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THERMAL EXCITATIONS IN COUPLED FIELDS

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1. Introduction

The problems of thermal excitations in the theory of coupled fields fall within the subject called briefly thermo-electro-magnetoelasticity and constitute an extension of the classical thermoelasticity to the phenomena of coupling with the electromagnetic field. Obviously, we may also speak more generally about thermoelectro-magnetoplasticity; at present, however, we lack serious papers in this field and consequently we confine ourselves to thermo-electro-magnetoelasticity. The richness of the equations, and hence the variety of solutions and physical phenomena, create a wide field of practical applications for various particular forms of the theory which is now in the initial stage of development and has so far comparatively few effective solutions of the fundamental problems; the same concerns to an even greater degree the practical problems.

Around the early sixties we observed a rather rapid and intensive development of the theory of coupled fields embracing problems of magnetoelasticity of media without and with spin. We witnessed also the emergence of electroelasticity (piezoelectricity) in connection on the one hand with the development of the nonlinear mechanics of continuous media, and on the other with the discovery of new practical possibilities in the theory of ultrasonics and hypersonics for the application of semi-conductors.

At the same time, there began a rather slower development of thermo-electro-magnetoelasticity, in the fields of both thermo-magnetoelasticity and thermo-electroelasticity.

The contribution of the Polish school to the development of the theory of coupled fields in particular to the thermo-electro-magnetoelasticity is considerable and sometimes pioneering. The purpose of the present paper is a brief outline of the existing achievements and an indication of the possible trends and new problems.

The thermoelectric, magnetoelectric effects and the fundamental physical relations of these effects on the elastic field have been known in physics for a long time (see e. g. [1]). we shall not, therefore, deal with them here. On the other hand, from the point of view of the field theory of thermal, elastic electromagnetic fields, the problem was elaborated

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as already indicated, early in the sixties. A systematic account of the equations of thermo-magnetoelasticity was presented in papers [2, 3].

The thermodynamic foundations of the thermo-electromagnetic processes on the basis of the thermodynamics of irreversible processes were systematized in the monograph [4], while the complete system of equations of thermo-electro-magnetoelasticity based on a thermodynamic analysis was given in [5].

In [5] we also deduced new equations of the so-called "wave" theory of thermo-electro-magnetoelasticity, in which, on the basis of a modification of the Fourier law, we constructed approximate phenomenological equations of thermo-electro-magneto-elasticity characterized by a finite velocity of propagation of thermal electromagnetic and elastic excitations.

The equations of thermo-piezoelectricity including a thermodynamic analysis were presented systematically in paper [6]. Further, in [7] C. ERINGEN deduced the equations on the nonlinear theory of the thermo-electro-magnetoelastic field with finite deformations.

In parallel with the fundamental papers concerning the construction of the equation of the coupled thermo-electro-magnetoelastic fields, there began to appear particular papers concerning either certain definite solutions or some more general theorems.

Thus, the fundamental one-dimensional problem for the elastic semispace subject to a thermal shock on its surface, in the case of a perfect and real conductor in the magnetic field, was solved in papers [8, 9]. In [10, 11, 12] particular problems were solved for the one-dimensional periodic plane waves in thermo-magnetoelasticity for perfect and real conductors, whereas in [19] the same problems were examined for thermo-piezoelectricity. In papers [13, 14, 15] solutions have been given for some two-dimensional stationary problems of thermo-magnetoelasticity for perfect and real conductors in magnetic field. In papers [16, 17, 18] were formulated reciprocity theorems. In papers [20, 21], variational theorems were deduced for perfect and real elastic conductors, while in [22, 23] the reciprocity theorems for the wave equations of thermo-magnetoelasticity and thermo-piezoelectricity were investigated. In paper [24] the problem of acceleration waves in nonlinear thermo-magnetoelasticity was examined.

Besides papers of a field-mathematical nature, a number of papers of a practical character appeared—e.g. [25], where the problem of conversion of the energy of laser radiation into heat and elastic energies was considered, and other papers of this type. Further, some papers were published of a physical nature concerning the character of the physical relations between the fields—for instance, [26, 27] where the construction of the equations of thermo-magneto-microelasticity was presented, for media without and with spin, and further papers [28, 29] in which it was proved that under the influence of the temperature gradient at sufficiently low temperatures there may exist new types of thermo-magnetic waves with specific stability properties. These problems are included into the scope of plasma dynamics (also in solids).

For obvious reasons, it is difficult in this paper to embrace this wide field of problems. Thus, we confine ourselves primarily to the papers on thermo-electro-magnetoelasticity developed on the basis of the mechanics of continuous media, in its specific language and methods. In the final section we refer qualitatively certain wider aspects of the problem and further trends.

The structure of the present paper is as follows.

After a general introduction section 2, we present the general linear system of equations of thermo-electro-magnetoelasticity including the thermodynamic foundations—i.e., the system of equations of thermo-magnetoelasticity, thermo-piezoelectricity, thermo-magneto-microelasticity and the wave equation of thermo-electro-magnetoelasticity.

In section 3, we deal with the generalization to the nonlinear case of the equations of thermo-magnetoelasticity.

In section 4, we review the fundamental solutions of the thermo-electro-magneto-elasticity and finally in section 5, we briefly consider further problems in thermo-electro-magnetoelasticity.

2. General Systems of Equations of Thermo-electro-magnetoelasticity. Thermodynamic foundations

In this section, we briefly present the equations and the relevant thermodynamic discussion of the linear theories of thermo-magnetoelasticity, thermo-piezoelectricity, extensions of these equations to wave phenomena, and the equations of thermo-magneto-microelasticity. We do not examine the assumptions, referring the reader to the papers [5, 26].

2.1. The equations of thermo-magnetoelasticity of conductors. According to [2, 3, 5], the equations of thermo-magnetoelasticity of real anisotropic conductors in a magnetic field have in RMKS units the form:

$$\operatorname{rot}\mathbf{h} = \mathbf{j}+\dot{\mathbf{D}}, \quad \operatorname{rot}\mathbf{E} = -\dot{\mathbf{b}}, \quad \operatorname{div}\mathbf{b} = 0, \quad \operatorname{div}\mathbf{D} = \varrho_e,$$

$$\varrho\ddot{u}_i = \sigma_{ik,k}+(\mathbf{j}\times\mathbf{B}_0)_i+\varrho_e E_i+P_i, \tag{2.1}$$

$$\beta\dot{T}+\lambda_{ij}\dot{e}_{ij}-(k_{ij}T_{,j})_{,i}+(\pi_{ik}j_k)_i = f.$$

Here,

$$b_i = \mu_{ik}h_k, \quad D_i = \varepsilon_{ik}\left[E_k+(\dot{\mathbf{u}}\times\mathbf{B}_0)_k-\frac{1}{c^2}(\dot{\mathbf{u}}\times\mathbf{H}_0)\right],$$

$$\sigma_{ik} = E_{ikmn}e_{mn}-\alpha_{ik}T-\int_0^t R_{ikmn}(t-\tau)\left[e_{mn}(\tau)-\alpha_{0ik}T(\tau)\right]d\tau, \tag{2.2}$$

$$j_i = \eta_{ik}E_k\kappa_{ik}T_{,k}+\eta_{ik}(\dot{\mathbf{u}}\times\mathbf{B}_0)_k+\varrho_e\dot{u}_i,$$

where $\alpha_{ik} = E_{ikmn}\alpha_{0mn}$, $e_{ik} = 1/2(u_{i,k}+u_{k,i})$.

