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THE AXIALLY SYMMETRICAL LAMB'S PROBLEM IN A SEMI-INFINITE MICROPOLAR ELASTIC SOLID

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1. Introduction

The subject of the present work is an axially symmetrical problem of wave propagation in an elastic micropolar semi-space on the boundary of which there is given a time varying loading axially symmetrical on the semi-space surface. In the classical elastokinetics, this is called the axially symmetrical Lamb's problem. On the ground of the non-symmetrical theory of elasticity, this problem is more complicated, since, in this case, the deformation of a body is described by two independent vectors—namely, the displacement vector $\mathbf{u}$, and the rotation vector $\boldsymbol{\omega}$. Because the semi-space boundary $z = 0$ is assumed to be axially symmetrically loaded, the investigation will be performed in the cylindrical coordinates (r, φ, z). It will be seen that in our modified axially symmetrical Lamb's problem, the loading of the boundary $z = 0$ can be divided into two groups. The first produces the displacement $\mathbf{u} = (u_r, 0, u_z)$ and the rotation $\boldsymbol{\omega} = (0, \omega_\varphi, 0)$, the second causes the displacement $\mathbf{u} = (0, u_\varphi, 0)$ and the rotation $\boldsymbol{\omega} = (\omega_r, 0, \omega_z)$. The integral Fourier-Hankel tranformation has been used to solve this problem.

2. Fundamental Equations

We shall consider an elastic isotropic homogeneous and centro-symmetrical medium. External loadings produce, in the medium, the field of displacement $\mathbf{u}(\mathbf{x}, t)$ and the rotation field $\boldsymbol{\omega}(\mathbf{x}, t)$ depending on the position $\mathbf{x}$ and time t.

The strain state is determined by two non-symmetrical tensors: the deformation tensor γ_{ji} and the curvature-twist tensor $\varkappa_{ji}$. These tensors are defined as follows [1-3]:

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\omega_k, \quad \varkappa_{ji} = u_{i,j}. \tag{2.1}$$

The stress state is determined by two non-symmetrical tensors—namely, the force-stress tensor $\boldsymbol{\sigma}$ and the couple-stress tensor $\boldsymbol{\mu}$. The relations between the stress and strain states are linear and expressed as

$$\begin{aligned} \sigma_{ji} &= (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ji}, \\ \mu_{ji} &= (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ji}. \end{aligned} \tag{2.2}$$

Here α, μ, β, γ, ε, λ are the material constants, ϵ_{kji} is a unit quasi-tensor.

By substituting (2.2) into the equations of motion

$$\sigma_{ji,j}+X_i = \varrho\ddot{u}_i, \quad \in_{ijk}\sigma_{jk}+\mu_{ji,j}+Y_i = J\ddot{\omega}_i, \tag{2.3}$$

and by expressing γ_{ji} and $\varkappa_{ji}$ in terms of the displacements u_i and rotations ω_i determined from (2.1), we obtain the following set of six equations which can be written in the vector from in the following way

$$\begin{aligned}(\mu+\alpha)\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\omega}+\mathbf{X} &= \varrho\ddot{\mathbf{u}},\\ (\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}+(\gamma+\beta-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega}-4\alpha\boldsymbol{\omega}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y} &= J\ddot{\boldsymbol{\omega}},\end{aligned} \tag{2.4}$$

where $\mathbf{X}$—vector of body forces, $\mathbf{Y}$—vector of body couples, ϱ—density, J—rotational inertia.

Equations (2.4) are coupled one to another. They can be decoupled by assuming $\alpha = 0$. In this case, we obtain:

$$\begin{aligned}\mu\nabla^2\mathbf{u}+(\mu+\lambda)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\mathbf{X} &= \varrho\ddot{\mathbf{u}},\\ (\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}+(\gamma+\beta-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega}+\mathbf{Y} &= J\ddot{\boldsymbol{\omega}}.\end{aligned} \tag{2.5}$$

The first of Eqs. (2.5) is an equation of the classical theory of elasticity. The second refers to a hypothetical medium in which only rotations are possible.

We have assumed that the loadings are axially symmetrical and, therefore, the investigation will be performed using the cylindrical coordinates (r, φ, z). After these coordinates have been introduced, the set of Eqs. (2.4) takes the following form:

$$(\mu+\alpha)\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}\right)+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}+2\alpha\left[\frac{1}{r}\frac{\partial\omega_z}{\partial\varphi}-\frac{\partial}{\partial z}(r\omega_\varphi)\right]+X_r = \varrho\ddot{u}_r$$

$$(\mu+\alpha)\left(\nabla^2 u_\varphi-\frac{u_\varphi}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\varphi}\right)+(\lambda+\mu-\alpha)\frac{1}{r}\frac{\partial e}{\partial\varphi}+2\alpha\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right)+X_\varphi = \varrho\ddot{u}_\varphi,$$

$$(\mu+\alpha)\nabla^2 u_z+(\lambda+\mu-\alpha)\frac{\partial e}{\partial z}+2\alpha\frac{1}{r}\left[\frac{\partial}{\partial r}(r\omega_\varphi)-\frac{\partial\omega_r}{\partial\varphi}\right]+X_z = \varrho\ddot{u}_z, \tag{2.6}$$

$$(\gamma+\varepsilon)\left((\nabla^2\omega_r-\frac{\omega_r}{r^2}-\frac{2}{r^2}\frac{\partial\omega_\varphi}{\partial\varphi}\right)-4\alpha\omega_r+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r}+2\alpha\left(\frac{1}{r}\frac{\partial u_z}{\partial\varphi}-\frac{\partial u_\varphi}{\partial z}\right)+Y_r = J\ddot{\omega}_r,$$

$$(\gamma+\varepsilon)\left(\nabla^2\omega_\varphi-\frac{\omega_\varphi}{r^2}+\frac{2}{r^2}\frac{\partial\omega_r}{\partial\varphi}\right)-4\alpha\omega_\varphi+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r\varphi}+2\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)+Y_\varphi = J\ddot{\omega}_\varphi,$$

$$(\gamma+\varepsilon)\nabla^2\omega_z-4\alpha\omega_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+2\alpha\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\varphi)-\frac{\partial u_r}{\partial\varphi}\right]+Y_z = J\ddot{\omega}_z,$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial u_z}{\partial z}, \quad \varkappa = \frac{1}{r}\frac{\partial}{\partial r}(r\omega_r)+\frac{1}{r}\frac{\partial\omega_\varphi}{\partial\varphi}+\frac{\partial\omega_z}{\partial z}.$$

