})\,dV-\kappa\int_\Sigma (\bar{\vartheta}\bar{\theta}'_{,n}-\bar{\vartheta}'\bar{\theta}_{,n})\,d\Sigma = 0. \tag{7.7}$$

The Eq. (7.6) has been derived by W. M. Maysel [43].

The Eq. (7.7) is the equation of reciprocity for the classical equation of thermal conductivity.

We shall, moreover, consider the case in which the causes $I = \{X_i, p_i, Q, \vartheta\}$ and consequences $C = \{u_i, \theta\}$ refer to a coupled problem of thermoelasticity, and the causes $I' = \{X'_i, p'_i, Q', \vartheta'\}$ and consequences $C' = \{u'_i, \theta'\}$—to an uncoupled problem. Taking into account the difference in the thermal conductivity equations for coupled and uncoupled problems

$$\bar{\theta}_{,kk} - \frac{p}{\kappa}\bar{\theta} - \eta p \bar{e} = -\frac{\bar{Q}}{\kappa},$$

(7.8)

$$\bar{\theta}'_{,kk} - \frac{p}{\kappa}\bar{\theta}' = -\frac{\bar{Q}'}{\kappa},$$

we obtain instead of the Eq. (7.8) the following equation:

$$\int_V (\bar{Q}'\bar{\theta} - \bar{Q}\bar{\theta}')\,dV + \eta\kappa p \int_V \bar{\theta} e'\,dV + \kappa \int_\Sigma (\bar{\vartheta}\bar{\theta}'_{,n} - \bar{\vartheta}'\bar{\theta}_{,n})\,d\Sigma = 0. \tag{7.9}$$

Eliminating the term $\int_V \bar{\theta}'\bar{e}\,dV$ from the Eqs. (7.1) and (7.9), we obtain the reciprocity theorem in the form:

$$\kappa\eta p\Big[\int_V (\bar{X}_i\bar{u}'_i - \bar{X}'_i\bar{u}_i)\,dV + \int_\Sigma (\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i)\,d\Sigma + \gamma\int_V \bar{\theta}\bar{e}'\,dV\Big] \tag{7.10}$$
$$= \kappa\gamma\int_\Sigma (\bar{\vartheta}'\bar{\theta}_{,n} - \bar{\vartheta}\bar{\theta}'_{,n})\,d\Sigma + \gamma\int_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV.$$

Let us now assume that only a concentrated and instantaneous heat source acts in the system with „primes" and boundary conditions are homogeneous. Inserting then into the Eq. (7.10)

$$Q' = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t), \quad X'_i = 0, \quad p'_i = 0, \quad \vartheta' = 0,$$

we have:

$$\bar{\theta}(\boldsymbol{\xi}, p) + \eta\kappa p \int_V \bar{\theta}(\mathbf{x}, p)\,\bar{e}'(\mathbf{x}, \boldsymbol{\xi}, p)\,dV(\mathbf{x}) = \bar{M}(\boldsymbol{\xi}, p), \tag{7.11}$$

where

$$\bar{M}(\boldsymbol{\xi}, p) = \int_V \bar{Q}(\mathbf{x}, p)\,\bar{\theta}'(\mathbf{x}, \boldsymbol{\xi}, p)\,dV(\mathbf{x}) - \kappa\int_\Sigma \bar{\vartheta}(\mathbf{x}, p)\,\bar{\theta}'_{,n}(\mathbf{x}, \boldsymbol{\xi}, p)\,d\Sigma(\mathbf{x})$$
$$-\frac{\eta\kappa p}{\gamma}\Big[\int_\Sigma \bar{p}_i(\mathbf{x}, p)\,\bar{u}_i(\mathbf{x}, \boldsymbol{\xi}, p)\,d\Sigma(\mathbf{x}) + \int_V \bar{X}_i(\mathbf{x}, p)\,\bar{u}'_i(\mathbf{x}, \boldsymbol{\xi}, p)\,dV(\mathbf{x})\Big].$$

Since the functions u'_i, θ' are known as solutions for the differential equations of the thermal stresses theory, and the functions $\bar{Q}$, $\bar{\vartheta}$, $\bar{p}_i$, $\bar{X}_i$ are given, then the function $\bar{M}(\boldsymbol{\xi}, p)$ is known. The Eq. (7.11) is a Fredholm nonhomogeneous integral equation of the second kind in which the temperature θ appears as an unknown function. Also, displacements can be obtained in a similar manner.

The procedure here presented was proposed by V. Ionescu-Cazimir [41] and applied for determining the Green's function in an unbounded elastic region [44, 45].

For the stationary and static problem, we have [43]:

$$\int_V X_i(\mathbf{x})\,u'_i(\mathbf{x})\,dV(\mathbf{x}) + \int_\Sigma p_i(\mathbf{x})\,u'_i(\mathbf{x})\,d\Sigma(\mathbf{x}) + \gamma\int_V \theta(\mathbf{x})\,\varepsilon'_{kk}(\mathbf{x})\,dV(\mathbf{x}) \tag{7.12}$$
$$= \int_V X'_i(\mathbf{x})\,u_i(\mathbf{x})\,dV(\mathbf{x}) + \int_\Sigma p'_i(\mathbf{x})\,u_i(\mathbf{x})\,d\Sigma(\mathbf{x}) + \gamma\int_V \theta'(\mathbf{x})\,\varepsilon_{kk}(\mathbf{x})\,dV(\mathbf{x}).$$

Let us consider a particular case of this theorem. Consider a bounded body fixed on the surface Σ_u and free of tractions on the surface Σ_σ: $\Sigma = \Sigma_u + \Sigma_\sigma$. The displacement $u_k(\xi)$ due to the heating of the body is obtained from the formula following from the theorem (7.12):

$$u_k(\xi) = \gamma \int_V \theta(\mathbf{x})\, U_{j,j}^{(k)}(\mathbf{x}, \xi)\, dV(\mathbf{x}). \tag{7.13}$$

Here $u'_j = U_j^{(k)}(\mathbf{x}, \xi)$ is the field of displacement occurring in a body of the same shape and the same boundary conditions, in the isothermal process $\theta' = 0$. The displacements $U_j^{(k)}$ result from the action of a concentrated force located at the point ξ and directed parallel to the x_k -axis. Formula (7.13) given by W. M. MAYSEL constitutes a method of solution of the equations of the theory of thermal stresses by means of the Green function. This method was applied by MAYSEL to the solution of a number of examples concerning thermal stresses in plates and shells. In these cases, the determination of the Green function for various shapes and boundary conditions does not encounter any serious difficulties.

The reciprocity theorem (7.12) yielded an interesting result concerning changes in the volume of a body. The increment of the volume of a simply connected body, heated and free of tractions on its surface Σ is given by the formula [46]:

$$\Delta V = 3\alpha_t \int_V \theta(\mathbf{x})\, dV(\mathbf{x}).$$

The formula (2.18) yields the statement that the mean values of the stress invariant vanish [47]:

$$\int_V \sigma_{kk}\, dV = 0.$$

8. Methods for Integrating Thermoelasticity Equations Following from the Reciprocity Theorem

In elastostatics, an expression is derived which interrelates displacement $u_i(\mathbf{x}, t)$, $\mathbf{x} \in V$, $t > 0$ inside a body with displacements u_i and loadings p_i on the body surface. Those relations are familiar as the Somiglian and Green theorem [48]. We shall present below such theorems extended for thermoelasticity problems.

Let us assume that causes producing deformations and temperature in the body are expressed solely by initial conditions. The initial conditions are assumed to be homogeneous. The equations describing the body motion are of the form:

$$\sigma_{ji,j} = \varrho \ddot{u}_i, \quad \theta_{,jj} - \frac{1}{\kappa}\dot{\theta} - \eta \dot{e} = 0, \quad \mathbf{x} \in V, \quad t > 0. \tag{8.1}$$

We add the equations of state to these equations:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma\theta)\,\delta_{ij}. \tag{8.2}$$

We consider the second system of equations with „primes" concerning an unbounded thermoelastic body:

$$(8.3) \qquad \sigma'_{ji,j} = \varrho \ddot{u}'_i, \quad \theta'_{,jj} - \frac{1}{\kappa}\dot{\theta}' - \eta \dot{e}' = -\frac{1}{\kappa}\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t), \quad \mathbf{x} \in V, \quad t > 0,$$

and the Duhamel-Neumann equations:

$$(8.4) \qquad \sigma'_{ij} = 2\mu\varepsilon'_{ij} + (\lambda\varepsilon'_{kk} - \gamma\theta')\delta_{ij},$$

In the Eqs. (8.1) to (8.4), we perform the Laplace transformation taking into accoun, homogeneous initial conditions, and next we add these equations appropriately and effect integration over the region V.

After a number transformations which are omitted here, we obtain finally [49]:

$$(8.5) \qquad \bar{\theta}(\mathbf{x}, p) = -\frac{\eta\kappa p}{\gamma}\int_{\Sigma}[\bar{p}_i(\boldsymbol{\xi}, p)\bar{u}'_i(\boldsymbol{\xi}, \mathbf{x}, p) - \bar{p}'_i(\boldsymbol{\xi}, \mathbf{x})\bar{u}_i(\boldsymbol{\xi}, p)]\, d\Sigma(\boldsymbol{\xi})$$
$$-\kappa\int_{\Sigma}[\bar{\vartheta}'(\boldsymbol{\xi}, \mathbf{x}, p)\bar{\theta}_{,n}(\boldsymbol{\xi}, p) - \bar{\vartheta}(\boldsymbol{\xi}, p)\bar{\theta}'_{,n}(\boldsymbol{\xi}, \mathbf{x}, p)]\, d\Sigma(\boldsymbol{\xi}).$$

This formula can also be derived from the reciprocity theorem (7.3), assuming $Q' = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)$, $X_i = 0$, $X'_i = 0$, $Q = 0$.