The system of Eqs. (2.1) constitutes a system of Maxwell equations, elasticity equations and the heat conduction, respectively, with the appropriate couplings, while the Eqs. (2.2) are equations of state and the relations between the generalized forces and fluxes, respectively. The expressions $\varrho_e E_i$, $\varrho_e\dot{u}_i$ may be disregarded in accordance with the linearization. The notations for the temperature, displacements and the field components are the usual ones. The tensors have the following meaning:

σ_{ik} stress tensor,
E_{ikmn}, R_{ikmn} tensors of elastic and relaxation moduli,
μ_{ik}, ε_{ik} tensors of magnetic and electric permeabilities,
η_{ik} tensor of electric conductivity,
α_{ik}, α_{0ik} tensors of thermal expansion,
k_{ik} tensors of thermal conduction,
λ_{ik} tensor describing the influence of the strain on the temperature field,
π_{ik} tensor describing the influence of the current intensity of the heat flux,
κ_{ik} tensor connecting the temperature gradient with the electric current,
p_i vector of body forces,
f density of thermal sources,
$\mathbf{H}_0$, $\mathbf{B}_0$ vectors of the initial magnetic field and the magnetic induction.

In the isotropic case, the system of Eqs. (2.1) and (2.2), after certain transformations, take the form:

$$\operatorname{rot}\mathbf{h} = \mathbf{j}+\varepsilon\mathbf{E}+\frac{\varepsilon\mu c^2-1}{c^2}(\ddot{\mathbf{u}}\times\mathbf{H}_0),$$

$$\operatorname{rot}\mathbf{E} = -\mu\dot{\mathbf{h}}, \tag{2.3}$$

$$\varrho\ddot{\mathbf{u}} = G\nabla^2\mathbf{u}+(\lambda_0+G)\operatorname{grad}\operatorname{div}\mathbf{u}+\varrho_e\mathbf{E}+\frac{1}{c}(\mathbf{j}\times\mathbf{B}_0)+\mathbf{P}-3\alpha_0 K\operatorname{grad}T,$$

$$\beta\dot{T}+\lambda\operatorname{div}\dot{\mathbf{u}}+\pi\operatorname{div}\mathbf{j}-k\nabla^2T = f,$$

$$\mathbf{j} = \eta[\mathbf{E}+(\dot{\mathbf{u}}\times\mathbf{B}_0)]-\kappa\operatorname{grad}T+\varrho_e\dot{\mathbf{u}},$$

where $K = \lambda_0+2/3\,G$.

In the coordinate system connected with the medium, the constitutive Eqs. (2.2) take the form:

$$b_i^0 = \mu_{ik}h_k^0,\quad D_i^0 = \varepsilon_{ik}E_k^0,\quad j_i^0 = \eta_{ik}E_k^0-\kappa_{ik}T_{,k}^0.$$

To derive the symmetry and energy relations, we briefly discuss the first and second laws of thermodynamics.

The energy equation for the thermo-magnetoelastic field is:

$$-\int_A q_i\,dA_i-\int_A N_i\,dA_i-\int_A \varrho w\dot{u}_i\,dA_i+\int_A \dot{u}_i\sigma_{ij}\,dA_j -\frac{\partial}{\partial t}\int_V \varrho w\,dV-\frac{\partial}{\partial t}\frac{1}{2}\int_V(H_iB_i+E_iD_i)\,dV = 0, \tag{2.4}$$

where $H_i = H_{0i}+h_i$, $B_i = B_{0i}+b_i$, $w = 1/2\,\dot{u}_i^2+w_0$,
w_0 is internal, mechanical energy per unit mass, N_i – Umov-Pointing vector, q_i – heat flux.

Making use of the Gauss formula, the expression for the Lorentz force and the Fourier law, we have:

$$q_i = -k_{ij}T_{,i}+\pi_{ik}j_k. \tag{2.5}$$

Bearing in mind that the energy equation of the electromagnetic field, independently of the contribution of the mechanical and thermal fields, has the form:

$$\int_A N_i\,dA_i+\frac{1}{2}\frac{\partial}{\partial t}\int_V(H_iB_i+E_iD_i)\,dV+\int_V E_ij_i\,dV = 0, \tag{2.6}$$

and taking into account the complete linearization we can represent the first law of thermodynamics for the thermo-magnetoelasticity in the form:

$$\varrho w_0 = E_i j_i + \sigma_{ij}\dot{e}_{ij} - (F_i + P_i)\dot{u}_i + (k_{ij}T_{,j})_{,i} - (\pi_{ik}j_k)_{,i}, \tag{2.7}$$

where $F_i = \varrho_e E_i + (\mathbf{j}\times\mathbf{B})_i$.

We now proceed to the second law:

$$\frac{Q}{T} \leqslant \frac{\partial}{\partial t}\int_V \varrho s\, dV + \int_A \varrho s\dot{u}_i\, dA_i. \tag{2.8}$$

Here s is the entropy per unit mass, Q quantity of heat entering V. Introducing the density of entropy production σ, we can write:

$$\int_V \dot{\sigma}\, dV - \int_A \frac{q_i}{\theta}\, dA_i = \frac{\partial}{\partial t}\int_V \varrho s\, dV + \int_A \varrho s\dot{u}_i\, dA_i, \tag{2.9}$$

where $\theta = T_0 + T, \quad T/T_0 \ll 1.$

Hence we obtain:

$$\dot{\sigma} - \left(\frac{q_i}{\theta}\right)_{,i} = \varrho\dot{s}, \tag{2.10}$$

and independently:

$$\varrho\dot{w}_0 = \varrho\theta\dot{s} + \sigma_{ij}\dot{e}_{ij}. \tag{2.11}$$

After appropriate transformations, (2.7) can be written in the form:

$$\varrho\dot{w}_0 = E_{0i}j_{0i} + \sigma_{ij}\dot{e}_{ij} + (k_{ij}T_{,j})_{,i} - (\pi_{ik}j_k)_{,i}, \tag{2.12}$$

(for $P_i = f = 0$), where

$$\mathbf{E}_0 = \mathbf{E} + (\dot{\mathbf{u}}\times\mathbf{B}_0), \qquad \mathbf{j}_0 = \mathbf{j} + \varrho_e\dot{\mathbf{u}} \approx \mathbf{j}_0. \tag{2.13}$$

Thus, from (2.11) and (2.12), we obtain:

$$\varrho\dot{s} = \frac{E_{0i}j_{0i}}{\theta} + \frac{(k_{ij}T_{,j})_{,i}}{\theta} - \frac{(\pi_{ik}j_{0k})_{,i}}{\theta} = \frac{E_{0i}j_{0i}}{\theta} - \left(\frac{q_i}{\theta}\right)_{,i} - \frac{q_iT_{,i}}{\theta^2}, \tag{2.14}$$

and hence, making use of (2.10):

$$\dot{\sigma} = \frac{E_{0i}j_{0i}}{\theta} - \frac{q_iT_{,i}}{\theta^2}, \tag{2.15}$$

where $E_{0i}j_{0i}$ is the term describing the Joule heat.

If in (2.15) we choose in an appropriate manner the forces and the fluxes, making use of the Onsager principle we arrive at the symmetry relations.