We shall consider a particular case in which external loadings, body forces and body couples, as well as the vectors of displacement $\mathbf{u}$ and of rotation $\boldsymbol{\omega}$ depend only on the coordinates r, z. In this case, the set of Eqs. (2.6) is decomposed into two mutually independent sets of equations:

$$(2.7)\qquad \begin{aligned} &(\mu+\alpha)\left(\nabla^2 u_r-\frac{u_r}{r^2}\right)+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}-2\alpha\frac{\partial\omega_\varphi}{\partial z}=\varrho\ddot{u}_r,\\ &(\mu+\alpha)\nabla^2 u_z+(\lambda+\mu-\alpha)\frac{\partial e}{\partial z}+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\varphi)=\varrho\ddot{u}_z,\\ &(\gamma+\varepsilon)\left(\nabla^2\omega_\varphi-\frac{\omega_\varphi}{r^2}\right)-4\alpha\omega_\varphi+2\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)=J\ddot{\omega}_\varphi,\end{aligned}$$

and

$$(2.8)\qquad \begin{aligned} &(\mu+\alpha)\left(\nabla^2 u_\varphi-\frac{u_\varphi}{r^2}\right)+2\alpha\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right)=\varrho\ddot{u}_\varphi,\\ &(\gamma+\varepsilon)\left(\nabla^2\omega_r-\frac{\omega_r}{r^2}\right)-4\alpha\omega_r+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r}-2\alpha\frac{\partial u_\varphi}{\partial z}=J\ddot{\omega}_r,\\ &(\gamma+\varepsilon)\nabla^2\omega_z-4\alpha\omega_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(ru_\varphi)=J\ddot{\omega}_z,\end{aligned}$$

where

$$e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z},\qquad \varkappa=\frac{1}{r}\frac{\partial}{\partial r}(r\omega_r)+\frac{\partial\omega_z}{\partial z},\qquad \nabla^2=\frac{\partial}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

In Eqs. (2.7) and (2.8) we have disregarded the body forces and body couples.

The sets of Eqs. (2.7) and (2.8) will be considered separately. First, we shall investigate the set of Eqs. (2.7). To the displacement vector $\mathbf{u}=(u_r, 0, u_z)$ and to the rotation vector $\boldsymbol{\omega}=(0, \omega_\varphi, 0)$, is ascribed the following state of force stresses and couple stresses

$$(2.9)\qquad \boldsymbol{\sigma}=\begin{vmatrix}\sigma_{rr} & 0 & \sigma_{rz}\\ 0 & \sigma_{\varphi\varphi} & 0\\ \sigma_{zr} & 0 & \sigma_{zz}\end{vmatrix},\qquad \boldsymbol{\mu}=\begin{vmatrix}0 & \mu_{r\varphi} & 0\\ \mu_{\varphi r} & 0 & \mu_{\varphi z}\\ 0 & \mu_{z\varphi} & 0\end{vmatrix},$$

where the particular components of the stress tensors have, according to (2.2), the following form:

$$(2.10)\qquad \begin{aligned} &\sigma_{rr}=2\mu\frac{\partial u_r}{\partial r}+\lambda e,\quad \sigma_{\varphi\varphi}=2\mu\frac{u_r}{r}+\lambda e,\quad \sigma_{zz}=2\mu\frac{\partial u_z}{\partial z}+\lambda e,\\ &\sigma_{rz}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)+2\alpha\omega_\varphi,\\ &\sigma_{zr}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)-\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-2\alpha\omega_\varphi,\\ &\mu_{r\varphi}=\gamma\left(\frac{\partial\omega_\varphi}{\partial r}-\frac{\omega_\varphi}{r}\right)+\varepsilon\left(\frac{\partial\omega_\varphi}{\partial r}+\frac{\omega_\varphi}{r}\right),\\ &\mu_{\varphi r}=\gamma\left(\frac{\partial\omega_\varphi}{\partial r}-\frac{\omega_\varphi}{r}\right)-\varepsilon\left(\frac{\partial\omega_\varphi}{\partial r}+\frac{\omega_\varphi}{r}\right),\\ &\mu_{\varphi z}=(\gamma-\varepsilon)\frac{\partial\omega_\varphi}{\partial z},\quad \mu_{z\varphi}=(\gamma+\varepsilon)\frac{\partial\omega_\varphi}{\partial z}.\end{aligned}$$

With the set of Eqs. (2.8), there is related the displacement field $\mathbf{u} = (0, u_\varphi, 0)$ and the rotation field $\boldsymbol{\omega} = (\omega_r, 0, \omega_z)$.