Let us consider, in turn, the second system of equations:

$$(8.6) \qquad \sigma^s_{ji,j} = \varrho\ddot{u}^s_i - \delta(\mathbf{x}-\boldsymbol{\xi})\delta_{is}\delta(t),$$

$$(8.7) \qquad \theta^s_{,jj} - \frac{1}{\kappa}\dot{\theta}^s - \eta\dot{e}^s = 0,$$

$$(8.8) \qquad \sigma^s_{ij} = 2\mu\varepsilon^s_{ij} + (\lambda\varepsilon^s_{kk} - \gamma\theta^s)\delta_{ij}.$$

The functions u^s_i, θ^s are assigned to an unbounded thermoelastic region. They are induced by action of an instantaneous concentrated force $X_i = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{is}$ directed towards the x_s -axis. Putting $X'_i = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t)\,\delta_{is}$, $X_i = 0$, $Q = 0$, $Q' = 0$, in the reciprocity theorem, we obtain the following expression for displacements u_s [50]:

$$(8.9) \qquad u_s(\mathbf{x}, p) = \int_{X}[\bar{p}_i(\boldsymbol{\xi}, p)\bar{u}^s_i(\boldsymbol{\xi}, \mathbf{x}, p) - \bar{p}^s_i(\boldsymbol{\xi}, \mathbf{x}, p)u_i(\boldsymbol{\xi}, p)]\, d\Sigma(\boldsymbol{\xi})$$
$$-\frac{\gamma}{\eta p}\int_{\Sigma}[\bar{\theta}_{,n}(\boldsymbol{\xi}, p)\,\bar{\theta}^s(\boldsymbol{\xi}, \mathbf{x}, p) - \bar{\vartheta}(\boldsymbol{\xi}, p)\,\bar{\theta}^s_{,n}(\boldsymbol{\xi}, \mathbf{x}, p)]\, d\Sigma(\boldsymbol{\xi}).$$

The Eqs (8.5) and (8.9) should be subjected additionally to the Laplace inverse transformation, which leads to convolution expressions which are omitted here.

The Eqs (8.5) and (8.9) constitute the generalization of Somiglian's equations for the thermoelasticity problems. Making use of them, we are able to express the functions $u_i(\mathbf{x}, t)$, $\theta(\mathbf{x}, t)$, $\mathbf{x} \in V$, $t > 0$ in terms of surface integrals in which the functions u_i, θ and their derivatives appear.

If the Green's functions $\bar{u}'_i$, $\bar{\theta}'$ and $\bar{u}^s_i$, $\bar{\theta}^s$ are so chosen that they refer to a body occuping the region V bounded by the surface Σ, and if it is assumed that the following boundary conditions should be satisfied on

$$\bar{u}_i' = 0, \quad \bar{\vartheta}' = 0, \quad \bar{u}_i^s = 0, \quad \bar{\theta}^s = 0 \quad \text{on } \Sigma,$$

then the Eqs (8.5) and (8.9) are simplified to the form:

$$(8.10) \quad \bar{\theta}(\mathbf{x}, p) = \kappa \int_\Sigma \bar{\vartheta}(\boldsymbol{\xi}, p)\, \bar{\theta}'_{,n}(\boldsymbol{\xi}, \mathbf{x}, p)\, d\Sigma(\boldsymbol{\xi}) + \frac{\eta\kappa p}{\gamma} \int_\Sigma \bar{p}_i'(\boldsymbol{\xi}, \mathbf{x}, p)\, \bar{u}_i(\boldsymbol{\xi}, p)\, d\Sigma(\boldsymbol{\xi}),$$

$$(8.11) \quad u_s(\mathbf{x}, p) = -\int_\Sigma \bar{p}_i^s(\boldsymbol{\xi}, \mathbf{x}, p)\, \bar{u}_i(\boldsymbol{\xi}, p)\, d\Sigma(\boldsymbol{\xi}) + \frac{\gamma}{\eta p} \int_\Sigma \bar{\vartheta}(\boldsymbol{\xi}, p)\, \bar{\theta}^s_{,n}(\boldsymbol{\xi}, \mathbf{x}, p)\, d\Sigma(\boldsymbol{\xi}).$$

These formulae constitute the solution of the first boundary problem in which displacements u_i and temperature θ are given on Σ. If the functions $\bar{u}_i'$, $\bar{\theta}'$ and $\bar{u}_i^s$, $\bar{\theta}^s$ are related to a body accupying the region V free from loadings and temperature on the surface Σ, it should be added to the Eqs. (8.5) and (8.9)

$$\bar{p}_i' = 0, \quad \bar{\vartheta}' = 0, \quad \bar{p}_i^s = 0, \quad \bar{\theta}^s = 0 \quad \text{on } \Sigma.$$

In this case, the formulae (8.5) and (8.9) assume the form:

$$(8.12) \quad \bar{\theta}(\mathbf{x}, p) = -\frac{\eta\kappa p}{\gamma} \int_\Sigma \bar{p}_i(\boldsymbol{\xi}, p)\, \bar{u}_i'(\boldsymbol{\xi}, \mathbf{x}, p)\, d\Sigma(\boldsymbol{\xi}) + \kappa \int_\Sigma \bar{\vartheta}(\boldsymbol{\xi}, p)\, \bar{\theta}'_{,n}(\boldsymbol{\xi}, \mathbf{x}, p)\, d\Sigma(\boldsymbol{\xi}),$$

$$(8.13) \quad \bar{u}_s(\mathbf{x}, p) = \int_\Sigma \bar{p}_i(\boldsymbol{\xi}, p)\, \bar{u}_i^s(\boldsymbol{\xi}, \mathbf{x}, p)\, d\Sigma(\boldsymbol{\xi}) + \frac{\gamma}{\eta p} \int_\Sigma \bar{\vartheta}(\boldsymbol{\xi}, p)\, \bar{\theta}^s_{,n}(\boldsymbol{\xi}, \mathbf{x}, p)\, d\Sigma(\boldsymbol{\xi}),$$

and constitute the solution of the second boundary problem in which loadings p_i and temperature θ are given on the surface Σ. However, the application of the formulae (8.10) to (8.13) is restricated owing to the difficulties associated with obtaining the Green's functions u_i', θ', u_i^s, θ^s satisfying the specified boundary conditions. In a manner analogous to that for the extended Somiglian's and Green's formulae, we can construct the solution of thermoelasticity equations for mixed boundary conditions. One such manner, which is the extension of W. M. MAYSEL's methods from the thermal problems theory to thermoelasticity problems, can be found in the previously cited work by V. IONESCU-CAZIMIR [41] It consists in using the Green's functions satisfying at once mixed boundary conditons. Another manner, devised by W. NOWACKI [50] consists in making use of the Green's auxiliary functions fulfilling continuous boundary conditions and reducing the problem to solution of the system of Fredholm's integral equations of the first kind.

9. Harmonic Waves

Discussion of a wave of the simplest type—i.e. the plane harmonic wave,—immediately reveals the essential properties of the propagation of elastic waves, their character, velocity of wave propagation, wave dispersion and damping. Also the fundamental differences between thermoelastic waves and elastic and thermal waves will be disclosed [51] and [52].

Let us consider a harmonic plane wave, moving in the direction of the x_1-axis, induced by a cause of mechanical or thermal nature. Since displacements u_j and temperature θ depend solely on the variables x_1 and t, the displacement equations and the thermal conductivity equations, assume taking into account that

(9.1) $$u_j = \operatorname{Re}[u_j^*(x_1, \omega) e^{i\omega t}], \quad \theta = \operatorname{Re}[\theta^*(x_1, \omega) e^{i\omega t}],$$

assume the form:

(9.2) $$(\partial_1^2 + \sigma^2) u_1^* = m\partial_1 \theta^*, \quad (\partial_1^2 + q)\theta^* + \eta\kappa p \partial_1 u_1^* = 0,$$
$$(\partial_1^2 + \tau^2) u_2^* = 0, \quad (\partial_1^2 + \tau^2) u_3^* = 0,$$

where $\sigma^2 = \dfrac{\omega^2}{c_1^2}$, $\tau^2 = \dfrac{\omega^2}{c_2^2}$, $q = \dfrac{i\omega}{\kappa}$.

Eliminating the temperature θ^* from the first two equations, we have:

(9.3) $$[(\partial_1^2 + \sigma^2)(\partial_1^2 + q) + q\varepsilon\partial_1^2] u_1^* = 0, \quad (\partial_1^2 + \tau^2) u_2^* = 0, \quad (\partial_1^2 + \tau^2) u_3^* = 0.$$

The first equation refers to a longitudinal wave; the two remaining ones—to transverse waves.

If we insert

$$u_1^* = u^0 e^{ikx_1}, \quad \theta^* = \theta^0 e^{ikx_1},$$

into the two first Eqs. (7.2), we obtain:

$$\frac{u^0}{\theta^0} = \frac{mik}{\sigma^2 - k^2}, \quad \frac{\theta^0}{u^0} = -\frac{\eta q \kappa i k}{q - k^2}.$$

After eliminating the quantity u^0/θ^0 from these relation, we obtain the following algebraic equation:

(9.4) $$k^4 - k^2(\sigma^2 + q(1+\varepsilon)) + q\sigma^2 = 0, \quad \varepsilon = \eta m \kappa,$$

from which, we determine the roots:

$$\left.\begin{matrix} k_1^2 \\ k_2^2 \end{matrix}\right\} = \frac{1}{2}\{\sigma^2 + q(1+\varepsilon) \pm [(\sigma^2 + q(1+\varepsilon))^2 - 4q\sigma^2]^{1/2}\}.$$

These roots are the functions of the parameter ε: $k_1 = k_1(\varepsilon)$, $k_2 = k_2(\varepsilon)$. For $\varepsilon = 0$, we have $k_1(0) = \lambda_1 = \sigma$, $k_2(0) = \lambda_2 = \sqrt{q}$.

The following functions are the solutions of the first two Eqs. (9.2)

(9.5)
$$u_1 = u_+^0 \exp[-i\omega t + ikx_1] + u_-^0 \exp[-i\omega t - ik_1 x_1] + \frac{mik_2}{\sigma^2 - k_2^2}\{\theta_+^0 \exp[-i\omega t + ik_2 x_1] - \theta_-^0 \exp[-i\omega t - ik_2 x_1]\},$$

$$\theta = \theta_+^0 \exp[-i\omega t + ik_2 x_1] + \theta_-^0 \exp[-i\omega t - ik_2 x_1] + \frac{\eta\kappa q i k_1}{k_1^2 - q}\{u_+^0 \exp[-i\omega t + ik_1 x_1] - u_-^0 \exp[-i\omega t - ik_1 x_1]\}.$$

The transverse waves are given by the relations

$$(9.6)\qquad \begin{aligned} u_2 &= B_+ \exp\left[-i\omega\left(t-\frac{x_1}{c_2}\right)\right] + B_- \exp\left[-i\omega\left(t+\frac{x_1}{c_2}\right)\right], \\ u_3 &= C_+ \exp\left[-i\omega\left(t-\frac{x_1}{c_2}\right)\right] + C_- \exp\left[-i\omega\left(t+\frac{x_1}{c_2}\right)\right]. \end{aligned}$$

They move with constant velocity $c_2 = (\mu/\varrho)^{1/2}$.

These waves do not cause volume change and do not produce a temperature field accompanying the wave motion.