We set

$$j_{0i} = \eta_{ik}\theta\frac{E_{0k}}{\theta} - \kappa_{ik}\theta^2\frac{T_{,k}}{\theta^2} = \bar{\eta}_{ik}\frac{E_{0k}}{\theta} - \kappa_{ik}\frac{T_{,k}}{\theta^2}, \tag{2.16}$$

$$q_i = \pi_{il}\eta_{lk}\theta\frac{E_{0k}}{\theta} - \theta^2(h_{ik} + \pi_{il}\kappa_{lk})\frac{T_{,k}}{\theta^2} = \bar{\gamma}_{ik}\frac{E_{0k}}{\theta} - \bar{\nu}_{ik}\frac{T_{,k}}{\theta^2}.$$

Then the symmetry relations take the form:

$$\bar{\eta}_{ik}(\mathbf{B}) = \bar{\eta}_{ki}(-\mathbf{B}), \quad \bar{\nu}_{ik}(\mathbf{B}) = \bar{\nu}_{ki}(-\mathbf{B}), \quad \bar{\gamma}_{ik}(\mathbf{B}) = \bar{\gamma}_{ki}(-\mathbf{B}). \tag{2.17}$$

For the rate of entropy we have the expression

$$\dot{\sigma} = \bar{\eta}_{ik}^{-1} j_{0i} j_{0k} + \frac{1}{\theta^4} (\bar{\nu}_{ik} - \bar{\gamma}_{li} \kappa_{sk} \bar{\eta}_{sl}^{-1}) T_{,i} T_{,k}, \tag{2.18}$$

which implies the conditions of its positiveness.

In the isotropic case, we have:

$$\dot{\sigma} = \frac{j_{0i}^2}{\bar{\eta}} + \frac{1}{\theta^4} \left(\bar{\nu} - \frac{\bar{\kappa}^2}{\bar{\eta}} \right) T_{,i}^2, \tag{2.19}$$

where $\nu - \theta\kappa^2/\eta = k$, which requires the positiveness of k and η.

2.2. Thermo-piezoelectricity. In the case of dielectrics, the coupling of which with the elastic and electric fields occurs by means of the piezo-effect the equations of the coupled fields of thermo-piezoelectricity in the linearized form take the form [5, 6]

$$\operatorname{rot}\mathbf{H} = \dot{\mathbf{D}}, \quad \operatorname{rot}\mathbf{E} = -\dot{\mathbf{B}}, \quad \operatorname{div}\mathbf{B} = 0, \quad \operatorname{div}\mathbf{D} = \varrho_e \tag{2.20}$$

$$\varrho \ddot{u}_i = \sigma_{ik,k}, \quad (k_{ij} T_{,j})_{,i} = T_0 \dot{s},$$

where

$$\begin{aligned} \sigma_{ik} &= E_{ikmn}^{ETH} e_{mn} - \alpha_{im}^{EH} T - r_{ikl}^{TH} E_l, \\ D_i &= r_{ikl}^{TH} e_{kl} + \varepsilon_{ij}^{TeH} E_j + p_i^{eH} T, \\ s &= \alpha_{ik}^{EH} e_{ik} + p_i^{EH} E_i + \beta_0^{eEH} T, \\ B_i &= u_{ik}^{TeH} H_k. \end{aligned} \tag{2.21}$$

The upper indices denote the thermodynamic constancy of the quantities for which the tensor is defined. In the above equations, the notations are analogous to those in the proceding section and, moreover, the following have been introduced:

r_{ikl}^{TH} tensor of piezoelectric constants at a constant temperature and magnetic field,

p_i vector connecting the electric induction with the temperature.

Substituting from (2.21), we write in full the system (2.20):

$$\begin{aligned} & E_{iklm} u_{m,ni} - r_{ikl} E_{l,i} - \alpha_{ik} T_{,i} = \varrho \ddot{u}_k, \\ & \varepsilon_{ijk} H_{k,j} = \frac{1}{2} r_{ikl} (\dot{u}_{k,l} + \dot{u}_{l,k}) + \varepsilon_{ij} \dot{E}_j + \varrho_i \dot{T}, \\ & \varepsilon_{ijk} E_{j,k} = \mu_{ik} \dot{H}_k, \\ & \alpha_{ij} \dot{u}_{i,j} - p_i \dot{E}_i + \beta_0 \dot{T} - T_0^{-1} (k_{ij} T_{,j})_{,i} = 0, \end{aligned} \tag{2.22}$$

where $\theta = T_0 + T, \quad T/T_0 \ll 1$.

The principle of conservation of energy yields:

$$-\int_A N_i dA_i - \int_A q_i dA_i + \int_A \sigma_{ij} \dot{u}_j dA_i = \int_V (\dot{K} + \dot{U}) dV, \tag{2.23}$$

where

$N_i = \varepsilon_{ijk} E_j H_k$ Umov-Pointing vector,

$K = \frac{1}{2} \varrho \dot{u}_i^2$ density of kinetic energy,

U density of internal energy.

Applying the Gauss formula to (2.23), after transformations we obtain:

$$\dot{U} = \sigma_{ij} \dot{e}_{ij} + (E_i \dot{D}_i + H_i \dot{B}_i) - q_{i,i}. \tag{2.24}$$

According to the definition of entropy

$$q_{i,i} = -\theta \dot{s} \approx -T_0 \dot{s},$$

and the Fourier law

$$q_i = -k_{ij} \theta_{,j} = -k_{ij} T_{,j}, \tag{2.25}$$

we obtain, introducing the free energy:

$$U_0 = U - T_0 s \tag{2.26}$$

$$\dot{U}_0 = \sigma_{ij} \dot{e}_{ij} + (E_i \dot{D}_i + H_i \dot{B}_i) + \dot{T} s = \dot{U}_0(\dot{e}_{ij}, D_i, B_i, s). \tag{2.27}$$

Hence,

$$\sigma_{ij} = \frac{\partial U_0}{\partial e_{ij}}, \quad H_i = \frac{\partial U_0}{\partial B_i}, \quad E_i = \frac{\partial U_0}{\partial D_i}, \quad T = \frac{\partial U_0}{\partial s}, \tag{2.28}$$

If we choose

$$U_0 = \frac{1}{2} E^{DsB}_{ikmn} e_{ij} e_{mn} + \frac{1}{2} \beta^{seB}_{ij} D_i D_j + \frac{1}{2} b^{eDB} s^2 - h^{sB}_{kij} D_k e_{ij} - g^{eB}_i s D_i - \gamma^{DB}_{ij} s e_{ij} + \frac{1}{2} (\mu_{ik})^{-1} B_i B_k, \tag{2.29}$$

where

$$\begin{gathered} E^{DsB}_{ikmn} = E^{DsB}_{klmn} = E^{DsB}_{iknm} = E^{DsB}_{mnik}, \\ \beta^{seB}_{ij} = \beta^{seB}_{ji}, \quad h^{sB}_{kij} = h^{sB}_{kji}, \quad \gamma^{DB}_{ij} = \gamma^{DB}_{ji}, \quad \mu_{ik}(\mu_{ik})^{-1} = 1, \end{gathered} \tag{2.30}$$

we arrive at the expressions for σ_{ij}, P, E_i, H_j.

Transforming these expressions by means of the coefficients used in (2.21), [5], we arrive exactly at the relations (2.21). In general in the equations of thermo-piezoelectricity, the influence of the magnetic induction may be disregarded, which results in a simplification of the equations and makes it possible to introduce the potential of electric field.

A discussion of the second law of thermodynamics implies

$$\dot{s} + \left(\frac{q_i}{\theta}\right)_{,i} \geqslant 0, \tag{2.31}$$

which, making use of the Fourier law leads to the condition:

$$-\frac{\theta_{,i} q_i}{\theta^2} = \frac{k_{ij} \theta_{,i} \theta_{,j}}{\theta^2} \approx \frac{k_{ij} T_{,i} T_{,j}}{\theta^2} \geqslant 0, \tag{2.32}$$

this, in turn, requires the symmetry $k_{ij} = k_{ji}$ and $|k_{ij}| > 0$;

$$\begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix} > 0, \quad k_{11} > 0.$$

2.3. The wave equations of thermo-electro-magnetoelasticity. It was proved in [5] that generalizing in an appropriate manner the Fourier law, we can extend the systems of equations of thermo-magnetoelasticity and thermo-piezoelectricity to wave equations—i.e., equations in which the disturbances of all fields, including the thermal field, are propagated with a finite velocity.