The following state of force stresses and couple stresses

$$(2.11)\qquad \boldsymbol{\sigma} = \begin{vmatrix} 0 & \sigma_{r\varphi} & 0 \\ \sigma_{\varphi r} & 0 & \sigma_{\varphi z} \\ 0 & \sigma_{z\varphi} & 0 \end{vmatrix}, \quad \boldsymbol{\mu} = \begin{vmatrix} \mu_{rr} & 0 & \mu_{rz} \\ 0 & \mu_{\varphi\varphi} & 0 \\ \mu_{zr} & 0 & \mu_{zz} \end{vmatrix},$$

is ascribed to this field. In (2.11), the following denotations are introduced:

$$\sigma_{r\varphi} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + \frac{\alpha}{r}\left[\frac{\partial}{\partial r}(u_\varphi r) - \frac{\partial u_r}{\partial\varphi}\right] - 2\alpha\omega_z,$$

$$\sigma_{\varphi r} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) - \frac{\alpha}{r}\left[\frac{\partial}{\partial r}(u_\varphi r) - \frac{\partial u_r}{\partial\varphi}\right] + 2\alpha\omega_z,$$

$$\sigma_{\varphi z} = \mu\left(\frac{\partial u_\varphi}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\varphi}\right) + \frac{\alpha}{r}\left[\frac{\partial u_z}{\partial\varphi} - \frac{\partial}{\partial z}(ru_\varphi)\right] - 2\alpha\omega_r,$$

$$(2.12)\qquad \sigma_{z\varphi} = \mu\left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial u_z}{r\partial\varphi}\right) - \frac{\alpha}{r}\left[\frac{\partial u_z}{\partial\varphi} - \frac{\partial}{\partial z}(ru_\varphi)\right] + 2\alpha\omega_r,$$

$$\mu_{rr} = 2\gamma\frac{\partial\omega_r}{\partial r} + \beta\varkappa, \quad \mu_{\varphi\varphi} = 2\gamma\frac{1}{r}\left(\frac{\partial\omega_\varphi}{\partial\varphi} + \omega_r\right) + \beta\varkappa, \quad \mu_{zz} = 2\gamma\frac{\partial\omega_z}{\partial z} + \beta\varkappa,$$

$$\mu_{rz} = \gamma\left(\frac{\partial\omega_r}{\partial z} + \frac{\partial\omega_z}{\partial r}\right) - \varepsilon\left(\frac{\partial\omega_r}{\partial z} - \frac{\partial\omega_z}{\partial r}\right),$$

$$\mu_{zr} = \gamma\left(\frac{\partial\omega_r}{\partial z} + \frac{\partial\omega_z}{\partial r}\right) + \varepsilon\left(\frac{\partial\omega_z}{\partial r} - \frac{\partial\omega_r}{\partial z}\right).$$

The components of the force-stress tensor and the couple-stress tensor described by the formulae (2.11) and (2.12) are functions of variables r, z and of time t. Now we proceed to solving the set of Eqs. (2.17) and then to the set of Eqs. (2.8).

3. General Solution to the Set of Eqs. (2.7)

The following mutually independent functions u_r, u_z, ω_φ are involved in the set of Eqs. (2.7). They arise in the semi-space under the action, on its boundary $z = 0$, of forces and moments: the loading normal and tangent to the boundary and the moment with a vector tangent to a circle of radius r (Fig. 1). The boundary conditions will be written in the following form:

$$(3.1)\qquad \sigma_{zz}(r, 0, t) = -f_1(r, t), \quad \sigma_{zr}(r, 0, t) = -f_2(r, t), \quad \mu_{z\varphi}(r, 0, t) = -f_3(r, t),$$

where $f_1 > 0$ represents the normal loading directed along the positive axis z, $f_2 > 0$ is a tangent loading lying in the plane $z = 0$ and $f_3 > 0$ is a moment with a vector tangent to a circle of radius r and lying in the plane $z = 0$.

We shall now introduce elastic potentials Φ, W and express the displacements u_r, u_z in terms of these potentials

$$(3.2)\qquad u_r = \frac{\partial\Phi}{\partial r} - \frac{\partial W}{\partial z}, \quad u_z = \frac{\partial\Phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rW).$$

We shall introduce, moreover, the functions Ψ, Γ defined as

$$W = -\frac{\partial \Psi}{\partial r}, \quad \omega_\varphi = -\frac{\partial \Gamma}{\partial r}. \tag{3.3}$$

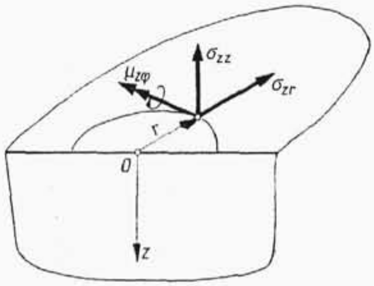

FIG. 1.

After substituting (3.2) and (3.3) into (2.7), we obtain the following set of wave equations:

$$\begin{gathered}\left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right)\Phi = 0,\\ \left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\Psi + p\Gamma = 0,\\ \left(\nabla^2 - \nu_0^2 - \frac{1}{c_4^2}\partial_t^2\right)\Gamma - s\nabla^2\Psi = 0,\end{gathered} \tag{3.4}$$

where the following denotations have been introduced

$$c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad c_2 = \left(\frac{\mu+\alpha}{\varrho}\right)^{1/2}, \quad c_4 = \left(\frac{\gamma+\varepsilon}{J}\right)^{1/2},$$

$$p = \frac{2\alpha}{\mu+\alpha}, \quad s = \frac{2\alpha}{\gamma+\varepsilon}, \quad \nu_0^2 = \frac{4\alpha}{\gamma+\varepsilon}.$$

Equation $(3.4)_1$ represents the propagation of longitudinal waves. Equations $(3.4)_2$ and $(3.4)_3$ are coupled one to another. After eliminating from them first Γ and then Ψ, we arrive at the following equation:

$$\left[\left(\nabla^2 - \nu_0^2 - \frac{1}{c_4^2}\partial_t^2\right)\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right) + \nu_1^2\nabla^2\right](\Psi, \Gamma) = 0, \tag{3.5}$$

where

$$\nu_1^2 = \frac{4\alpha^2}{(\gamma+\varepsilon)(\mu+\alpha)},$$

which describes the propagation of modified transverse waves.