The set of Eqs. (9.5) will be called equations of thermoelastic waves. The first Eq. (9.5) presents a longitudinal wave, the second—the temperature accompanying to these waves. Denoting by v_β ($\beta = 1, 2$) the phase velocity, and by ϑ_β the damping coefficient and combining them with the roots of the Eq. (9.4) by means of the relations

$$v_\beta = \frac{\omega}{\operatorname{Re}(k_\beta)} \qquad \vartheta_\beta = \operatorname{Im}(k_\beta), \qquad \beta = 1, 2,$$

we transform the Eq. (9.5) into the form:

$$(9.7)\qquad \begin{aligned} u_1 &= u_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right) - \vartheta_1 x_1\right] + u_-^0 \exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right) + \vartheta_1 x_1\right] \\ &\quad + \frac{mik_2}{\sigma^2 - k_2^2}\left\{\theta_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right) - \vartheta_2 x_1\right] - \theta_-^0 \exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right) + \vartheta_2 x_1\right]\right\}, \\ \theta &= \theta_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right) - \vartheta_2 x_1\right] + \theta_-^0 \exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right) + \vartheta_2 x_1\right] \\ &\quad + \frac{\eta\kappa q i k_1}{k_1^2 - q}\left\{u_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right) - \vartheta_1 x_1\right] - u_-^0 \exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right) + \vartheta_1 x_1\right]\right\}. \end{aligned}$$

It is seen that both waves are damped and subjected to dispersion because the phase velocities v_β depend on frequencies ω. The physical meaning of the waves (9.7) is clear if we compare them with waves in a hypothetical medium characterized by the zero value of linear expansion α_t. For $\alpha_t = 0$, and then for $\eta = 0$, $m = 0$ the two first of Eqs. (9.2) become

$$(9.8)\qquad (\partial_1^2 + \sigma^2)\hat{u}_1^* = 0, \quad (\partial_1^2 + q)\hat{\theta}^* = 0.$$

The solutions for these equations take the form

$$(9.9)\qquad \begin{aligned} \hat{u}_1^* &= u_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{c_1}\right)\right] + u^0 \exp\left[-i\omega\left(t+\frac{x_1}{c_1}\right)\right], \\ \hat{\theta}^* &= \theta_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{\hat{v}_2}\right) - \hat{\vartheta}_2 x_1\right] + \theta_-^0 \exp\left[-i\omega\left(t+\frac{x_1}{\hat{v}_2}\right) + \hat{\vartheta}_2 x_1\right] \end{aligned}$$

where $\hat{v}_2 = (2\kappa\omega)^{1/2}$, $\quad \hat{\vartheta}_2 = (\omega/2\kappa)^{1/2}$, $\quad c_1 = \varrho^{-1/2}(\lambda_T + 2\mu_T)^{1/2}$

Here $\hat{u}_1^*$ represents a purely elastic wave moving in the direction of the x_1-axis or $-x_1$-axis with constant velocity $\hat{v}_1 = c_1$. These waves are subjected neither to damping nor to dispersion. The second of the Eqs (9.9) represents a wave purely thermal wave undergoing damping and dispersion. The damping is characterized by the coefficient $\vartheta_2 = \mathrm{Im}(\lambda_2) = (\omega/2\kappa)^{1/2}$.

Dispersion here takes place since the phase velocity $\hat{u}_2 = \omega/\mathrm{Re}(\lambda_2) = (2\kappa\omega)^{1/2}$ is a function of the frequency ω. The Eqs. (9.7) describe the modified longitudinal wave and the modified thermal wave. Comparison of (9.7) and (9.9), it results that the root $k_1(\varepsilon)$ characterizes the quasi-elastic form of a thermoelastic wave, since $k_1(0) = \sigma = \omega/c_1$ refers to the purely elastic wave. Similarly, the root $k_2(\varepsilon)$ characterizes the form of a quasi-thermal wave, whereas $k_2(0) = \lambda_2 = \sqrt{q}$ concerns the purely thermal waves in a hypothetic medium. It is of interest that in the modified elastic wave (the first equation of the group (9.9), there appear close one to one other the quasi-elastic terms:

$$u_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right)-\vartheta_1 x_1\right], \quad u_-^0 \exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right)+\vartheta_1 x_1\right],$$

and the quasi-thermal terms:

$$\theta_+^0 \exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right)-\vartheta_2 x_1\right], \quad \theta_-^0 \exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right)+\vartheta_2 x_1\right].$$

A similar situation exists in the modified thermal wave. Moreover, we should discuss the roots k_1, k_2 or the quantities v_β, ϑ_β, $\beta = 1, 2$. Introducing the new denotations

$$\zeta = \frac{c_1}{\omega^*}k, \quad \omega^* = \frac{c_1^2}{\kappa}, \quad \chi = \frac{\omega}{\omega^*}$$

we lead the Eq. (9.4) takes takes the simple form:

$$\zeta^4 - \zeta^2[\chi^2 + i\chi(1+\varepsilon)] + i\chi^3 = 0. \tag{9.10}$$

The roots ζ_1, ζ_2 of this equation are the functions of the parameters ε and $\chi = \omega/\omega^*$. The quantity $\varepsilon = \eta m\kappa$ is a constant depending on the thermal and mechanical properties of materials (whereas the χ changes together with a change in frequency ω). The quantity ω^* is a characteristic quantity for the given material.

The frequency of forced vibrations ω is limited by the quantity

$$\omega_c = 2\pi(c_1)_s\left(\frac{3\varrho}{4\pi M}\right)^{1/2}$$

resulting from the Debye spectrum for longitudinal waves [53]. In this formula, M denotes the atomic masse of a material constituting an elastic body, and $(c_1)_s = \varrho^{-1/2}(\lambda_s + 2\mu_s)^{1/2}$, where λ_s, μ_s are the Lamé's constants for an adiabatic state.

The fundamental values for four metals are set in the Table [52].

	Aluminium	Copper	Steel	Lead
$(c_1)_s$ cm/sek	$6.32\cdot 10^5$	$4.36\cdot 10^5$	$5.80\cdot 10^5$	$2.14\cdot 10^5$
ε	$3.56\cdot 10^{-2}$	$1.68\cdot 10^{-2}$	$2.97\cdot 10^{-4}$	$7.33\cdot 10^{-2}$
ω^* sek^{-1}	$4.66\cdot 10^{11}$	$1.73\cdot 10^{11}$	$1.75\cdot 10^{12}$	$1.91\cdot 10^{11}$
ϑ_1^∞ cm^{-1}	$1.31\cdot 10^4$	$3.29\cdot 10^3$	$4.48\cdot 10^2$	$3.27\cdot 10^4$
ω_c sek^{-1}	$9.80\cdot 10^{13}$	$7.55\cdot 10^{13}$	$9.95\cdot 10^{13}$	$3.69\cdot 10^{13}$

In the Table, is also given the damping coefficient ϑ_1^∞ for $\chi = \infty$, where $\vartheta_1^\infty = \frac{1}{2}\varepsilon\omega^*/(c_1)_T$.

Let us note that ω_c is considerably greater than ω^*. In the laboratory tests performed by means of ultrasonic vibration of very high frequency, we have:

$$\omega_c > \omega^* \gg \omega,$$

so that for mechanical vibration encountered in practice it can be assumed that $\chi = \omega/\omega^* \ll 1$.

The phase velocity v_1 is greater than $(c_1)_T$ and tends to this value as $\chi \to \infty$. The damping coefficient ϑ_1 increases together with χ, and at small frequencies it is proportional to χ^2, approaching the asymptotic value ϑ_1^∞. In the neighbourhood of $\chi = 1$ $(\omega = \omega^*)$, the quantities v_1, ϑ_1 change abruptly. But for the practical application of the theory we take into account only a small region of variability of $\chi = \omega/\omega^*$. Therefore, for $\chi \ll 1$, the roots ζ_1, ζ_2 can be expanded into power series in χ, and we can employ the relation,

$$\zeta_\beta = (c_1)_T\left(\frac{\chi}{v_\beta} + i\frac{\vartheta_\beta}{\omega^*}\right), \quad \beta = 1, 2.$$

In this way, we can obtain approximate values of phase velocities and damping coefficients. We present them according to P. CHADWICK: [54]

$$\begin{aligned}
v_1 &= c_1(1+\varepsilon)^{1/2}\left[1 - \frac{\chi^2\varepsilon(4-3\varepsilon)}{8(1+\varepsilon)^4} + O(\chi^4)\right],\\
\vartheta_1 &= \frac{\omega^*}{c_1(1+\varepsilon)^{1/2}}\left[\frac{\chi^2\varepsilon}{2(1+\varepsilon)^2} + O(\chi^4)\right],\\
v_2 &= c_1\left(\frac{2\chi}{1+\varepsilon}\right)^{1/2}\left[1 - \frac{\chi\varepsilon}{2(1+\varepsilon)^2} + \frac{\chi^2\varepsilon(4+\varepsilon)}{8(1+\varepsilon)^4} + \frac{\chi^3\varepsilon(8-20\varepsilon+\varepsilon^2)}{16(1+\varepsilon)^6} + O(\chi^4)\right],\\
\vartheta_2 &= \frac{\omega^*}{c_1}\left(\frac{\chi}{2}(1+\varepsilon)\right)^{1/2}\left[1 - \frac{\chi\varepsilon}{2(1+\varepsilon)^2} + \frac{\chi^2\varepsilon^2(4-\varepsilon)}{8(1+\varepsilon)^4} + \frac{\chi^3\varepsilon(8-12\varepsilon+\varepsilon^2)}{16(1+\varepsilon)^6} + O(\chi^4)\right].
\end{aligned} \tag{9.11}$$

It seen that for $\chi \ll 1$ the phase velocity $v_1 \approx c_1(1+\varepsilon)^{1/2}$ can be considered as a constant value slightly greater than $c_1 = (c_1)_T$, and the quasi elastic longitudinal wave can be treated as damped but not subjected to dispersion.

We shall present below the solution of a very simple example of a plane wave when a plane heat source acts with the intensity Q_0. This source changes harmonically in time and operates in the plane $x_1 = 0$. We obtain here:

$$\begin{aligned}
u_1 &= \frac{mQ_0}{2\kappa}\operatorname{Re}\left\{\frac{1}{k_1^2-k_2^2}\left[\exp\left(-i\omega\left(t-\frac{x_1}{v_1}\right)-\vartheta_1 x_1\right)\right.\right.\\
&\qquad\left.\left.-\exp\left(-i\omega\left(t-\frac{x_1}{v_2}\right)-\vartheta_2 x_1\right)\right]\right\}, \quad x_1 > 0,\\
\theta &= \frac{Q_0}{2\kappa}\operatorname{Re}\left\{\frac{1}{k_1^2-k_2^2}\left[\frac{k_2^2-\sigma^2}{ik_2}\exp\left(-i\omega\left(t-\frac{x_1}{v_2}\right)-\vartheta_2 x_1\right)\right.\right.\\
&\qquad\left.\left.-\frac{k_1^2-\sigma^2}{ik_1}\exp\left(-i\omega\left(t-\frac{x_1}{v_1}\right)-\vartheta_1 x_1\right)\right]\right\}, \quad x_1 > 0.
\end{aligned} \tag{9.12}$$

The phase velocities v_β and the damping coefficients ϑ_β are taken from the formulae (9.11).