This effect cannot be obtained (see [5]) by taking into account relativistic effects in the systems of equations. In view of exigencies of space, we do not quote here the considerations and modifications of the thermodynamics of irreversible processes, referring the reader to [5], but we write down the final wave equations. These equations therefore contain strongly non-stationary processes by contrast with the above considered parabolic-hyperbolic equations which are true for stationary or weakly nonstationary processes. The wave equations of thermo-magnetoelasticity have, according to [5], the form:

$$(2.33)\quad \begin{aligned} &\operatorname{rot}\mathbf{h} = \dot{\mathbf{j}}+\mathbf{D}, \quad \operatorname{rot}\mathbf{E} = -\dot{\mathbf{b}}, \quad \operatorname{div}\mathbf{b} = 0, \quad \operatorname{div}\mathbf{D} = \varrho_e, \\ &\varrho\ddot{u}_i = \sigma_{ik,k}+(\mathbf{j}\times\mathbf{B}_0)_i+\varrho_e E_i+P_i, \\ &\tau\dot{q}_{i,i}+q_{i,i}+(\mathrm{k}_{ij}T_{,j})_{,i}-(\pi_{ik}j_k)_{,i} = -f, \\ &b_i = \mu_{ik}h_k, \quad D_i = \varepsilon_{ik}[E_k+(\dot{\mathbf{u}}\times\mathbf{B}_0)_k]-\frac{1}{c^2}(\dot{\mathbf{u}}\times\mathbf{H}_0)_i, \\ &\sigma_{ik} = E_{ikmn}\varepsilon_{mn}-\alpha_{ik}T \quad \text{(the relaxation has been disregarded)} \\ &j_i = \eta_{ik}E_k+\kappa_{ik}[k_{kj}^{-1}q_j-k_{kj}^{-1}\pi_{js}j_s]+\eta_{ik}(\dot{\mathbf{u}}\times\mathbf{B}_0)_k+\varrho_e\dot{u}_i, \end{aligned}$$

where the constant τ has the character of the relaxation constant and follows from the generalized Fourier law:

$$(2.34\quad \tau\dot{q}_{i,i}+q_{i,i} = -(k_{ij}T_{,j})_{,i}+(\pi_{jk}j_k)_{,i},$$

since $\tau \to 0$, the Eqs. (2.33) are transformed into the equations of (2.1).

In the case of an isotropic medium, the Eqs. (2.33) take the form:

$$2.35)\quad \begin{aligned} &\operatorname{rot}\mathbf{h} = \mathbf{j}+\varepsilon[\dot{\mathbf{E}}+(\ddot{\mathbf{u}}\times\mathbf{B}_0)], \quad \operatorname{rot}\mathbf{E} = -\dot{\mathbf{b}}, \quad \operatorname{div}\mathbf{b} = 0, \quad \operatorname{div}\mathbf{D} = 0 \quad (\varrho_e = 0), \\ &\varrho\ddot{\mathbf{u}} = G\nabla^2\mathbf{u}+(\lambda+G)\operatorname{grad}\operatorname{div}\mathbf{u}-\alpha\operatorname{grad}T+(\mathbf{j}\times\mathbf{B}_0), \\ &\tau\beta\varphi\ddot{T}+\beta\dot{T}+\tau\lambda\varphi\operatorname{div}\ddot{\mathbf{u}}+\lambda\operatorname{div}\dot{\mathbf{u}}-k\varphi\nabla^2T = 0, \\ &\mathbf{j}+\tau\varphi\dot{\mathbf{j}} = \eta\{\mathbf{E}+\tau\dot{\mathbf{E}}+(\dot{\mathbf{u}}+\tau\ddot{\mathbf{u}})\times\mathbf{B}_0\}-\kappa\operatorname{grad}T, \end{aligned}$$

where $\lambda = T_0\alpha, \quad \beta = \beta_0 T_0, \quad \varphi = 1+\kappa\pi/\eta$.

The corresponding equations of thermo-piezoelectricity have the form:

$$(2.36)\quad \begin{aligned} &E_{ikmn}u_{m,ni}-r_{ikl}E_{l,i}-\alpha_{ik}T_{,i} = \varrho\ddot{u}_k, \\ &\varepsilon_{ijk}H_{kj} = \frac{1}{2}r_{ikl}(\dot{u}_{k,l}+\dot{u}_{l,k})+\varepsilon_{ij}\dot{E}_j+p_i\dot{T}, \\ &\varepsilon_{ijk}E_{j,k} = \mu_{ik}\dot{H}_k, \end{aligned}$$

$$T_0 \alpha_{ij}(\tau \ddot{u}_{i,j} + \dot{u}_{i,j}) + T_0 p_i(\tau \ddot{E}_i + \dot{E}_i) + T_0 \beta (\tau \ddot{T} + \dot{T}) - (k_{ij} T_{,j})_{,i} = 0 .$$

Since $\tau = 0$, they are transformed into (2.22).

It can readily be verified that both (2.33) and (2.36) constitute hyperbolic systems of equations (see [5]). The idea of an experimental verification of the wave phenomenon on the basis of the Ĉerenkov effect in coupled fields was presented in [30].

In our considerations, we have not dealt with the boundary conditions. In general, they follow directly from the physical nature of the relations on the boundary and can be expressed in terms of the field components and the constitutive relations. It has been necessary to confine our review mainly to the equations and the physical effects of field couplings.

2.4 Thermo-magneto-microelasticity. In the case of medium without spin, we can easily generalize the equations of thermo-microelasticity (with 6 local degrees of freedom) to the equations of thermo-magneto-microelasticity [26]. In view of exigencies of space, we omit here also the thermodynamic considerations, quoting only the final system of equations. In the particular case of a centrosymmetric body (more general look [26]) we find:

$$\operatorname{rot} \mathbf{h} = \dot{\mathbf{j}} + \varepsilon \dot{\mathbf{E}} - \frac{\varepsilon \mu c^2 - 1}{c^2} (\ddot{\mathbf{u}} \times \mathbf{H}_0), \qquad \operatorname{rot} \mathbf{E} = -\mu \dot{\mathbf{h}},$$

$$\varrho \ddot{\mathbf{u}} = (\mu + \alpha) \nabla^2 \mathbf{u} + (\lambda + \mu - \alpha) \operatorname{grad} \operatorname{div} \mathbf{u} + 2\alpha \operatorname{rot} \boldsymbol{\varphi} + \varrho_e \mathbf{E} + (\mathbf{j} \times \mathbf{B}_0) - 3\alpha_0 K \operatorname{grad} T + \mathbf{P},$$

$$m \ddot{\boldsymbol{\varphi}} = (\gamma + \varepsilon) \nabla^2 \boldsymbol{\varphi} + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \boldsymbol{\varphi} + 2\alpha \operatorname{rot} \mathbf{u} - 4\alpha \boldsymbol{\varphi} + \mathbf{M}, \tag{2.37}$$

$$\beta \dot{T} + \lambda \operatorname{div} \dot{\mathbf{u}} + \pi \operatorname{div} \mathbf{j} - k \nabla^2 T = f,$$

$$\mathbf{j} = \eta [\mathbf{E} + (\dot{\mathbf{u}} \times \mathbf{B}_0)] - \kappa \operatorname{grad} T + \varrho_e \dot{\mathbf{u}}$$

(the terms $\varrho_e \mathbf{E}$, $\varrho_e \dot{\mathbf{u}}$ drop out in view of the linearization; we have written them to indicate the structure of the equations).