After the Fourier-Hankel transformation has been performed on Eqs. $(3.4)_1$ and (3.5) defined as [4]

$$(3.6)\qquad \begin{aligned}(\tilde{\Phi}, \tilde{\Psi}, \tilde{\Gamma}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\zeta t} dt \int_{0}^{\infty} r \mathscr{J}_0(\eta r)(\Phi, \Psi, \Gamma) dr,\\ (\Phi, \Psi, \Gamma) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_{0}^{\infty} \eta \mathscr{J}_0(\eta r)(\tilde{\Phi}, \tilde{\Psi}, \tilde{\Gamma}) d\eta,\end{aligned}$$

where $\tilde{\Phi}$, $\tilde{\Psi}$, $\tilde{\Gamma}$ are the functions of z, η, ξ, we obtain the following set of ordinary differential equations with respect to the variable z

$$(3.7)\qquad \begin{aligned}[\partial_z^2 - (\eta^2 - \sigma_1^2)]\tilde{\Phi} &= 0,\\ (\partial_z^2 - \lambda_1^2)(\partial_z^2 - \lambda_2^2)(\tilde{\Psi}, \tilde{\Gamma}) &= 0,\end{aligned}$$

where

$$\begin{aligned}(\lambda_1^2 - \eta^2) + (\lambda_2^2 - \eta^2) &= \nu_0^2 - \sigma_2^2 - \sigma_4^2,\\ (\lambda_1^2 - \eta^2)(\lambda_2^2 - \eta^2) &= \sigma_2^2(\sigma_4^2 - \nu_0^2),\\ \sigma_1^2 = \frac{\zeta}{c_1}, \quad \sigma_2 = \frac{\zeta}{c_2}, &\quad \sigma_4 = \frac{\zeta}{c_4}.\end{aligned}$$

Similarly as in the classical Lamb's problem, we assume that the loading acting on the boundary is bounded. We assume, moreover, that the functions Φ, Ψ, Γ should tend to zero for $R = |r^2 + z^2|^{1/2} \to \infty$. In this connection, we seek solutions to Eqs. (3.4) in the form:

$$(3.8)\qquad \begin{aligned}\tilde{\Phi} &= Ae^{-\sigma z}, \quad \sigma = (\eta^2 - \sigma_1^2),^{1/2}\\ \tilde{\Psi} &= Be^{-\lambda_1 z} + Ce^{-\lambda_2 z},\\ \tilde{\Gamma} &= B_1 e^{-\lambda_1 z} + C_1 e^{-\lambda_2 z}.\end{aligned}$$

The quantities B_1 and C_1 are related with the quantities B and C by means of Eqs. $(3.4)_2$ and $(3.4)_3$. They can be determined, for example, from Eq. $(3.4)_2$ as

$$(3.9)\qquad B_1 = \varkappa_1 B, \quad C_1 = \varkappa_2 C,$$

where

$$\varkappa_i = \frac{1}{p}(\eta^2 - \lambda_i^2 - \sigma_2^2), \quad i = 1, 2.$$

The quantities A, B, C involved in Eq. (3.8) will be determined from the boundary conditions (3.1) which can be written, on the basis of (2.10), in the form:

$$(3.10)\qquad \begin{aligned}\left|2\mu \frac{\partial u_z}{\partial z} + \lambda e\right|_{z=0} &= -f_1(r, t),\\ \left|(\mu+\alpha)\frac{\partial u_r}{\partial z} + (\mu-\alpha)\frac{\partial u_z}{\partial r} - 2\alpha\omega_\varphi\right|_{z=0} &= -f_2(r, t),\\ \left|(\gamma+\varepsilon)\frac{\partial \omega_\varphi}{\partial z}\right|_{z=0} &= -f_3(r, t).\end{aligned}$$

We express the displacements u_r, u_z in Eqs. (3.10) in terms of the potentials Φ, Ψ, and perform the Fourier-Hankel transformation on these equations. Thus, we obtain the set of nonhomogeneous linear equations:

$$(3.11)\quad \begin{aligned} [2\mu\sigma^2+\lambda(\sigma^2-\eta^2)]A-2\mu\eta^2\lambda_1 B-2\mu\eta^2\lambda_2 C &= -\tilde{f}_1(\eta,\zeta),\\ 2\mu\sigma\eta A-\eta a_1 B-\eta a_2 C &= -\tilde{f}_2(\eta,\zeta),\\ \lambda_1\eta(\gamma+\varepsilon)\varkappa_1 B+\lambda_2\eta(\gamma+\varepsilon)\varkappa_2 C &= \tilde{f}_3(\eta,\zeta), \end{aligned}$$

from which we can determine the quantities A, B, C:

$$(3.12)\quad \begin{aligned} A &= \alpha_{11}\tilde{f}_1+\alpha_{12}\tilde{f}_2+\alpha_{13}\tilde{f}_3,\\ B &= \alpha_{21}\tilde{f}_1+\alpha_{22}\tilde{f}_2+\alpha_{23}\tilde{f}_3,\\ C &= \alpha_{31}\tilde{f}_1+\alpha_{32}\tilde{f}_2+\alpha_{31}\tilde{f}_3, \end{aligned}$$