If we disregard the coupling of deformation and temperature fields—i.e., if in the thermal conductivity equation we delete the term $\eta\dot{\varepsilon}_{kk}$, then inserting $k_1(0) = \sigma$, $k_2(0) = \sqrt{q}$ instead of $k_1(\varepsilon)$, $k_2(\varepsilon)$, we obtain from (9.12) the approximate solution of the thermal stresses theory:

$$\theta = \frac{Q_0}{2\kappa}\operatorname{Re}\left\{\frac{i}{\sqrt{q}}\exp\left(-i\omega\left(t-\frac{x_1}{\sqrt{2\kappa\omega}}\right)-x_1\sqrt{\frac{\omega}{2\kappa}}\right)\right\},$$

$$(9.13)\qquad u_1 = \frac{mQ_0}{2\kappa}\operatorname{Re}\left\{\frac{1}{\sigma^2-q}\left[\exp\left(-i\omega\left(t-\frac{x_1}{c_1}\right)\right) - \exp\left(-i\omega\left(t-\frac{x_1}{\sqrt{2\kappa\omega}}\right)-x_1\sqrt{\frac{\omega}{2\kappa}}\right)\right]\right\}.$$

The displacement u_1 is composed of two parts: the undamped elastic wave moving with velocity c_1, and of diffusion wave damped and subjected to dispersion.

So far, a number of particular problems concerning the propagation of plane waves in elastic space and semi-space have been solved. I. N. Sneddon [55] studied the propagation of a wave in a semi-infinite and infinite rod with the assumption of various boundary conditions, and consequently various causes inducing waves. The author considered forced vibration for a finite rod. W. Nowacki [56] considered the action of plane body forces in an unbounded space and the action of plane heat sources causing vibration in the thermoelastic layer [56].

The interesting result is here that the phenomenon of resonance does not arise for forced vibration. It follows from the character of wave motion which is damped. For forced vibration we have amplitudes with finite values. Namely, for the case of a layer of thickness a, which is free of stresses and temperature in the planes bounding the layer $x_1 = 0, a$ subjected to the action of heat sources $Q = Q^*\cos\omega t$, we obtain the following expression for the stress:

$$(9.14)\qquad \sigma_{11} = \frac{m\varrho\omega^2}{\kappa}\sum_{n=1}^{\infty}\frac{Q_n^*\{\alpha_n^2(\alpha_n^2-\sigma^2)\cos\omega t - \zeta[\alpha_n^2(1+\varepsilon)-\sigma^2]\sin\omega t\}}{\alpha_n^4(\alpha_n^2-\sigma^2)^2+\zeta^2[\alpha_n^2(1+\varepsilon)-\sigma^2]^2}\sin\alpha_n x_1$$

where

$$\zeta = \frac{\omega}{\kappa}, \qquad \alpha_n = \frac{n\pi}{a}, \qquad Q_n^* = \frac{2}{a}\int_0^a Q^*(x_1)\sin\alpha_n x_1\,dx_1.$$

We shall not here obtain resonance, since the denominator under the sum sign is always positive. In the particular case $\alpha_r^2 = \sigma^2$ corresponding to the resonance for the uncoupled problem, the r-th term of this series can be written as:

$$(9.15)\qquad \sigma_{11}^{(r)} = -\frac{\varrho\omega m}{\varepsilon}\sin\omega t\,\frac{Q_r^*\sin\alpha_r x_1}{\alpha_r^2}.$$

This term possesses a finite value although the magnitude of stress $\sigma_{11}^{(r)}$ will be considerable because the ε is for metals of the order of several percent.

10. Spherical and Cylindrical Waves

Let us consider the wave equation characterizing longitudinal thermoelastic waves which was derived in Sec. 3 (formulae (3.9) and 3.11)):

$$\Box_1^2 \Phi = m\theta, \tag{10.1}$$

$$D\theta - \eta \nabla^2 \dot{\Phi} = 0. \tag{10.2}$$

If we assume that the wave motion changes harmonically in time, then if

$$\Phi(\mathbf{x}, t) = \Phi^*(\mathbf{x})\, e^{i\omega t}, \quad \theta(\mathbf{x}, t) = \theta^*(\mathbf{x})\, e^{i\omega t},$$

then from the Eqs (10.1) and (10.2), we obtain the following equations

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)(\Phi^*, \theta^*) = 0, \tag{10.3}$$

where the quantities k_1, k_2 are the roots of the Eq. (9.4) discussed in the preceding section.

Let us consider those solutions for the Eq. (10.3) which are characterized by singularity at the point $\boldsymbol{\xi}$, and which depend on radius r, distance between the points $\mathbf{x}$ and $\boldsymbol{\xi}$. These solutions which will be denoted by $\varphi_\alpha^*(r)$, satisfy the equations:

$$\frac{d^2\varphi_\alpha^*}{dr^2} + \frac{n-1}{r}\frac{d\varphi_\alpha^*}{dr} + k_\alpha^2 \varphi_\alpha^* = 0, \quad \alpha = 1, 2. \tag{10.4}$$

Here, $n = 3$ refers to a three-dimensional problem, $n = 2$ to a two-dimensional problem. In the Eq. (10.4), the summation with respect to the index α should not be performed.

The general solution of the Eq. (10.4) takes the form

$$\varphi_\alpha^*(r) = \frac{1}{r^m}\left[A H_m^{(1)}(k_\alpha r) + B H_m^{(2)}(k_\alpha r)\right], \tag{10.5}$$

$$m = \frac{n-2}{2}.$$

Here, $H_m^{(1)}$ and $H_m^{(2)}$ are the Hankel functions of m-th order and of the first and second kind.

For $n = 3$ (then for $m = 1/2$), we have

$$H_{1/2}^{(1)}(k_\alpha r) = -i\sqrt{\frac{2}{\pi k_\alpha}}\,\frac{e^{ik_\alpha r}}{r},$$

$$H_{1/2}^{(2)}(k_\alpha r) = i\sqrt{\frac{2}{\pi k_\alpha}}\,\frac{e^{-ik_\alpha r}}{r}, \quad \alpha = 1, 2,$$

and the function

$$\varphi_\alpha^*(r) = A_1\frac{e^{-ik_\alpha r}}{r} + A_2\frac{\bar{e}^{ik_\alpha r}}{r}, \quad r^2 = (x_j - \xi_j)(x_j - \xi_j), \quad j = 1, 2, 3, \tag{10.6}$$

becomes the solution of the Eq. (10.4).

In the unbounded thermoelastic space, we take into account only the first term of the Eq. (10.6), since the solution:

$$\mathrm{Re}\left[e^{-i\omega t}\frac{e^{ik_\alpha r}}{r}\right] = \frac{e^{\vartheta_2 r}}{r}\cos\omega\left(t - \frac{r}{v_\alpha}\right),$$

$$v_\alpha = \frac{\omega}{\mathrm{Re}(k_\alpha)}, \quad \vartheta_\alpha = \mathrm{Im}(k_\alpha), \quad \alpha = 1, 2,$$

represents a divergent wave propagating with the phase adopted from the origin of the system $r = 0$ to infinity. Only this solution has physical sense. For a cylindrical wave for $n = 2$ and $m = 0$ we obtain:

(10.7)
$$\varphi_\alpha^*(r) = AH_0^{(1)}(k_\alpha r) + BH_0^{(2)}(k_\alpha r),$$
$$r^2 = (x_j - \xi_j)(x_j - \xi_j), \quad j = 1, 2.$$

Here, we take into account only the first term of (10.7) for an unbounded medium, since for high values of the argument, we obtain:

(10.8)
$$\operatorname{Re}\,[e^{-i\omega t} H_0^{(1)}(k_\alpha r)] \to \sqrt{\frac{2}{\pi r k_\alpha}} \cos\left(k_\alpha r - \frac{\pi}{4} - \omega t\right)[1 + O(r^{-1})],$$

representing a divergent wave propagating in the direction of increasing r.

In the expression (10.8) the symbol $O(r^{-\alpha})$ denote a value x which is such that the ratio x/r^α remains bounded as $r \to \infty$. The solutions here presented $\frac{e^{ik_\alpha r}}{r}$, $H_o^{(1)}(k_\alpha r)$ should satisfy at infinity what are called the radiation conditions ([56–58]):

(10.9)
$$n = 3: \quad \frac{\partial}{\partial r}\left(\frac{e^{ik_\alpha r}}{r}\right) - ik_\alpha \frac{e^{ik_\alpha r}}{r} = e^{ik_\alpha r} O(r^{-2}), \quad \vartheta_\alpha > 0,$$
$$n = 2: \quad \frac{\partial}{\partial r}(H_0^{(1)}(k_\alpha r)) - ik_\alpha H_0^{(1)}(k_\alpha r) = e^{ik_\alpha r} O(r^{3/2}), \quad \vartheta_\alpha > 0,$$
$$\alpha = 1, 2.$$

These formulae provide information about the behaviour of fundamental solutions in the neighbourhood of a point infinitely removed.

If we consider for the Eqs. (10.3) a class of solutions which behave at infinity in a manner similar to the fundamental solutions $e^{ik_\alpha r}/r$, $H_0^{(1)}(k_\alpha r)$, then we should require that the functions $\Phi^* = \Phi_1^* + \Phi_2^*$ satisfy the following conditions at infinity:

(10.10)
$$n = 3: \quad \frac{\partial \Phi_\alpha^*}{\partial r} - ik_\alpha \Phi_\alpha^* = e^{ik_\alpha r} O(r^{-2}), \quad \vartheta_\alpha \geqslant 0,$$
$$n = 2: \quad \frac{\partial \Phi_\alpha^*}{\partial r} - ik_\alpha \Phi_\alpha^* = e^{ik_\alpha r} O(r^{-3/2}), \quad \vartheta_\alpha \geqslant 0,$$
$$\alpha = 1, 2.$$

To these solutions we should add the conditions concerning a finite value of the function

$$\Phi_\alpha^* = O(1) \quad \text{for} \quad r \to \infty,$$

where the symbol $O(1)$ denotes a value arbitrarily small.

Longitudinal spherical waves are obtained only for a special choice of disturbances. They arise owing to the action of heat sources and body forces of potential origin, in both an infinite and a bounded medium with a spherical cavity with the boundary conditions being characterized by symmetry with respect to a point.