The equations of thermo-magneto-microelasticity for a medium with a spin are far more complicated. In fact, in this case the equation of micromoments and the Landau spin equation require a deeper physical analysis in order to determine the nature of the couplings [27]. Similarly, the equations of micro-piezoelectricity are more complicated; here the basic problem is the connection between the polarization and the micromoments.

We have omitted here the boundary conditions, referring the reader for the details to [26]. However, these conditions follow either from the variational equations or from the fundamental physical considerations if we make use of the appropriate state relations for the stresses, moments, inductions, etc. and the components of the displacement, microrotations, field vectors, etc.

3. Nonlinear Equations of Thermo-magnetoelasticity

In view of exigencies of space, we do not intend to discuss the nonlinear (finite deformations) equations of thermo-piezoelectricity, or more generally, the thermo-electroelasticity of dielectrics. These equations can easily be derived on the basis of the nonlinear equations [31] for the general dynamics of dielectrics, by completing them by the equations of heat transport and thermal couplings.

Consider now briefly the nonlinear equations of thermo-magnetoelasticity [24], the investigation of which has begun only recently. Here, similarly to the linear theory we can consider jointly the problems of thermo-electroelasticity and thermo-magnetoelasticity, but in view of the physical nature of the problem and different ranges of applications of dielectrics and conductors, a separate consideration is simpler and more expedient. Similarly to the linear case, we do not deal here with the media with spin. These problems will be examined in the final section.

Denote the natural, initial and current coordinates by x_α, X_k, y_i, respectively. The deformations at an arbitrary instant of time are described by the relations:

$$\mathbf{y} = \mathbf{y}(\mathbf{x}, t) \tag{3.1}$$

We follow here the notations of TRUESDELL and his collaborators.

Disregarding the mechanical body forces, the system of nonlinear equations of thermo-magnetoelasticity takes the following form:

1. The Maxwell equations:

$$\begin{aligned} &\varepsilon_{ijk}\frac{\partial e_k}{\partial y_j} + B_i^* = 0, \qquad \frac{\partial B_i}{\partial y_i} = 0, \\ &\varepsilon_{ijk}\frac{\partial \mu_k}{\partial y_i} - D_i^* = j_i, \qquad \frac{\partial D_i}{\partial y_i} = q_e, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &e_i = E_i + \varepsilon_{irs} v_r B_s, \quad \mu_i = H_i - \varepsilon_{irs} v_r D_s, \quad D_i = \varepsilon E_i + \alpha\varepsilon_{irs} v_r H_s, \\ &B_i = \mu H_i - \alpha\varepsilon_{irs} v_r E_s, \quad j_i = e_i \eta, \quad \alpha = \mu\varepsilon - \mu_0\varepsilon_0; \end{aligned} \tag{3.3}$$

the vectors have the same meaning as before, and ε, μ, ε_0, μ_0 denote the electric and magnetic permeabilities in the medium and in vacuum; ε_{ijk} is the unit pseudotensor. We have also denoted:

$$f_i^* = \frac{\partial f_i}{\partial t} + v_j \frac{\partial f_i}{\partial y_j} - f_j \frac{\partial v_i}{\partial y_i} + f_i \frac{\partial v_j}{\partial y_i}. \tag{3.4}$$

2. The equations of motion:

$$\frac{\partial t_{ij}}{\partial y_j} + q_e e_i + \varepsilon_{irs} j_r B_s = \varrho \dot{v}_i. \tag{3.5}$$

3. The equations of energy balance:

$$\varrho\dot{\Sigma} = t_{ij}\frac{\partial v_i}{\partial y_j} + e_i j_i - \frac{\partial q_i}{\partial y_i}, \tag{3.6}$$

where t_{ij} is the Cauchy tensor and Σ is the internal energy. The above equations have to be completed by the equations of state and transport:

$$t_{ij} = t_{ij}(p_{i\alpha}, \theta), \qquad q_i = q_i(\theta_{,j}, p_{j\alpha}, \theta), \tag{3.7}$$

where θ is the temperature

$$\theta_{,i} = \frac{\partial\theta}{\partial y_i}, \quad p_{i\alpha} = \frac{\partial y_i}{\partial x_\alpha}.$$

The functions t_{ij}, $\mathbf{q}$ have to be chosen in such a way that the symmetry conditions are satisfied, and, moreover, that the Clausius-Duhem inequality

$$\varrho\dot{s} - \frac{e_i j_i}{\theta} - \frac{\partial}{\partial y_i}\left(\frac{q_i}{\theta}\right) \geqslant 0 \tag{3.8}$$

holds true.

A shortcoming of the above equations consists in the fact that some derivatives are with respect to $\mathbf{x}$ while some are with respect to $\mathbf{y}$.

The equations considered can be simplified by referring them to the natural configuration by means of the transformation:

$$\begin{aligned}
b_\alpha(\mathbf{x},t) &= JX_{\alpha,i}B_i(\mathbf{x},t), & h_\alpha(\mathbf{x},t) &= p_{i\alpha}H_i(\mathbf{x},t),\\
d_\alpha(\mathbf{x},t) &= Jx_{\alpha,i}D_i(\mathbf{x},t), & \bar{e}_\alpha(\mathbf{x},t) &= p_{i\alpha}E_i(\mathbf{x},t),\\
g_\alpha &= Jx_{\alpha,i}j_i(\mathbf{x},t), & \theta_e(\mathbf{x},t) &= Jq_e(\mathbf{x},t),\\
\bar{\varepsilon}_{\alpha\beta\gamma} &= J^{-1}p_{i\alpha}p_{j\beta}p_{k\gamma}\varepsilon_{ijk}, & v_\alpha(\mathbf{x},t) &= x_{\alpha,i}v_i(\mathbf{x},t),\\
T_{\alpha i}(\mathbf{x},t) &= Jx_{\alpha,j}t_{ij}(\mathbf{x},t), \quad \varrho_0 = J\varrho, & \theta_\alpha(\mathbf{x},t) &= Jx_{\alpha,i}q_i(\mathbf{x},t),
\end{aligned} \tag{3.9}$$

where

$$x_{\alpha,i} = \frac{\partial x_\alpha}{\partial y_i}, \quad J = |p_{i\alpha}|.$$

Then the equations take the form:

$$\begin{aligned}
\bar{\varepsilon}_{\alpha\beta\gamma}\frac{\partial e_j}{\partial x_\beta} + \dot{b}_\alpha &= 0, & \frac{\partial b_\alpha}{\partial x_\alpha} &= 0,\\
\bar{\varepsilon}_{\alpha\beta\gamma}\frac{\partial \mu_\gamma}{\partial x_\beta} - \dot{d}_\alpha &= q_\alpha, & \frac{dd_\alpha}{dx_\alpha} &= Q_e.
\end{aligned} \tag{3.10}$$

Here we have

$$\begin{aligned}
e_\alpha &= \bar{e}_\alpha + \bar{\varepsilon}_{\alpha\beta\gamma}v_\beta b_\gamma, \quad \mu_\alpha = h_\alpha - \varepsilon_{\alpha\beta\gamma}v_\beta d_\gamma,\\
d_\alpha &= Ic_{\alpha\beta}^{-1}(\varepsilon\bar{e}_\beta + \alpha I\bar{\varepsilon}_{\beta\mu\gamma}v_\mu c_{\gamma\omega}^{-1}h_\omega),\\
b_\alpha &= Ic_{\alpha\beta}^{-1}(\mu h_\beta - \alpha I\bar{\varepsilon}_{\beta\mu\gamma}v_\mu c_{\gamma\omega}^{-1}\bar{e}_\omega),\\
g_\alpha &= \eta Ic_{\alpha\beta}^{-1}e_\beta,
\end{aligned} \tag{3.11}$$

where

$$c_{\alpha\beta}^{-1} = \frac{\partial x_\alpha}{\partial y_s}\frac{\partial x_\beta}{\partial y_s},$$

and

$$\frac{\partial T_{\alpha i}}{\partial x_\alpha} = \mu_{\alpha,i}\{Q_e e_\alpha + \bar{\varepsilon}_{\alpha\beta\gamma}g_\beta b_\gamma\} = \varrho_0\dot{v}_i,$$

$$\varrho_0 \dot{\sum} = T_{\alpha i} \frac{\partial v_i}{\partial x_\alpha} + e_\alpha g_\alpha - \frac{\partial Q_\alpha}{\partial x_\alpha}, \tag{3.12}$$

$$\varrho_0 \dot{s} - \frac{e_\alpha q_\alpha}{\theta} - \frac{\partial}{\partial x_\alpha}\left(\frac{Q_\alpha}{Q}\right) \geqslant 0.$$