where

$$(3.13)\quad \begin{aligned} &\alpha_{11}=\frac{1}{\Delta}(a_2\lambda_1\varkappa_1-a_1\lambda_2\varkappa_2), \qquad \alpha_{12}=\frac{2}{\Delta}\mu\eta\lambda_1\lambda_2(\varkappa_2-\varkappa_1),\\ &\alpha_{13}=-\frac{2\mu\eta}{(\gamma+\varepsilon)\Delta}(\lambda_1 a_2-a_1\lambda_2), \qquad \alpha_{21}=-\frac{2}{\Delta}\mu\sigma\lambda_2\varkappa_2,\\ &\alpha_{22}=\frac{1}{\eta\Delta}\lambda_2\varkappa_2[2\mu\sigma^2+\lambda(\sigma^2-\eta^2)],\\ &\alpha_{23}=-\frac{1}{\eta\Delta(\gamma+\varepsilon)}\{[2\mu\sigma^2+\lambda(\sigma^2-\eta^2)]a_2-4\mu^2\eta^2\sigma\lambda_2\},\\ &\alpha_{31}=\frac{2}{\Delta}\mu\sigma\lambda_1\varkappa_1, \qquad \alpha_{32}=-\frac{1}{\eta\Delta}\lambda_1\varkappa_1[2\mu\sigma^2+\lambda(\sigma^2-\eta^2)],\\ &\alpha_{33}=\frac{1}{\eta\Delta(\gamma+\varepsilon)}\{[2\mu\sigma^2+\lambda(\sigma^2-\eta^2)]a_1-4\mu^2\eta^2\lambda_1\sigma\}, \end{aligned}$$

in which we have adopted the following denotations:

$$(3.14)\quad \Delta = [2\mu\sigma^2+\lambda(\sigma^2-\eta^2)](\lambda_2\varkappa_2 a_1-\lambda_1\varkappa_1 a_2)-4\mu^2\eta^2\sigma\lambda_1\lambda_2(\varkappa_2-\varkappa_1)$$

and

$$a_i=(\mu+\alpha)\lambda_i^2+\eta^2(\mu-\alpha)+2\alpha\varkappa_i, \qquad i=1,2.$$

Now, we subject Eqs. (3.2) and (3.3) to the Fourier-Henkel transformation. Then we make use of (2.8) and the relations (3.9), and thus obtain:

$$(3.15)\quad \begin{aligned} \tilde{u}_r &= -\eta[Ae^{-\sigma z}+\lambda_1 Be^{-\lambda_1 z}+\lambda_2 Ce^{-\lambda_2 z}],\\ \tilde{u}_z &= -\sigma Ae^{-\sigma z}+\eta^2(Be^{-\lambda_1 z}+Ce^{-\lambda_2 z}),\\ \tilde{\omega}_\varphi &= \eta(B\varkappa_1 e^{-\lambda_1 z}+C\varkappa_2 e^{-\lambda_2 z}). \end{aligned}$$

The quantities A, B, C involved in (3.15) are determined by the formulae (3.11). We perform the inverse Fourier-Hankel transformation on Eqs. (3.15) and thus obtain the expressions for the displacements u_r, u_z and for rotation ω_φ in the form

$$(3.16)\quad \begin{aligned} u_r &= -\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int\limits_0^{\infty}\eta^2[Ae^{-\sigma z}+\lambda_1 Be^{-\lambda_1 z}+\lambda_2 Ce^{-\lambda_2 z}]\mathscr{I}_1(\eta r)\,d\eta,\\ u_z &= -\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int\limits_0^{\infty}\eta^2[\sigma Ae^{-\sigma z}-\eta^2(Be^{-\lambda_1 z}+Ce^{-\lambda_2 z})]\mathscr{I}_0(\eta r)\,d\eta,\\ \omega_\varphi &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int\limits_0^{\infty}\eta^2(B\varkappa_1 e^{-\lambda_1 z}+C\varkappa_2 e^{-\lambda_2 z})\mathscr{I}_1(\eta r)\,d\eta. \end{aligned}$$

We have already determined the displacements and rotations, and therefore we can utilize the formulae (2.1) and (2.2) to find the state of strain and the state of stress in the semi-space.

We shall now consider the particular case $\alpha = 0$. In this case, Eqs. (2.7) can be decoupled and we obtain the following set of equations:

$$\mu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + (\lambda+\mu)\frac{\partial e}{\partial r} = \varrho \ddot{u}_r, \quad \left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right)\Phi = 0,$$

$$\mu\nabla^2 u_z + (\lambda+\mu)\frac{\partial e}{\partial z} = \varrho \ddot{u}_z, \quad \left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\Psi = 0, \tag{3.17}$$

$$(\gamma+\varepsilon)\left(\nabla^2\omega_\varphi - \frac{\omega_\varphi}{r^2}\right) = J\ddot{\omega}_\varphi, \quad \left(\nabla^2 - \frac{1}{c_4^2}\partial_t^2\right)\Gamma = 0,$$

and the wave Eqs. (3.4) assume the form:

$$\left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right)\Phi = 0, \quad \left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\Psi = 0, \quad \left(\nabla^2 - \frac{1}{c_4^2}\partial_t^2\right)\Gamma = 0. \tag{3.18}$$

After the Fourier-Hankel integral transformation has been carried out on the wave Eqs. (3.18), the solution to them is given by the functions:

$$\tilde{\tilde{\Phi}}(z) = A_0 e^{-\sigma z}, \quad \tilde{\tilde{\Psi}}(z) = B_0 e^{-\lambda_0 z}, \quad \tilde{\tilde{\omega}}_3(z) = C_0 e^{-\gamma_0 z}, \tag{3.19}$$

where

$$\sigma = (\eta^2 - \sigma_1^2)^{1/2}, \quad \lambda_0 = (\eta^2 - \hat{\sigma}_2^2)^{1/2}, \quad \gamma_0 = (\eta^2 - \sigma_4^2)^{1/2}, \quad \hat{\sigma}_2 = \frac{\zeta}{\hat{c}_2^2}, \quad \hat{c}_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}.$$