Let us consider one of those cases—namely, the action of the concentrated heat source $Q_0 e^{-i\omega t} \delta(r)$. We assume the following form of the solution for the Eq. (10.3):

$$\Phi^* = \frac{1}{r}(A_1 e^{ik_1 r} + A_2 e^{ik_2 r}), \tag{10.11}$$

where the constants A_1, A_2 will be determined from the condition of requirement that the heat flow through the surface of a sphere $r \to 0$ be equal to the heat source intensity, and in order that $u_r^* = \partial\Phi^*/\partial r$ be equal to zero for $r = 0$.

In consequence, we obtain for the functions Φ^*, θ^* the followling formulae [59]:

$$\Phi^* = \frac{mQ_0}{4\pi\kappa r(k_2^2 - k_1^2)} \left\{ \exp\left[-i\omega\left(t - \frac{r}{v_1}\right) - \vartheta_1 r \right] - \exp\left[-i\omega\left(t - \frac{r}{v_2}\right) - \vartheta_2 r \right] \right\},$$

(10.12)

$$\theta^* = \frac{Q_0}{4\pi\kappa r(k_2^2 - k_1^2)} \left\{ (k_2^2 - \sigma^2) \exp\left[-i\omega\left(t - \frac{r}{v_2}\right) - \vartheta_2 r \right] \right.$$

$$\left. - (k_1^2 - \sigma^2) \exp\left[-i\omega\left(t - \frac{r}{v_1}\right) - \vartheta_1 r \right] \right\}.$$

Here ϑ_α is a damping coefficient, v_α a phase velocity of the wave. The functions Φ^*, θ^* are damped, subjected to dispersion, satisfy radiation conditions and exhibit a singularity at the point $r = 0$.

Knowing the function Φ^* we are able to determine radial displacement $u_r = \partial\Phi/\partial r$. For $Q_0 = 1$, the formulae (10.12) become the Green's functions for the potential $\hat{\Phi}^*$ and temperature $\hat{\theta}^*$. If the distribution of sources $Q(\mathbf{x}, t) = Q^*(\mathbf{x}) e^{-i\omega t}$ is given in a bounded region V_1, the potential is expressed by the formula:

$$\Phi^*(\mathbf{x}, \omega) = \int_{V_1} Q^*(\boldsymbol{\xi}) \hat{\Phi}^*(\mathbf{x}, \boldsymbol{\xi}, \omega)\, dV(\boldsymbol{\xi}). \tag{10.13}$$

Sofar a number of particular cases have been solved referring to spherical waves. They concern the action of a compression centre in an unbounded region and a space with a cavity, various boundary conditions characterized by spherical symmetry [59, 56] being assumed.

A number of theorems have been developed for spherical waves. They can be regarded as an extension of the Helmholtz theorem for elastokinetics and the analogous theorem of the thermal conductivity theory for problems of thermoelasticity [49]. The idea of his theorem is as follows. The system of equations is given

$$(\nabla^2 + \sigma^2) u^* - mv^* = 0, \qquad (\nabla^2 + q) v^* + \frac{q\varepsilon}{m} \nabla^2 u^* = 0, \tag{10.14}$$

which is regular in the region B considered. Here, u^* denotes the potential of thermoelastic displacement, v^*—temperature. The elimination of the functions u^* or v^* from the Eqs (10.14) lead to on equation of type (10.3).

It can be demonstrated that if the functions $u^*, v^*, \partial u^*/\partial n, \partial v^*/\partial n$ are given on the boundary Σ of the region B, then the function v^* at a point $\mathbf{x} \in B$ can be written as:

$$(10.14') \quad v^*(\mathbf{x}) = \kappa \int_\Sigma \left[\theta^*(\boldsymbol{\xi}, \mathbf{x}) \frac{\partial v^*(\boldsymbol{\xi})}{\partial n} - v^*(\boldsymbol{\xi}) \frac{\partial \theta^*(\boldsymbol{\xi}, \mathbf{x})}{\partial n} \right] d\Sigma(\boldsymbol{\xi}) + \frac{\varepsilon q \sigma^2 \kappa}{m^2} \int_\Sigma \left[\Phi^*(\boldsymbol{\xi}, \mathbf{x}) \frac{\partial u^*(\boldsymbol{\xi})}{\partial n} - u^*(\boldsymbol{\xi}) \frac{\partial \Phi^*(\boldsymbol{\xi}, \mathbf{x})}{\partial n} \right] d\Sigma(\boldsymbol{\xi}),$$

$$\boldsymbol{\xi} \in \mathrm{B}.$$

In this case, the functions $\Phi^*(\mathbf{x}, \boldsymbol{\xi})$, $\theta^*(\mathbf{x}, \boldsymbol{\xi})$ are the solutions of the equations

$$(10.15) \quad (\nabla^2 + \sigma^2)\Phi^* - m\theta^* = 0, \quad (\nabla^2 + q)\theta^* + \frac{q\varepsilon}{m} \nabla^2 \Phi^* = -\frac{1}{\kappa} \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \boldsymbol{\xi} \in B,$$

where

$$(10.16) \quad \Phi^* = \frac{m}{4\pi\kappa (k_2^2 - k_1^2) r} (e^{ik_1 r} - e^{ik_2 r}), \qquad \theta^* = \frac{1}{4\pi\kappa (k_2^2 - k_1^2) r} (n_2 e^{ik_2 r} - n_1 e^{ik_1 r}),$$

$$n_\alpha = k_\alpha^2 - \sigma^2, \quad \alpha = 1, 2.$$

For $\mathbf{x} \in \mathcal{C} - B$, where $\mathcal{C}$ is a whole space, we have $v^*(\mathbf{x}) = 0$. For an uncoupled problem ($\varepsilon = 0$)—i.e. for the theory of thermal stresses, the second integral of the Eq. (10.14) disappears. In consequence, we obtain the equation:

$$(10.17) \quad v^*(\mathbf{x}) = \frac{1}{4\pi} \int_\Sigma \left[v^*(\boldsymbol{\xi}) \frac{\partial}{\partial n} \left(\frac{e^{ir\sqrt{q}}}{r} \right) - \frac{e^{ir\sqrt{q}}}{r} \frac{\partial v^*(\boldsymbol{\xi})}{\partial n} \right] d\Sigma(\boldsymbol{\xi}),$$

$$r = r(\mathbf{x}, \boldsymbol{\xi}),$$

which is a theorem familiar in the thermal conductivity theory. For the function $u^*(\mathbf{x})$, we obtain the following formula

$$(10.18) \quad u^*(\mathbf{x}) = \kappa \int_\Sigma \left[\Phi^*(\boldsymbol{\xi}, \mathbf{x}) \frac{\partial v^*(\boldsymbol{\xi})}{\partial n} - v^*(\boldsymbol{\xi}) \frac{\partial \Phi^*(\boldsymbol{\xi}, \mathbf{x})}{\partial n} \right] d\Sigma(\boldsymbol{\xi}) + \frac{\kappa}{m} \int_\Sigma \left[\Box_k^2 \Phi^*(\boldsymbol{\xi}, \mathbf{x}) \frac{\partial u^*(\boldsymbol{\xi})}{\partial n} - u^*(\boldsymbol{\xi}) \frac{\partial}{\partial n} \Box_k^2 \Phi^*(\boldsymbol{\xi}, \mathbf{x}) \right] d\Sigma(\boldsymbol{\xi}), \quad \mathbf{x} \in B$$

$$u^*(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{C} - B.$$

The symbol $\Box_k^2 = \nabla^2 + k_1^2 + k_2^2 - \sigma^2$ is introduced in this formula. The formula (10.18) is expressed through the function $u^*(\mathbf{x})$ inside the region B by means of the function

$$u^*(\boldsymbol{\xi}), \quad \frac{\partial u^*(\boldsymbol{\xi})}{\partial n}, \quad v^*(\boldsymbol{\xi}), \quad \frac{\partial v^*(\boldsymbol{\xi})}{\partial n},$$

on the surface Σ. Proceeding from thermoelasticity to elastokinetics, we obtain from (10.18), after a number of transformations, the Helmholtz's familiar theorem [60]:

$$u^*(\mathbf{x}) = \left\{ \frac{1}{4\pi} \int_{\Sigma} \left[\frac{e^{i\sigma r}}{r} \frac{\partial u^*}{\partial n} - u^* \frac{\partial}{\partial n}\left(\frac{e^{i\sigma r}}{r}\right)\right] dA(\boldsymbol{\xi}), \quad \mathbf{x} \in B, \right. \tag{10.19}$$

$$u^*(\mathbf{x}) = 0 \quad \text{if} \quad \mathbf{x} \in \mathcal{C} - B.$$

Here, $\sigma = \omega/(c_1)_s$, where $(c_1)_s = \varrho^{-1/2}(\lambda_s+2\mu)/^{1/2}$. Cylindrical waves can arise in the case of a linear heat source of a linear compression centre, or in an unbounded thermoelastic medium with a cylindrical cavity, on the boundary of which heating, pressure or deformation takes place and is distributed in an axial-symmetrical way.

Of numerous solutions [56–61], we consider to a linear heat source $\theta(r, t) = \theta_0 \, e^{-i\omega t} \, \delta(r)/2\pi r$, $r = (x_1^2+x_2^2)^{1/2}$. For the amplitudes of thermoelastic potential of displacement and for temperature, we obtain the following formulae [59]:

$$\begin{aligned} \Phi^* &= \frac{Q_0 \, mi}{4\kappa(k_2^2-k_1^2)} \left[H_0^{(1)}(k_1 r) - H_0^{(1)}(k_2 r)\right], \\ \theta^* &= \frac{Q_0 \, i}{4\kappa(k_2^2-k_1^2)} \left[(\sigma^2-k_1^2) H_0^{(1)}(k_1 r) - (\sigma^2-k_2^2) H_0^{(1)}(k_2 r)\right]. \end{aligned} \tag{10.20}$$

These functions satisfy the radiation conditions. They are damped and subjected to dispersion.