The equations are considerably simplified if we assume $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, and disregard the effects connected with the velocity of light. A further simplification is obtained for perfect conductors. For details, the reader is referred to [24]. The latter paper contains also a consideration of the problem of propagation of the acceleration waves for the equations derived in this section.

4. Review of the Fundamental Solutions of Magneto-thermoelasticity

4.1. One-dimensional problem of magneto-thermoelasticity. In this field, there have appeared papers concerning the propagation of plane magneto-thermoelastic waves. They deal mainly with real conductors. The problem of propagation of a plane wave in an infinite medium was first considered in a paper by G. PARIA [10] and then, under wider assumptions, by A. J. WILLSON [11]. PARIA assumed orthogonality of the initial vector of the magnetic field to the direction of propagation of the plane wave; A. J. WILLSON, on the other hand, assumed that the initial magnetic field has also a component in the direction of propagation of the longitudinal wave. The initial field is described by the vector $H = (H_1, H_2, 0)$, and all quantities changing with the deformation, temperature and electromagnetic field depend on the variables x_1 and t.

If we assume that $H_1 = 0$, $H_2 \neq 0$, we find that the transverse wave is not coupled with the temperature and electromagnetic fields; the coupling appears only in the longitudinal wave. If $H_1 \neq 0$, $H_2 = 0$ there exists a coupling of the displacement and temperature fields in the longitudinal wave and a coupling of the deformation and electromagnetic fields in the transverse wave. Clearly, in the case $H_1 \neq 0$, $H_2 \neq 0$, we have coupling of the deformation, temperature and electromagnetic fields in both the longitudinal and transverse waves.

Two particular cases of propagation of a plane wave were examined by W. NOWACKI [12]. They are produced by the action of either a plane heat source of the type $Q(x, t) = Q_0\,\delta(x_1)\,e^{-i\omega t}$ or body forces $P(x, t) = P_0\,\delta(x_1)\,e^{-i\omega t}$. The equation for the longitudinal wave in the case of a perfect conductor has the form:

$$\left[\left(\partial_1^2 - \frac{1}{\kappa}\,\partial_t\right)\left(\partial_1^2 - \frac{1}{c_0^2}\,\partial_t^2\right) - \eta m_0\,\partial_t\,\partial_1^2\right] u_1 = 0. \tag{4.1}$$

This equation differs from the equation of propagation of the thermoelastic plane wave by the terms c_0 and $m_0 = m/c_0^2$, where $c_0 = c_1^2(1+\alpha)$, $\alpha = a_0^2/c_1^2$. Here $a_0^2 = H_2^2\,\mu_0/4\pi\varrho$, and a_0 is the Alfven velocity. The electromagnetic excitation is described by the quantity α.

It follows from the structure of the Eq. (4.1) that the plane wave undergoes dispersion and damping.

Another one-dimensional problem soluble in a closed form is the propagation of a plane wave in the elastic semi-space $x_1 \geqq 0$, due to a sudden application of temperature of the plane $x_1 = 0$ bounding the semi-space. At the instant $t = 0$, the temperature was applied and held at this value. Under the action of the thermal shock $\theta(0, t) = \theta_0 H(t)$ a magneto-thermoelastic wave is propagated in the medium, depending on two variables x_1 and t. The problem under consideration was discussed in two papers by S. KALISKI and W. NOWACKI [8, 9]—in the former for a perfect conductor and in the latter for a real conductor. To derive results sufficiently simple to be examined analytically, the coupling between the deformation and temperature fields was disregarded. Finally, it was assumed that the boundary $x_1 = 0$ is free of tractions. The closed expression for the displacement $u_1(x_1, t)$, the stress $\sigma_{11}(x_1, t)$, and the field component $h_3(x_1, t)$ were deduced. It turns out that the stress $\sigma_{11}(x_1, t)$ consists of two parts, first having the nature of an elastic wave moving with a phase velocity $c_0 = c_1(1+\alpha)^{1/2}$, and the second of a diffusional nature. When the elastic wave passes through the plane $x_1 = \text{const}$ at the instant $t = x_1/c_0$, there appears a jump in the stress of constant value $\gamma\theta_0/\vartheta(1+\beta)$, where $\gamma = (3\lambda+2\mu)\,\alpha_t$; here, λ, μ are the Lamé constants in the adiabatic case and α_t is the coefficient of linear thermal expansion $\vartheta = c_0^2/c_1$ and $\beta = n\beta_0/\alpha$, where

$$\beta_0 = \frac{\kappa H_3}{4\pi\varrho c_0^2}, \qquad n = \frac{\mu_0 c_0^3 H_3}{\kappa c^2}.$$

The modified electromagnetic wave is propagated with the same velocity as the modified elastic wave, and in the plane $x_1 = \text{const}$ at the instant $t = x_1/c_0$ has a discontinuity. There is a wave h_0 moving with the light velocity c radiated out into the vacuum. If we assume that the initial magnetic field $\mathbf{H} = (0, 0, H_3)$ vanishes, then the results become those obtained by J. V. DANILOVSKAYA in the theory of thermal stresses.

4.2. Two-dimensional problem of magneto-thermoelasticity. In this field, only particular cases were considered, concerning the propagation of a cylindrical wave in a perfect and real conductor (W. NOWACKI), [13, 14, 15]. If the plane in which the excitation is propagated is the plane x_1, x_2, then the production of cylindrical waves is possible only when the initial magnetic field has the direction of the x_3—axis. Assuming that $\mathbf{H} = (0, 0, H_3)$, we arrive at the following system of equations of magneto-thermoelasticity for a perfect conductor:

$$\mu\nabla_1^2 u_\alpha + (\lambda+\mu+a_0^2\varrho)\,u_{\beta,\beta\alpha} + X_\alpha = \varrho\ddot{u}_\alpha + \gamma\theta_{,\alpha}, \qquad \alpha = 1,2 \tag{4.2}$$

$$\left(\nabla^2 - \frac{1}{\kappa}\partial_t\right)\theta - \eta\partial_t \operatorname{div}\dot{\mathbf{u}} = -\frac{Q}{\kappa}, \tag{4.3}$$

where $a_0^2 = H_3^2\mu_0/4\pi\varrho$, $\quad \nabla_1^2 = \partial_1^2 + \partial_2^2$.

The Eqs. (4.2) are the displacement equations in which the influence of the electromagnetic field is expressed by the term $a_0^2\,\varrho$. The Eq. (4.3) is the heat conduction equation. When $a_0 = 0$, the Eqs. (4.2) and (4.3) become the equations of thermoelasticity.