The constants A_0, B_0, C_0 will be determined directly from (3.12) with the following assumptions

$$A_0 = \alpha_{11}^0 \tilde{f}_1 + \alpha_{12}^0 \tilde{f}_2, \quad B_0 = \alpha_{21}^0 \tilde{f}_1 + \alpha_{22}^0 \tilde{f}_2, \quad C_0 = \alpha_{33}^0 \tilde{f}_3, \tag{3.20}$$

where

$$\alpha_{11}^0 = \frac{1}{\Delta_0}(\lambda_0^2+\eta^2)\eta, \quad \alpha_{12}^0 = -\frac{2\mu\eta^2\sigma}{\Delta_0},$$

$$\alpha_{21}^0 = \frac{2}{\Delta_0}\mu\sigma\eta, \quad \alpha_{22}^0 = -\frac{1}{\Delta_0}[(2\mu+\lambda)\sigma^2 - \lambda\eta^2], \quad \alpha_{33}^0 = \frac{1}{\gamma(\gamma+\varepsilon)}, \tag{3.21}$$

$$\Delta_0 = \eta(\lambda_0^2+\eta^2)[(2\mu+\lambda)\sigma^2 - \lambda\eta^2] - 4\mu^2\sigma\eta^3\lambda_0.$$

Finally, the displacements and rotations are expressed by the formulae:

$$u_r = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^\infty \eta[A_0 e^{-\sigma z} + \lambda_0 B_0 e^{-\lambda_0 z}]\mathscr{J}_1(\eta r)\, d\eta,$$

$$u_z = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^\infty \eta[\sigma A_0 e^{-\sigma z} - \eta^2 B_0 e^{-\lambda_0 z}]\mathscr{J}_0(\eta r)\, d\eta, \tag{3.22}$$

$$\omega_\varphi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^\infty \eta C_0 e^{-\gamma_0 z}\mathscr{J}_1(\eta r)\, d\eta.$$

It is obvious from (3.22) and (3.20) that the displacements u_r, u_z can be produced only by the loading on the semi-space boundary $\sigma_{zz}(r, 0, t) = -f_1(r, t)$, $\sigma_{zr}(r, 0, t) = -f_r(r, t)$. But the rotations can be caused only by the moment $\mu_{z\varphi}(r, 0, t) = -f_3(r, t)$ applied to the semi-space surface. The formulae $(3.22)_1$ and $(3.22)_2$ describe displacements in a classical elastic medium, whereas the third of Eqs. (3.22) refers to a hypothetical medium in which only moment rotations and stresses can arise.

Now we proceed to solving the set of Eqs. (2.8).

4. The General Solution to the Set of Eqs. (2.8)

The solutions (2.8) constitute a set of differential equations with respect to mutually independent functions: the displacement u_φ and the rotations ω_r, ω_z. The displacement u_φ and the rotations ω_r, ω_z arise in the semi-space due to the following loadings acting on its boundary $z = 0$: the tangent stress $\sigma_{r\varphi}$ and the moments μ_{rr} and μ_{rz}. The boundary conditions for the set of Eqs. (2.8) have the form:

$$\sigma_{z\varphi}(r, 0, t) = -h_1(r, t), \quad \mu_{zz}(r, 0, t) = -h_2(r, t), \quad \mu_{zr}(r, 0, t) = -h_3(r, t), \tag{4.1}$$

where $h_1 > 0$ is a loading tangent to a circle of radius r on the plane $z = 0$; $h_2 > 0$ is a moment with a vector directed along the positive axis z; $h_3 > 0$ is a moment with a vector lying on the plane $z = 0$, and this vector is directed along the radius r (Fig. 2).

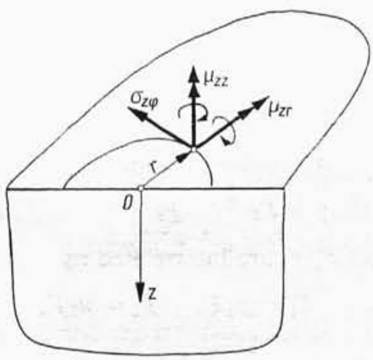

FIG. 2.

Similarly as in Sec. 3, we shall now introduce the elastic potentials Ξ, Π, and use them to express the rotations:

$$\omega_r = \frac{\partial \Xi}{\partial r} - \frac{\partial \Pi}{\partial z}, \quad \omega_z = \frac{\partial \Xi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\Pi). \tag{4.2}$$

Moreover, we shall introduce the functions Λ, Θ expressed as

$$u_\varphi = -\frac{\partial \Lambda}{\partial r}, \quad \Pi = -\frac{\partial \Theta}{\partial r}. \tag{4.3}$$

After substituting (4.2) and (4.3) into (2.8), we obtain the following set of equations

$$\left(\nabla^2 - \nu_2^2 - \frac{1}{c_3^2}\partial_t^2\right)\Xi = 0,$$

$$\left(\nabla^2-\nu_0^2-\frac{1}{c_4^2}\partial_t^2\right)\Theta+s\Lambda=0, \tag{4.4}$$
$$\left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)\Lambda-p\nabla^2\Theta=0,\quad \nu_2^2=\frac{4\alpha}{\beta+2\gamma}.$$

The first of Eqs. (4.4) describes the propagation of a twist wave and the next two are coupled one to another. We find from them first the Λ and then the Θ; we arrive at the equation:

$$\left[\left(\nabla^2-\nu_0^2-\frac{1}{c_4^2}\partial_t^2\right)\left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)+\nu_1^2\nabla^2\right](\Lambda,\Theta)=0, \tag{4.5}$$

which is identical with Eq. (3.5) describing the modified transverse waves.