11. Green's Functions for an Infinite Thermoelastic Medium. The Singular Integral Equation of Thermoelasticity

In the preceding section, we presented Green's functions for a point linear heat source. They satisfy the equations:

$$\begin{aligned} &\hat{\sigma}_{ji,j} = -\omega^2 \varrho \hat{u}_i, \\ &\hat{\theta}_{,kk} + h_3^2 \hat{\theta} + \frac{\gamma}{\alpha} \hat{u}_{k,k} = -\frac{1}{\kappa} \delta(\mathbf{x}-\boldsymbol{\xi}), \\ &\alpha = \frac{m\gamma}{\varepsilon h_3^2}, \qquad h_3 = i\left(\frac{i\omega}{\kappa}\right)^{1/2} \end{aligned} \tag{11.1}$$

By $\hat{u}$, $\hat{\theta}$ we denote here the amplitudes of displacements and temperature. In turn, we should determine Green's functions for a concentrated force. At the point $\boldsymbol{\xi}$ of an unbounded region, let there act the concentrated force $X_i = \delta(\mathbf{x}-\boldsymbol{\xi}) \, \delta_{i1} \, e^{-i\omega t}$ which is directed towards the x_1-axis. The action of this forces produces both longitudinal and transverse waves. We should solve the system of equations:

$$\begin{aligned} &\sigma_{ji,j}^{(1)} = -\omega^2 \varrho u_1^{(1)} - \delta(\mathbf{x}-\boldsymbol{\xi}) \, \delta_{i1}, \\ &\theta_{,kk}^{(1)} + h_3^2 \theta^{(1)} + \frac{\gamma}{\alpha} u_{k,k}^{(1)} = 0 \end{aligned} \tag{11.2}$$

in which we have denoted by $\sigma_{ji}^{(1)}, u_i^{(1)}; \theta^{(1)}$ the amplitudes of stresses, displacements, and the temperature caused by the action of the concentrated force applied at the point $\boldsymbol{\xi}$ and directed towards the x_1-axis. The system of Eqs. (11.2) can be replaced by the system of wave equations:

$$(11.3)\qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)\Phi^{(1)} = -\frac{1}{c_1^2}(\nabla^2+q)\vartheta\,,$$

$$(11.4)\qquad (\nabla^2+\tau^2)\psi_i^{(1)} = -\frac{1}{c_2^2}\chi_i, \quad \tau=\frac{\omega}{c_2} \quad i=1,2,3\,.$$

These equations follow from the Eqs. (11.2) under the assumption that

$$(11.5)\qquad \mathbf{u}^{(1)} = \operatorname{grad}\Phi+\operatorname{rot}\boldsymbol{\psi}, \quad \mathbf{X} = \varrho(\operatorname{grad}\vartheta+\operatorname{rot}\boldsymbol{\chi})\,.$$

The amplitudes of body forces are determined from the formulae [62]:

$$(11.6)\qquad \begin{aligned} \vartheta(\mathbf{x}) &= -\frac{1}{4\pi\varrho}\int_V \mathbf{X}(\mathbf{x}')\cdot\operatorname{grad}_x\left(\frac{1}{r(\mathbf{x}',\mathbf{x})}\right)dV(\mathbf{x}'),\\ \boldsymbol{\chi}(\mathbf{x}) &= -\frac{1}{4\pi\varrho}\int_V \mathbf{X}(\mathbf{x}')\times\operatorname{grad}_x\left(\frac{1}{r(\mathbf{x}',\mathbf{x})}\right)dV(\mathbf{x}'). \end{aligned}$$

For the case here considered of concentrated force directed towards the x_1-axis, we find:

$$\vartheta = -\frac{1}{4\pi\varrho}\,\partial_1\left(\frac{1}{r}\right), \quad \chi_1 = 0\,,$$

$$\chi_2 = \frac{1}{4\pi\varrho}\,\partial_3\left(\frac{1}{r}\right), \quad \chi_3 = -\frac{1}{4\pi\varrho}\,\partial_2\left(\frac{1}{r}\right).$$

From the solution of the Eqs. (11.4) we obtain:

$$(11.6'')\qquad \psi_1 = 0\,, \quad \psi_2 = \frac{1}{4\pi\varrho\omega^2}\,\partial_3 F_0(r,\omega)\,, \quad \psi_3 = -\frac{1}{4\pi\varrho\omega^2}\,\partial_2 F_0(r,\omega)\,,$$

where

$$F_0(r,\omega) = \frac{1}{r}(e^{i\tau r}-1), \quad r^2 = (x_i-\xi_i)(x_i-\xi_i), \quad i=1,2,3\,.$$

From the solution of the Eq. (11.3) taking into account the fact that the function $\Phi^{(1)}$ is characterized by an axial symmetry with respect to the x_1-axis, we obtain [62, 63]:

$$(11.7)\qquad \Phi^{(1)} = -\frac{1}{4\pi\varrho\omega^2}\,\partial_1 F(r,\omega)\,,$$

where

$$F(r,\omega) = A_1 I_1 - A_2 I_2 - I_0\,, \quad I_0 = \frac{1}{r}\,, \quad I_\beta = \frac{e^{ik_\beta r}}{r}\,, \quad \beta = 1,2,$$

$$A_1 = \frac{(k_1^2-q)\sigma^2}{k_1^2(k_1^2-k_2^2)}\,, \quad A_2 = \frac{(k_2^2-q)\sigma^2}{k_2^2(k_1^2-k_2^2)}\,.$$

The temperature $\theta^{(1)}$ is determined from the formula

$$\theta^{(1)} = \frac{1}{m}(\nabla^2+\sigma^2)\Phi^{(1)} + \frac{1}{c_1^2 m}\vartheta. \tag{11.8}$$

Employing the formulae (11.5) and (11.8) we arrive at:

$$u_j^{(1)} = -\frac{1}{4\pi\varrho\omega^2}\partial_1\partial_j[F(r,\omega)-F_0(r,\omega)] + \frac{1}{4\pi\varrho c_2^2}\delta_{1j}\frac{e^{i\tau r}}{r}, \tag{11.9}$$

$$\theta^{(1)} = \frac{q\varepsilon}{4\pi\varrho m c_1^2(k_1^2-k_2^2)}\partial_1[I_1(r,\omega)-I_2(r,\omega)]. \tag{11.10}$$

These functions have a singularity at the point $\boldsymbol{\xi}$, and satisfy at infinity the radiation conditions. If a concentrated force acts in the direction of the x_s-axis, we have the following expression for Green's displacement tensor u_j^s, and for temperature θ^s:

$$u_j^s = -\frac{1}{4\pi\varrho\omega^2}\{\partial_j\partial_s[F(r,\omega)-F_0(r,\omega)]-\tau^2\delta_{js}e^{i\omega\tau}\}, \tag{11.11}$$

$$\theta^s = \frac{q\varepsilon}{4\pi\varrho m c_1^2(k_1^2-k_2^2)}\partial_s[I_1(r,\omega)-I_2(r,\omega)], \quad j,s=1,2,3. \tag{11.12}$$

From the solutions found for a concentrated force, we can obtain further singularities—the expressions u_i^s, θ^s for a double force, for a concentrated moment and for a centre of compression.

For a two-dimensional problem, we obtain for the force concentrated and directed towards the x_3-axis, the following Green's functions [64]:

$$u_j^s = -\frac{i}{4\varrho\omega^2}\{\partial_j\partial_s[A_1H_0^{(1)}(k_1r)-A_2H_0^{(1)}(k_2r)-H_0^{(1)}(\tau r)]-\tau^2\delta_{js}H_0^{(1)}(\tau r)\}, \tag{11.13}$$

$$\theta^s = \frac{iq\varepsilon}{4\varrho m c_1^2(k_1^2-k_2^2)}\partial_s[H_0^{(1)}(k_1r)-H_0^{(1)}(k_2r)], \tag{11.14}$$

$$r^2 = (x_j-\xi_j)(x_j-\xi_j), \quad j,s=1,2.$$

Knownig the displacement functions and temperature for the section of a concentrated heat source and a concentrated force, we are able to construct methods for integrating the thermoelasticity equations for a bounded body [49].

We introduce the thermoelastic surface potentials analogous to the elastokinetics potentials [57]:

$$\begin{aligned} V_s(\mathbf{x}) &= 2\int_\Sigma d\Sigma(\boldsymbol{\xi})\varphi_k(\boldsymbol{\xi})u_s^k(\boldsymbol{\xi},\mathbf{x}) + 2\alpha\int_\Sigma d\Sigma(\boldsymbol{\xi})\psi(\boldsymbol{\xi})\theta^s(\boldsymbol{\xi},\mathbf{x}), \\ V(\mathbf{x}) &= 2\int_\Sigma d\Sigma(\boldsymbol{\xi})\psi(\boldsymbol{\xi})\hat{\theta}(\boldsymbol{\xi}) + \frac{2}{\alpha}\int_\Sigma d\Sigma(\boldsymbol{\xi})\varphi_k(\boldsymbol{\xi})\hat{u}_k(\boldsymbol{\xi},\mathbf{x}). \end{aligned} \tag{11.15}$$

Here, $\varphi_k(\xi)$, $\psi(\xi)$ are the unknown densities of the corresponding regularity. The functions $\hat{u}_k$, $\hat{\theta}$, u_k^s, θ^s are the Green's functions satisfying the Eqs. (11.1) and (11.2)—i.e., they are the known functions. The following system is called the thermoelastic potential of a double layer:

$$(11.16)\qquad \begin{aligned} W_s(\mathbf{x}) &= 2\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,p_k^s(\xi,\mathbf{x}) + 2\alpha\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\frac{\partial\theta^s(\xi,\mathbf{x})}{\partial n},\\ W(\mathbf{x}) &= 2\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\frac{\partial\hat{\theta}(\xi,\mathbf{x})}{\partial n} + \frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,\hat{p}_k(\xi,\mathbf{x}).\end{aligned}$$

The following denotations are introduced here:

$$p_k^s(\xi,\mathbf{x}) = [2\mu\varepsilon_{kj}^s + (\lambda u_{p,p}^s - \gamma\theta^s)\,\delta_{kj}]\,n_j,$$
$$\hat{p}_k(\xi,\mathbf{x}) = [2\mu\hat{\varepsilon}_{kj} + (\lambda\hat{u}_{p,p} - \gamma\hat{\theta})\,\delta_{kj}]\,n_j.$$

Finally we can utilize the thermoelastic potential, being the combination of the potential of the single and the double layer:

$$(11.17)\qquad \begin{aligned} M_s(\mathbf{x}) &= 2\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,p_k^s(\xi,\mathbf{x}) + 2\alpha\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\theta^s(\xi,\mathbf{x}),\\ M(\mathbf{x}) &= 2\int_\Sigma d\Sigma(\xi)\,\psi(\xi)\,\hat{\theta}(\xi,\mathbf{x}) + \frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\,\varphi_k(\xi)\,\hat{p}_k(\xi,\mathbf{x}).\end{aligned}$$

It is clear that the potentials $V_s(\mathbf{x})$, $V(\mathbf{x})$ are the continuous functions of the points $x \in \Sigma$ But the potentials of double layer $W_s(\mathbf{x})$, $W(\mathbf{x})$ demonstrate the discontinuity of this surface. For we have

$$(11.18)\qquad \begin{aligned} W_s^{(i)}(\xi_0) &= -\varphi_s(\xi_0) + W_s(\xi_0), \qquad W^{(i)}(\xi_0) = -\psi(\xi_0) + W(\xi_0),\\ W_s^{(e)}(\xi_0) &= \varphi_s(\xi_0) + W_s(\xi_0), \qquad W^{(e)}(\xi_0) = \psi(\xi_0) + W(\xi_0).\end{aligned}$$

The functions $W_s(\xi_0)$, $W_s^{(i)}(\xi_0)$ and $W_s^{(e)}(\xi_0)$ denote correspondingly the limit of the vector $W_s(\xi)$ as $\xi \to \xi_0 \in \Sigma$ of the surface Σ, $W_s^{(i)}(\xi)$ as $\xi \to \xi_0 \in \Sigma$ from the interior of the region V and $W_s^{(e)}(\xi)$ as $\xi \to \xi_0 \in \Sigma$ for $\xi \in \mathcal{C} - V$. It is clear that the first surface integral in the formulae (11.16) represents a discontinuous function, the second integral—a continuous function.