Decomposing the displacement field and the body forces into the potential and solenoidal parts

$$\begin{aligned} u_1 &= \partial_1\Phi - \partial_2\Psi, & u_2 &= \partial_2\Phi + \partial_1\Psi, \\ X_1 &= \varrho(\partial_1\vartheta - \partial_2\chi), & X_2 &= \varrho(\partial_2\vartheta + \partial_1\chi), \end{aligned} \tag{4.4}$$

and eliminating the temperature θ, we obtain the following wave equations:

$$(\Box_1^2 \Box_3^2 - \eta m_0 \partial_t \nabla_1^2)\Phi = \frac{m_0 Q}{\kappa} - \frac{1}{c_0^2} \Box_3^2 \vartheta, \tag{4.5}$$

$$\Box_2^2 \Psi = -\frac{1}{c_2^2} \chi. \tag{4.6}$$

We have introduced here the notations:

$$\Box_1^2 = \nabla_1^2 - \frac{1}{c_0^2}\partial_t^2, \quad \Box_2^2 = \nabla_1^2 - \frac{1}{c_2^2}\partial_t^2, \quad \Box_3^2 = \nabla_1^2 - \frac{1}{\kappa}\partial_t,$$

$$c_0^2 = c_1(1+\alpha), \quad \alpha = \frac{a_0^2}{c_1^2}.$$

The Eq. (4.5) represents the longitudinal wave undergoing a dispersion and damping. In the infinite space, the factors producing the longitudinal wave are the heat sources and body forces of the form $X_\alpha = \varrho \vartheta_{,\alpha}$, $\alpha = 1, 2$.

The transverse waves are produced in the infinite space by the body forces $X_1 = -\varrho \partial_2 \chi$, $X_2 = \varrho \partial_1 \chi$. They are not dispersed or damped and are propagated with acoustic valocity c_2. These waves in the infinite space are not accompanied by temperature field. However, there exists an electromagnetic field, for

$$\mathbf{E} = \frac{\mu_0 H_3}{c}(-\partial_1 \dot{\psi}, \partial_2 \dot{\psi}, 0) \quad \mathbf{h} = 0, \quad \mathbf{j} = 0.$$

The system of Eqs. (4.2), (4.3) can also be uncoupled by using three functions: the vector $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, 0)$ and the scalar ξ.

This method constitutes a generalization of the method applied by M. Iacovache to the problems of dynamic elasticity and the B. G. Galerkin method in static elasticity. This paper contains two particular problems: the action of a linear heat source $Q(r, t) = Q_0 \, \delta(r)/2\pi r \, e^{-i\omega t}$, and a linear center of pressure $\vartheta_0 = \vartheta_0 \, \delta(r) \,/2\pi r e^{-i\omega t}$.

There also a two-dimensional problem was examined for a medium with finite conductivity. The system of displacement equations

$$\begin{aligned} &\mu \nabla_1^2 u_1 + (\lambda+\mu)\partial_1 e + X_1 - \gamma \partial_1 \theta - \frac{\mu_0 H_3}{4\pi}\partial_1 h_3 = \varrho \ddot{u}_1, \\ &\mu \nabla_1^2 u_2 + (\lambda+\mu)\partial_2 e + X_2 - \gamma \partial_2 \theta - \frac{\mu_0 H_3}{4\pi}\partial_2 h_3 = \varrho \ddot{u}_2, \end{aligned} \tag{4.7}$$

and the heat conduction equation

$$\left(\nabla_1^2 - \frac{1}{\kappa}\partial_t\right)\theta - \eta \partial_t e = -\frac{Q}{\kappa}, \quad e = \partial_1 u_1 + \partial_2 u_2, \tag{4.8}$$

are completed by the field equation:

$$\nabla_1^2 h_3 - \beta \partial_t h_3 = -\beta H_3 \partial_t e, \quad \beta = \frac{4\pi \lambda_0 \mu_0}{c^2}. \tag{4.9}$$

The above representations reduce the system (4.7)–(4.9) to the following wave equations:

$$\left(\nabla_1^2 - \frac{1}{c_1^2}\partial_t^2\right)\Phi - m\vartheta - \frac{\mu_0 H_3 h_3}{4\pi\varrho c_1^2} = -\frac{1}{c_1^2}\vartheta, \tag{4.10}$$

$$\left(\nabla_1^2 - \frac{1}{c_2^2}\partial_t^2\right)\Psi = -\frac{1}{c_2^2}\chi, \tag{4.11}$$

$$\left(\nabla_1^2 - \frac{1}{\kappa}\partial_t\right)\theta - \eta\partial_t\nabla^2\Phi = -\frac{Q}{\kappa}, \tag{4.12}$$

$$(\nabla_1^2 - \beta\partial_t)h_3 - \beta H_3\partial_t\nabla_1^2\Phi = 0. \tag{4.13}$$

Here, the Eq. (4.10) represents the longitudinal wave, the Eq. (4.11)—the transverse wave. The Eq. (4.12) is the heat conduction equation and (4.13) the field equation. It is evident that in an elastic space, the equation of the longitudinal wave is independent of the other equations. Eliminating h_3 and θ from the above system, we arrive at a complex equation for the longitudinal wave:

$$D\Box_1^2\Box_3^2 - \frac{1}{\kappa}\partial_t\nabla_1^2(D\varepsilon_T + \varepsilon_H\Box_3^2)\Phi = -\frac{m}{\kappa}DQ - \frac{1}{c_1^2}D\Box_3^2\vartheta, \tag{4.14}$$

where $D = \nabla_1^2 - \beta\partial_t, \quad \varepsilon_T = \eta\kappa m, \quad \varepsilon_H = \alpha\beta\kappa, \quad \alpha = \dfrac{a_0^2}{c_1^2}.$

In this equation, the coefficient ε_T describes the coupling between the deformation and temperature fields, while the coefficient ε_H couples the deformation and electromagnetic fields. The longitudinal waves undergo dispersion and damping.

Well known simplifications are obtained by assuming an adiabatic process (for $Q = 0$), which leads to the wave equation:

$$(D\Box_1^2 - \alpha\beta\partial_t\nabla_1^2)\Phi = -\frac{1}{c_1^2}D\vartheta, \tag{4.15}$$

the structure of which resembles that of the wave equation in the coupled thermoelasticity. The Eq. (4.14) is considerably simplified if $\alpha = a_0^2/c_1^2 \ll 1$—i.e., when the initial magnetic field $\mathbf{H} = (0, 0, H_3)$ is small. In this case, the perturbation method may be of considerable use.

4.3. General theorems of magneto-thermoelasticity. One of the most interesting theorems of the theory of elasticity is the Betti recipirocty theorem. This is a very general theorem, and offers the possibility of introducing methods of solving the equations of the elasticity theory by means of the Green function.

The reciprocity theorem was extended by V. Casimir-Jonescu to the problems of coupled thermoelasticity. The generalization of the reciprocity theorem to the problems of magnetothermoelasticity was given by S. Kaliski and W. Nowacki in the three papers [16, 17, 18]. In the first of these the theorem was examined for a perfect isotropic conductor, in the second for a real isotropic conductor, and in the third for a real anisotropic conductor. Assuming that the motion of the body begins at the instant $t = 0$, and that the initial conditions are homogeneous, we obtain from the constitutive relations the identity:

$$\bar{\sigma}_{ij}\bar{\varepsilon}'_{ij} - \bar{\sigma}'_{ij}\bar{\varepsilon}_{ij} + \gamma(\bar{\theta}\bar{e}' - \bar{\theta}'\bar{e}) = 0. \tag{4.16}$$

Here $\bar{\sigma}_{ij}$, $\bar{\varepsilon}_{ij}$, etc. denote the Laplace transforms of the functions σ_{ij}, ε_{ij}, etc., Integrating (4.16) over the volume of the body, and applying the Gauss transformations we have:

$$(4.17)\qquad \int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\, dV + \int_A (\bar{p}_i \bar{u}'_i - \bar{p}_i \bar{u}_i)\, dA + \gamma \int_V (\bar{\theta}\bar{e}' - \bar{\theta}'\bar{e})\, dV = \int_V (\bar{T}_{ij}\bar{\varepsilon}'_{ij} - \bar{T}'_{ij}\bar{\varepsilon}_{ij})\, dV.$$

Here, $T_{ij} = \mu_0/4\pi(h_i H_j + H_i h_j - \delta_{ij}(h_k H_k))$ are the components of the Maxwell tensor and $p_i = \sigma_{ij} n_j$, is the traction on the surface A. From the heat conduction equation written down for the two states, we obtain:

$$(4.18)\qquad \int_v (\bar{e}\bar{\theta}' - \bar{e}'\bar{\theta})\, dV = \frac{1}{\kappa\eta p} \int_v (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\, dV + \frac{1}{\eta p} \int_A (\bar{\theta}'\bar{\theta}_{,n} - \bar{\theta}\bar{\theta}'_{,n})\, dV,$$

where p is the Laplace transform parameter.