Similarly as previously, we perform the Fourier-Hankel transformation on Eqs. $(4.4)_1$ and (4.5)

$$(\tilde{\Xi},\tilde{\Lambda},\tilde{\Theta})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\zeta t}dt\int_0^{\infty}r\mathscr{J}_0(\eta r)(\Xi,\Lambda,\Theta)\,dr, \tag{4.6}$$

and obtain in this way the following set of equations:

$$(\partial_z^2-\sigma_0^2)\tilde{\Xi}=0,\quad (\partial_z^2-\lambda_1^2)(\partial_z^2-\lambda_2^2)(\tilde{\Lambda},\tilde{\Theta})=0, \tag{4.7}$$

where

$$\sigma_0=(\eta^2+\nu_2^2-\sigma_3^2)^{1/2},\quad \sigma_3=\frac{\zeta}{c_3},\quad \nu_2^2=\frac{4\alpha}{2\gamma+\beta},$$
$$\lambda_{1,2}^2=-\frac{1}{2}\left(\sigma_2^2+\sigma_4^2-2\eta^2+\nu_0^2-\nu_1^2\mp\sqrt{(\sigma_2^2-\sigma_4^2+\nu_0^2-\nu_1^2)^2+4\nu_1^2\sigma_2^2}\right).$$

The solution to the set of Eqs. (4.7) will be sought in the form

$$\tilde{\Xi}(z)=De^{-\sigma_0 z},\quad \tilde{\Theta}(z)=Ee^{-\lambda_1 z}+Fe^{-\lambda_2 z},\quad \tilde{\Lambda}(z)=E_1e^{-\lambda_1 z}+F_1e^{-\lambda_2 z}, \tag{4.8}$$

where the quantities E, E_1 and F, F_1 are interrelated by

$$E_1=\bar{\varkappa}_1E,\quad F_1=\bar{\varkappa}_2F, \tag{4.9}$$

where

$$\bar{\varkappa}_i=\frac{1}{s}(\eta^2-\lambda_i^2+\nu_0^2-\sigma_4^2),\quad i=1,2,$$

which result from the requirement that the functions Θ, Λ should satisfy Eqs. $(4.4)_2$. We should, moreover, determine the remaining quantities D, E and F. They will be found from the boundary conditions (4.1). By virtue of (2.10), the boundary conditions (4.1) take the form:

$$\left|(\mu+\alpha)\frac{\partial u_\varphi}{\partial z}+2\alpha\omega_r\right|_{z=0}=-h_1(r,t),$$
$$\left|2\gamma\frac{\partial\omega_z}{\partial z}+\beta\varkappa\right|_{z=0}=-h_2(r,t), \tag{4.10}$$
$$\left|(\gamma+\varepsilon)\frac{\partial\omega_z}{\partial r}+(\gamma-\varepsilon)\frac{\partial\omega_r}{\partial z}\right|_{z=0}=-h_3(r,t).$$

By expressing the displacement u_φ and the rotations ω_r, ω_z in (4.10) in terms of the potentials Ξ, Θ, and performing the Fourier-Hankel integral transformation on these equations, we obtain the following set of equations:

$$\begin{gathered} -2\alpha\eta D-\eta\lambda_1 r_1 E-\eta\lambda_2 r_2 F = -\tilde{h}_1(\eta,\zeta), \\ 2\gamma\sigma_0\eta D-\eta q_1 E-\eta q_2 F = -\tilde{h}_2(\eta,\zeta), \\ [(2\gamma+\beta)\sigma_0^2-\beta\eta^2]D-2\gamma\lambda_1\eta^2 E-2\gamma\lambda_2\eta^2 F = -\tilde{h}_3(\eta,\zeta), \end{gathered} \tag{4.11}$$

where

$$r_i = \lambda_i[(\mu+\alpha)\bar{\varkappa}_i-2\alpha], \quad q_i = (\gamma+\varepsilon)\eta^2+(\gamma-\varepsilon)\lambda_i^2, \quad i = 1, 2.$$

This is a set of algebraic nonhomogeneous linear equations with respect to D, E, F. From the solution to (4.11) we obtain:

$$\begin{gathered} D = \beta_{11}\tilde{h}_1+\beta_{12}\tilde{h}_2+\beta_{13}\tilde{h}_3, \\ E = \beta_{21}\tilde{h}_1+\beta_{22}\tilde{h}_2+\beta_{23}\tilde{h}_3, \\ F = \beta_{31}\tilde{h}_1+\beta_{32}\tilde{h}_2+\beta_{33}\tilde{h}_3, \end{gathered} \tag{4.12}$$

where

$$\beta_{11} = \frac{1}{\Delta}2\gamma\eta(q_1\lambda_2-q_2\lambda_1), \quad \beta_{12} = \frac{1}{\Delta}2\gamma\lambda_1\lambda_2(r_2-r_1), \quad \beta_{13} = \frac{1}{\Delta}(r_1\lambda_1 q_2-r_2\lambda_2 q),$$

$$\beta_{21} = \frac{1}{\Delta}\left(4\gamma^2\sigma_0\lambda_2\eta-\frac{1}{\eta}nq_2\right), \quad \beta_{22} = \frac{1}{\Delta}\left(4\gamma\alpha\lambda_2\eta+\frac{1}{\eta}nr_2\lambda_1\right),$$

$$\beta_{23} = -\frac{2}{\Delta}(\alpha q_2+\gamma\sigma_0 r_2\lambda_2), \quad \beta_{31} = -\frac{1}{\Delta}\left(4\gamma^2\sigma_0\lambda_1\eta-\frac{n}{\eta}q_1\right),$$

$$\beta_{32} = -\frac{1}{\Delta}\left(4\gamma\alpha\lambda_1\eta-nr_1\frac{1}{\eta}\right), \quad \beta_{33} = \frac{2}{\Delta}(\alpha q_1+\gamma\sigma_0\lambda_1 r_1), \tag{4.13}$$

$$\Delta = 4\alpha\gamma\eta^2(\lambda_2 q_1-\lambda_1 q_2)-\lambda_1 r_1(nq_2-4\gamma^2\sigma_0\lambda_2\eta^2)+r_2\lambda_2(nq_1-4\gamma^2\lambda_1\sigma_0\eta^2),$$

$$n = (2\gamma+\beta)\sigma_0^2-\beta\eta^2.$$

To determine the displacement u_φ and the rotations ω_r, ω_z, we shall perform the Fourier-Hankel transformation on (4.2) and (4.3), taking into account (4.8) and (4.9). We obtain then

$$\begin{gathered} \tilde{\omega}_r = -\eta(De^{-\sigma_0 z}+\lambda_1 Ee^{-\lambda_1 z}+\lambda_2 Fe^{-\lambda_2 z}), \\ \tilde{\omega}_z = -\sigma_0 De^{-\sigma_0 z}+\eta^2(Ee^{-\lambda_1 z}+Fe^{-\lambda_2 z}), \\ \tilde{u}_\varphi = \eta(\bar{\varkappa}_1 Ee^{-\lambda_1 z}+\bar{\varkappa}_2 Fe^{-\lambda_2 z}). \end{gathered} \tag{4.14}$$