We next introduce the denotations:

$$(11.19)\qquad \hat{p}_i(\mathbf{x}) = [2\mu V_{(i,j)} + \lambda(V_{k,k} - \gamma V)\,\delta_{ij}]\,n_j(\mathbf{x}), \qquad \hat{\theta}(\mathbf{x}) = V_{,k}n_k(\mathbf{x}),$$

where V_s, V are defined by the formulae (11.15). It may be shown that

$$(11.20)\qquad \begin{aligned} \hat{p}_k^{(i)}(\xi_0) &= \varphi_0(\xi_0) + \hat{p}_k(\xi_0), \qquad \hat{\theta}^{(i)}(\xi_0) = \psi(\xi_0) + \hat{\theta}(\xi_0),\\ \hat{p}_k^{(e)}(\xi_0) &= -\varphi_k(\xi_0) + \hat{p}_k(\xi_0), \qquad \hat{\theta}^{(e)}(\xi_0) = -\psi(\xi_0) + \hat{\theta}(\xi_0).\end{aligned}$$

The thermoelastic potentials (11.15) to (11.17) and the relations concerning discontinuities of these potentials, make it posible to reduce the fundamental boundary problems to solving a system of singular integral equations.

Let us consider the case in which the displacements $u_s(\xi_0) = f_s(\xi_0)$ and temperature $\theta(\xi_0) = g(\xi_0)$ are given on the boundary Σ, then we seek for solutions in the form:

$$U_s(\mathbf{x}) = W_s(\mathbf{x}), \quad \theta(\mathbf{x}) = W(\mathbf{x}),$$

where the functions $U_s(\mathbf{x}), \theta(\mathbf{x})$ are given by the formulae (10.16). We can easily verify that inside the region V, the equations:

$$(11.21) \qquad L_{sk} U_k - \gamma \partial_s \theta = 0, \quad (\nabla^2 + q)\theta + \frac{\gamma}{\varkappa} \partial_k U_k = 0, \quad x \in V,$$

are satisfied, where

$$L_{sk} = (\mu \partial_p \partial_p + \omega^2 \varrho)\delta_{sk} + (\lambda + \mu)\partial_s \partial_k .$$

Taking into account the relations (11.18) for the functions $\varphi_k(\xi)$, $\psi(\xi)$, we arrive at the following system of coupled integral equations:

$$(11.22) \qquad \begin{aligned} &\varphi_s(\xi_0) - 2\int_\Sigma \varphi_k(\xi)\, p_k^s(\xi, \xi_0)\, d\Sigma(\xi) - 2\alpha \int_\Sigma \psi(\xi) \frac{\partial}{\partial n} \theta^s(\xi, \xi_0)\, d\Sigma(\xi) = -f_s(\xi_0), \\ &\psi(\xi_0) - 2\int_\Sigma \psi(\xi) \frac{\partial}{\partial n} \hat{\theta}(\xi, \xi_0)\, d\Sigma(\xi) - \frac{2}{\alpha}\int_\Sigma \varphi_k(\xi)\, \hat{p}_k(\xi, \xi_0)\, d\Sigma(\xi) = -g(\xi_0). \end{aligned}$$

These equations have the form of singular integral equations of the second kind, and the integrals occurring in them should be regarded of in the sense of major values. If, on the boundary Σ the displacements $u_i(\xi_0) = f_i(\xi_0)$ and heat flow $\partial\theta/\partial n|_{\xi=\xi} = S(\xi_0)$ are given, then we seek solutions in the form:

$$U_s(\mathbf{x}) = M_s(\mathbf{x}), \quad \theta(\mathbf{x}) = M(\mathbf{x}), \quad \mathbf{x} \in V,$$

where the functions M_s, M are given by the formulae (11.17). We can easily verify that inside the region V, the Eqs. (11.21) are satisfied, and the unknown densities fulfill the system of integral singular equations

$$(11.23) \qquad \begin{aligned} &\varphi_s(\xi_0) - 2\int_\Sigma \varphi_k(\xi)\, p_k^s(\xi, \xi_0)\, d\Sigma(\xi) - 2\alpha \int_\Sigma \psi(\xi)\, \theta^s(\xi, \xi_0)\, d\Sigma(\xi) = -f_s(\xi_0), \\ &\psi(\xi_0) + 2\int_\Sigma \psi(\xi) \frac{\partial}{\partial n_0} \hat{\theta}(\xi, \xi_0)\, d\Sigma(\xi) + \frac{2}{\alpha}\int_\Sigma \varphi_k(\xi) \frac{\partial}{\partial n_0} \hat{p}_k(\xi, \xi_0)\, d\Sigma(\xi) = g(\xi_0), \end{aligned}$$

where $\dfrac{\partial}{\partial n_0} \hat{\theta}(\xi, \xi_0) = \lim\limits_{\mathbf{x}\to\xi_0} \dfrac{\partial}{\partial n_x} \hat{\theta}(\xi, \mathbf{x}), \quad \mathbf{x} \in \Sigma.$

The quantity $\partial \hat{p}_k(\xi, \xi_0)/\partial n_0$ is defined abalogously. Let us note finally, that if loading $p_i = p_i(\xi_0)$ and heat flow $S = S(\xi_0)$ are given on Σ, then the solution should be sought for using the potentials of a single layer $V_s(\mathbf{x})$, $V(\mathbf{x})$. The investigation of the existence and uniqueness of the singular equations obtained is carried out in a manner similar to that used in elastodynamics. The system of singular integral equations presented here comprise particular cases related to thermal stresses theory, thermal conductivity theory and elastodynamics.

When developing the general theory of propagation of thermoelastic waves changing harmonically in time, a number of particular problems, ware solved simultaneously admissing them to a form useful for discussion. They are mostly the problems typical for classical elastokinetics which in fa mework of thermoelasticity were extended and generalized. A great deal of attention has been devoted to surface waves. This problem was first discussed in the work by F. J. LOCKETT [65, 66] and then, in a broader and more thorough manner, by P. CHADWICK and D. W. WINDLE [67].

When derivind surface waves in a plane state of deformation, we start from the wave equations (for longitudinal and transverse wave) and from the thermal conductivity equation. The wave travels parallel to the plane bounding the semi-space and vanishes in greater depth. It is assumed that stresses and temperature, or stresses and heat flow disappear in the plane bounding the space. An algebraic equation of the third order with complex coefficients is obtained from the determinant of the system of equations expressing homogeneous boundary conditions. One of roots of this equation satisfying prescribed inequalities provides a phase velocity of the surface wave. It is found that the surface wave undergoes damping and dispersion, and its velocity is smaller than the velocity of longitudinal and transverse waves.

W. NOWACKI and M. SOKOŁOWSKI [69] have investigated, in a similar way, the propagation of a harmonic wave in a thermoelastic layer. The authors considered there both symmetric and antisymmetric (elastic wave) forms of wave for two thermal conditions on the boundary: $\theta = 0$ and $\theta_{,n} = 0$. Owing to the small value of the parameter characterizing the thermoelastic medium, the approximate solution of the transcedental equation was presented using the perturbation method.

The propagation of harmonic waves in an infinite circular cylinder and thick-walled pipe was studied by F. J. LOCKETT [68] who gave the transcedental equations relevant to this problem. J. IGNACZAK and W. NOWACKI [70] have considered the forced vibration of an infinite cylinder with rectangular cross-section. Heating of the cylinder surface and the action of heat sources were here the cause exciting vibration. The same authors presented in [71] a method for solving and the actual solution of the problem of forced longitudinal vibration in discs and of flexural vibrations produced by loadings and heating in plates. The paper by P. CHADWICK [72] is devoted to analogous problems.

The propagation of a thermoelastic plane wave in an infinite medium in a spherical and cylindrical wave [68] is the next problem solved. The idea is as follows. A plane wave induced by the action of a plane heat source moves in an unbounded space and encounters a spherical or a cylindrical cavity. Flowing around this cavity the temperature field undergoes a disturbance, and concentration of temperature and stresses takes place in the neighbourhood of the cavity. The partial solution obtained here is in a closed form and the residual solution is expressed as an infinite system of algebraic equations with complex coefficients.

A considerable group of solutions corresponds to what is known as Lamb's problem of classical elastokinetics. The question consists in considering the influence of loadings and heatings acting on a thermoelastic semi-space. Two typical problems have been solved here—namely, when loading or heating is axially symmetric and when loading and heating produce a plane state of deformation [61]. Further problems concerning the action in source of heat (concentrated or linear) in an elastic semi-space [59] have something in common with the above subjects. However, the solutions of this group are of formal character only.

12. The Aperiodic problems of Thermoelasticity

The domain of investigation discussed here is the branch of thermoelasticity developed most weakly. This anses from the great mathematic difficultes encountered in obtaining solutions.

In general, three ways are used for solving the aperiodic problems of thermoelasticity. The first consists in eliminating the time t from the differential equations of thermoelasticity:

$$\begin{gathered} \mu u_{i,jj}+(\lambda+\mu)\,u_{j,ji}+X_i=\varrho\ddot{u}_i+\gamma\theta_{,i}\,, \\ \theta_{,jj}-\frac{1}{\kappa}\dot{\theta}-\eta\dot{e}=-\frac{Q}{\kappa}\,, \end{gathered} \tag{12.1}$$

by subjecting these equations to Laplace's transformations or Fourier's transformation with respect to time. The former transformation is most frequently applied in view of numerous inverse transformations.