Eliminating the common terms from (4.17) and (4.18), we obtain for anisotropic body the following form of the reciprocity theorem:

$$(4.19)\qquad \kappa\eta p \Big[\int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\, dV + \int_A (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\, dA\Big] + \gamma \int_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\, dV + \gamma\kappa \int_A (\bar{\theta}'\bar{\theta}_{,n} - \bar{\theta}\bar{\theta}'_{,n})\, dA = \eta\kappa p \int_V (\bar{T}_{ij}\bar{\varepsilon}'_{ij} - \bar{T}'_{ij}\bar{\varepsilon}_{ij})\, dV.$$

For an unbounded body V, the Eq. (4.19) is considerably simplified. The integrals drop out also in a bounded body, in the case of homogeneous boundary conditions. It can easily be proved that then the Eq. (4.19) is decomposed into two parts:

$$(4.20)\qquad \begin{aligned} &\eta\kappa p \int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\, dV + \gamma \int_V (\bar{Q}\bar{\theta}' - \bar{\theta}\bar{\theta}')\, dV = 0, \\ &p \int_V (\bar{T}_{ij}\bar{\varepsilon}'_{ij} - \bar{T}'_{ij}\bar{\varepsilon}_{ij})\, dV = 0. \end{aligned}$$

The validity of this decomposition follows from the symmetry of the system of the displacement equations and, hence, the symmetry of the Green function.

Inverting the Laplace transform in the Eqs. (4.20), we arrive at the final form of the reciprocity theorem for a perfect isotropic conductor:

$$(4.21)\qquad \eta\kappa\Big\{\int_0^t d\tau \int_V \Big[X(x, t-\tau)\frac{\partial u'_i(x,\tau)}{\partial\tau} - X'(x,\tau)\frac{\partial u_i(x, t-\tau)}{\partial\tau}\Big] dV + \gamma \int_0^t d\tau \int_V [Q(x,\tau)\theta'(x, t-\tau) - \theta'(x,\tau)\, Q'(x, t-\tau)]\, dV\Big\} = 0,$$

$$(4.22)\qquad \int_0^t d\tau \int_V \Big[T_{ij}(x,\tau)\frac{\partial\varepsilon'_{ij}(x, t-\tau)}{\partial\tau} - T'_{ij}(x, t-\tau)\frac{\partial\varepsilon_{ij}(x,\tau)}{\partial\tau}\Big] dV = 0.$$

On the basis of the above theorem, a number of practical formulae can be deduced, possessing many applications. In particular, we can obtain an extension of the Somigliana theorem of the problems of magneto-thermoelasticity; this theorem can be used to construct integral equations for certain boundary value problems.

We shall not dwell here on reciprocity theorems for a medium with a finite electric conductivity, isotropic and anisotropic. Of course, the formulae here will be much more complicated. Finally, variational theorems were deduced for perfect and real conductors, in which the displacement, entropy and field underwent a variation [20, 21].

These theorems make it possible to deduce the fundamental energy theorems, which can be used to prove the uniqueness of the solution of the differential equations of thermo-magnetoelasticity.

4.4. Thermo-electroelasticity in piezoelectrics. In conclusion, a few words about the general theorems of thermo-electroelasticity in piezoelectrics.

The general constitutive equation and the fundamental energy theorems were given by R. D. Mindlin [6]. The variational principle and the reciprocity theorem was presented by W. Nowacki in [19]. The reciprocity theorem for the wave equations of piezoelectricity is contained in the paper by S. Kaliski [22]. Similar theorems for the wave equations of thermo-magnetoelasticity were given by S. Kaliski in [23].

5. Remarks on the Problems of Thermo-electro-magnetoelasticity of More Complicated Media

The problems considered in the preceding four sections concerned only problems of thermo-magnetoelasticity and thermo-electroelasticity for the simplest cases—i.e., in the first case for simple media without spin in which the coupling with the electromagnetic field appears only through the Lorentz force in the equations of motion, and the coupling due to the internal mechanisms is present only for the temperature field. This excludes all ferromedia—i.e., media is a spin. Omitting discussion of the transport phenomena of heat, for media with the spin the equations were derived in [32]. The heat problems are here very complicated not only as a result of the conductivity and the couplings but also in view of the production of heat in the field processes due to the effects of nonlinear irreversibility (the hysteresis loop); this fact, similarly to the thermo-magnetoplasticity, introduces a very serious difficulty. Not only are there no present solutions to these problems, but we lack sufficiently precise formulations (in the media with a spin the boundary conditions question is a problem in itself)(1). A similar, though somewhat simpler situation occurs in the case of the second group of problems (ferrodielectrics, etc.).

Another group of problems omitted in this paper are the problems of semiconductors and semimetals in which, in view of their present importance and engineering applicability, a phenomenological construction of the theory of coupled thermo-electro-magnetoelastic fields is of fundamental engineering importance, while the complexity of the equations and couplings contains a variety of physical phenomena. In order to realize the importance of these investigations for measurement technique in electronics, ultrasonics and in the

(1) A general paper about the equations of electro-magneto-elastic-spin field theory with correct boundary conditions is in preparation to print by S. Kaliski.

methods of investigation of physics of solids, suffice it to mention by way of example the possibility of amplification of thermal, acoustic and electric impulses in piezosemiconductors and semimetals, and couplings of these fields with the spin waves in ferrobodies, and further a number of phenomena connected with thermo-electromagnetic waves in low temperatures [28, 29] or a number of magnetocaloric phenomena. Recent applications of laser radiation to the investigation of the propagation of shock and heat waves, propagation of microcracks, problems of conversion of heat, mechanical and electromagnetic energies in these processes (see e.g. [25]) constitute further examples of and suggest perspectives in the theory of coupled thermo-electro-magnetoelastic fields. A separate problem is comprised of the microquantum solutions for lattice dynamics, taking into account the effects of coupled fields, problems of dynamics of dislocations in piezoferromedia, etc.

The scope of this paprer permits mention of only the most impotant problems.

Bearing in mind further scientific possibilities, it seems that the following several trends should be mentioned.

1. Mathematical solutions in the field theory of linear equations of thermo-magneto and thermo-electroelasticity, and practical applications.

2. Investigations of nonlinear equations (including plastic effects).

3. Constructions of the theory of thermo-electro-magnetoelasticity for media with spin (in general, ferrobodies).

4. Investigations of thermo-electro-magnetoelastic fields in semiconductors and semimetals.

5. Microscopic invetsigations (lattice dynamics, dislocations, etc. of the above phenomena in coupled fields).

6. All engineering problems, making use of the possibilities created by the application of laser radiation to the investigation of physical media.

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