Subjecting, then (12.1) to the Laplace'a transformation defined by the relation

$$\mathcal{L}\,(u_i,\theta)=(\bar{u}_i,\bar{\theta})=\int\limits_0^\infty(u_i,\theta)\,e^{-pt}\,dt\,,\qquad p>0\,,$$

and assuming the homogeneity of initial conditions, we obtain from (12.1) the following transformed solutions:

$$\begin{gathered} \mu\bar{u}_{i,jj}+(\lambda+\mu)\,\bar{u}_{j,ji}+\bar{X}_i=\varrho p^2\bar{u}_i+\gamma\theta_{,i}\,, \\ \bar{\theta}_{,jj}-\frac{p}{\kappa}\bar{\theta}-\eta p\bar{u}_{j,j}=-\frac{\bar{Q}}{\kappa}\,. \end{gathered} \tag{12.2}$$

Here, the unknown functions $\bar{u}_i$, $\bar{\theta}$ depend on position and the transformation parameter p. Solving the Eqs. (12.2) is not very onerous for many particular problems; the difficulties are of the same order as in problems of vibration harmonically changing in time. The essential difficulty consists in performing Laplace's inverse transformation for the solutions obtained $\bar{u}_i(\mathbf{x}, p)$, $\bar{\theta}(\mathbf{x}, p)$.

The second manner of solution consists in subjecting the Eqs. (12.1) to the Fourier triple integral transformation with respect to the variables x_i. Thus, the Eqs. (12.1) become a system of ordinary differential equations in which time appears as an independent variable. After solving this equation, the Fourier triple inverse transformation is accomplished [75].

The third way, used for thermoelastic space and semi-space consists in applying the Fourier quadruple transformation. The system of Eqs. (12.1) leads to a system of four algebraic equations for the transforms $\bar{\Phi}$, $\bar{\psi}_i$. The quadruple inverse transformation yields the final result [76, 77].

Each of these ways is accompanied by considerable mathematical difficulties—so considerable, in fact that so far no solution has been obtained in a closed form.

We shall consider more exactly the wave Eq. (3.9) and (3.11) to be obtained from the Eqs. (12.1). If we use the first way of investigation and apply the Laplace transformation for the wave equations with assuming homogeneous initial conditions, then we arrive at the system of equations:

$$
\begin{aligned}
&\left[\left(\nabla^2 - \frac{p^2}{c_1^2}\right)\left(\nabla^2 - \frac{p}{\kappa}\right) - \frac{\varepsilon p}{\kappa}\nabla^2\right]\bar{\Phi} = -\frac{m}{\kappa}\bar{Q} - \frac{1}{c_1^2}\left(\nabla^2 - \frac{p}{\kappa}\right)\bar{\vartheta}, \\
&\left(\nabla^2 - \frac{p^2}{c_2^2}\right)\bar{\psi}_i = -\frac{1}{c_2^2}\bar{\chi}_i,
\end{aligned} \tag{12.3}
$$

$$
\bar{\theta} = \frac{1}{m}\left(\nabla^2 - \frac{p^2}{c_1^2}\right)\bar{\Phi}, \qquad \varepsilon = \eta m \kappa
$$

The longitudinal wave equation for $Q = 0$, $\vartheta = 0$ can be presented in the form

$$
(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2)\bar{\Phi} = 0, \tag{12.4}
$$

where λ_1, λ_2 are the roots of the bi-quadratic equation:

$$
\lambda^4 - \lambda^2 p\left(\frac{p}{c_1^2} + \frac{1}{\kappa}(1+\varepsilon)\right) + \frac{p^3}{\kappa c_1^2} = 0.
$$

Since the roots of this equation:

$$
\left.\begin{matrix}\lambda_1^2\\ \lambda_2^2\end{matrix}\right\} = \frac{1}{2}\left\{\frac{p}{\kappa}(1+\varepsilon) + \frac{p^2}{c_1^2} \pm \left[\left(\frac{p}{\kappa}(1+\varepsilon) + \frac{p^2}{c_1^2}\right)^2 - \frac{4p^3}{\kappa c_1^2}\right]^{1/2}\right\},
$$

are expressed in a markebly complicated manner as functions of the parameter ε, it i clear that applying the Laplace inverse transformation for the functions Φ, θ encounters great difficulties. Therefore, we are forced to employ approximate solutions.

In general, two ways of approximate solution are used. The first consists in taking ad vantage of the fact that the quantity $\varepsilon = \eta m \kappa$ is a small parameter [54]. Writing, then the functions Φ, θ as a power series in

$$
\Phi = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \dots, \qquad \theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \dots \tag{12.5}
$$

we lead the Eq. (12.3) becomes the system of equations:

$$
\begin{aligned}
&D_1 D_2 \bar{\Phi}_0 = -\frac{m}{\kappa}\bar{Q} - \frac{1}{c_1^2} D_2 \bar{\vartheta}, \\
&D_1 D_2 \bar{\Phi}_1 = \frac{p}{\kappa}\nabla^2 \bar{\Phi}_0, \\
&\dots\dots\dots\dots\dots\dots,
\end{aligned} \tag{12.6}
$$

where $D_1 = \nabla^2 - p^2/c_1^2$, $\quad D_2 = \nabla^2 - p/\kappa$.

For the temperature $\bar{\theta}$, we obtain

$$
\bar{\theta} = \frac{1}{m} D_1(\bar{\Phi}_0 + \varepsilon\bar{\Phi}_1 + \varepsilon^2\bar{\Phi}_2 + \dots). \tag{12.7}
$$

When we use the perturbation method, it is adwandageous for practical purposes to retain only two terms of the series (12.5). Let us note, moreover, that the functions Φ_0, θ_0 concerns an uncoupled problem.

Another variant of the perturbation method consists in solving the Eqs (12.3) and next expanding the functions containing the quantities $k_1(\varepsilon, p)$, $k_2(\varepsilon, p)$ into a power series in the parameter ε. This variant was successfully applied by R. B. HETNARSKI [78, 79] for solving a number of problems referred to a thermoelastic space and a semi-space.

The second way for approximate solution consists in determining the function Φ, θ for small times. Solutions of this type are very useful since an essential difference exists between the dynamic and the quasi-static problem for small times t. This difference vanishes as time passes.

According the Abel's theorem

$$\lim_{t\to 0} f(t) = \lim_{p\to\infty} p\,\mathcal{L}[f(t)],$$

to small times correspond large values of the parameter p in the Laplace transforms Therefore, in the solutions for the Eqs (12.2) or the Eqs (12.3), the term containing the quantities $k_1(\varepsilon, p)$, $k_2(\varepsilon, p)$ should be expanded in powers of $1/p$, and several terms of this expansion should be retained. Performing the Laplace's inverse transformation provides finally an approximate solution of the problem.

Works on the propagation of aperiodic waves are not numerous and deal with the simplest systems, they refer to an elastic space and a semi-space. The problem of the action of an instantaneous and continuous concentrated source of heat in an unbounded thermoelastic space was investigated by R. B. HETNARSKI [78, 79] who applied the method of perturbation and small times. The problem of the action of instantaneous and concentrated force in a space was considered by E. SOÓS [17]. W. NOWACKI studied the influence of initial conditions on the propagation of thermoelastic waves in an infinite space [64].

The problem of determining the field of deformation and temperature around a spherical cavity in infinite space is allied with problems presented here.

The problem of sudden loading of the body boundary was the subject of two works. In the first, M. LESSEN [80] applied the perturbation method, in the second P. CHADWICK [54] presents the application of the asymptotic method for small times.

The problem of sudden heating of the boundary of a body with a spherical cavity by application of perturbation was investigated by G. A. NARIBOLI [81]. It results from the approximate solutions obtained that thermoelastic waves are damped and dispersed. The influence of coupling deformation and temperature fields is small. The solutions slightly differ quantatively from the solutions obtained within the framework of the theory of thermal stresses.

The second important problem to which several works are devoted in the propagation of a plane wave in a thermoelastic semi-space caused by sudden heating of the plane bounding a space. The question consists in the generalization of the „Danilovskaya problem" familiar in the theory of thermal stresses. This subject was undertaken by R. B. HETNARSKI [79, 82] applying the perturbation method and making use of Abel's theorem for small times. The same problem was investigated by B. A. BOLEY and I. S. TOLINS [83] as also as by R. MUKI and S. BREUER [84]. The action of the points heating of a thermoelastic semi-space was the subject of work by G. PARIA [85].

The papers by I. N. SNEDDON [55] and J. IGNACZAK [75] were devoted to the propagation of a longitudinal wave in an elastic semi-space and in an infinite and semi-infinite rod. In this publication, the Fourier's transformation with respect to the position variable was first applied, and an ordinary differential equation of the third order with respect to time was solved. Solving this equation and performing the Fourier's inverse transformation, led to a final result.

There is an extensive literature concerning dynamic problems of the theory of thermal stresses. So far, numerous one-dimensional and two-dimensional problems have been solved. The first of these solutions here given by V. I. DANILOVSKAYA, date to 1950 [73, 74]. They concern the propagation of thermoelastic waves in an elastic semi-space, due to a sudden heating of the bounding surface. Here, too, we find the two types of waves, elastic and diffusional. On the front of the elastic wave, there occurs a jump of stress (change of sign). The problem of propagation of a spherical wave in an infinite space was discussed by W. NOWACKI [87] while the cylindrical wave was dealt with by H. PARKUS [88]. A practically important case of a sudden heating of the boundary of a spherical cavity in an infinite elastic space was investigated by E. STERNBERG and J. C. CHAKRAVARTY [89]. J. IGNACZAK examined the action of a concentrated instantaneous heat source in an infinite elastic space with a spherical cavity [9]. A concentration of stresses around a spherical and cylindrical cavities was dealt with by J. IGNACZAK and W. NOWACKI [91]. The problem of heat sources moving with a constant velocity in an infinite elastic space was the subject of a paper by M. ŻÓRAWSKI [92]. Finally we mention the paper of B. A. BOLEY and A. D. BARBER [93] concerned the vibrations of a thin plate produced by a sudden heating or cooling.

To conclude this survey, reference should be made to the further developing directions of thermoelasticity. It seems that further general theorems will be obtained which will constitute a generalization of the theorems familiar in elastodynamics. We mean the generalization of Kirchhoff's, Weber's and Volterra's theorems. Attempts are being made [86] to obtain further and wider variational theorems. The next efforts will be directed towards eliminating the restrictions on small deformations, and thereby towards developing nonlinear geometrically thermoelasticity. Other direction intends to removing the restriction $|\theta/T_0| \ll 1$— i.e. to investigate bodies with higher temperature when thermal and mechanical coefficients are functions of temperature.

Recently, investigations have been initiated in the field of combining the fields of deformation, temperature and electric fields in piezoelectric materials [94–96]. The initiated direction of magneto-thermoelasticity now embarked on is also of interest [71, 101].

The question consists in investigating the deformation field and temperature field in electrical conductors in the presence of a strong primery magnetic field.

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