In turn, we shall subject the above equations to the inverse Fourier-Hankel transformations. Thus, we obtain the expression for the quantities sought ω_r, ω_z and u_φ:

$$\omega_r = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int_0^\infty \eta^2(De^{-\sigma_0 z}+\lambda_1 Ee^{-\lambda_1 z}+\lambda_2 Fe^{-\lambda_2 z})\mathscr{J}_1(\eta r)d\eta,$$

$$\omega_z = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int_0^\infty \eta\sigma_0 De^{-\sigma_0 z}-\eta^3(Ee^{-\lambda_1 z}+Fe^{-\lambda_2 z})]\mathscr{J}_0(\eta r)d\eta, \tag{4.15}$$

$$u_\varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^{\infty} \eta^2 (\bar{\varkappa}_1 E e^{-\lambda_1 z} + \bar{\varkappa}_2 F e^{-\lambda_2 z}) \mathscr{J}_1(\eta r) d\eta,$$

the quantities D, E, F being given by the formulae (4.12). Now, making use of the formulae (2.1) and (2.2), we are able to determine the strain and stress fields in the semi-space.

We shall now proceed to consideration of the particular case in which $\alpha = 0$. In this case, the set of Eqs. (2.8) can be reduced to the following set:

$$\mu\left(\nabla^2 u_\varphi - \frac{u_\varphi}{r^2}\right) = \varrho \ddot{u}_\varphi,$$

$$(4.16) \qquad (\gamma+\varepsilon)\left(\nabla^2 \omega_r - \frac{\omega_r}{r^2}\right) + (\gamma-\varepsilon)\frac{\partial \varkappa}{\partial r} = J\ddot{\omega}_r,$$

$$(\gamma+\varepsilon)\nabla^2 \omega_z + (\gamma-\varepsilon)\frac{\partial \varkappa}{\partial z} = J\ddot{\omega}_z,$$

and the set of wave Eq. (4.4) will have the form:

$$(4.17) \qquad \left(\nabla^2 - \frac{1}{c_3^2}\partial_t^2\right)\Xi = 0, \quad \left(\nabla^2 - \frac{1}{c_4^2}\partial_t^2\right)\Theta = 0, \quad \left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\Lambda = 0.$$

After the Fourier-Hankel integral transformation has been carried out on Eqs. (4.17), the solution to these equations will be sought in the form:

$$(4.18) \qquad \Xi(z) = D_0 e^{-\sigma_0 z}, \quad \Lambda(z) = F_0 e^{-\lambda_0 z}, \quad \Theta(z) = E_0 e^{-\gamma_0 z},$$

where

$$\sigma_0 = (\eta^2 - \sigma_3^2)^{1/2}, \quad \lambda_0 = (\eta^2 - \hat{\sigma}_2^2)^{1/2}, \quad \gamma_0 = (\eta^2 - \sigma_4^2)^{1/2}.$$

The constants D_0, E_0, F_0 will be obtained from (4.12) under assumption that $\alpha = 0$:

$$(4.19) \qquad D_0 = \beta_{12}^0 \tilde{h}_2 + \beta_{13}^0 \tilde{h}_3, \quad E_0 = \beta_{22}^0 \tilde{h}_2 + \beta_{23}^0 \tilde{h}_3, \quad F_0 = \beta_{31}^0 \tilde{h}_1,$$

where

$$(4.20) \qquad \beta_{22}^0 = \frac{1}{\Delta_0}[(2\gamma+\beta)\sigma_0^2 - \beta\eta^2], \quad \beta_{23}^0 = -\frac{1}{\Delta_0} 2\gamma\eta\sigma_0, \quad \beta_{31}^0 = \frac{\lambda_0 \eta}{\Delta_0 \mu},$$

$$\Delta_0 = -4\gamma^2\eta^3\gamma_0\sigma_0 - \eta[(2\gamma+\beta)\sigma_0^2 - \beta\eta^2][(\gamma+\varepsilon)\eta^2 - (\gamma-\varepsilon)\gamma_0^2].$$

The displacements and rotation will be determined from the formulae:

$$\omega_r = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^{\infty} \eta [D_0 e^{-\sigma_0 z} - E_0 \gamma_0 e^{-\gamma_0 z}] \mathscr{J}_1(\eta r) d\eta,$$

$$(4.21) \qquad \omega_z = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^{\infty} [\sigma_0 D_0 e^{-\sigma_0 z} - \gamma_0^2 E_0 e^{-\gamma_0 z}] \eta \mathscr{J}_0(\eta r) d\eta,$$

$$u_\varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^{\infty} \eta F_0 e^{-\lambda_0 z} \mathscr{J}_1(\eta r) d\eta,$$

the constants D_0, E_0, F_0 being determined by the formulae (4.19). It is easy to find that the rotations ω_r, ω_z can be caused only by loading the semi-space boundary with the moment stresses $\mu_{zz}(r, 0, t) = -h_2(r, t)$, $\mu_{zr}(r, 0, t) = -h_3(r, t)$ [cf. the boundary conditions (4.1)]. On the other hand, the displacement u_φ can be produced only by the tangent stress $\sigma_{z\varphi}(r, 0, t) = -h_1(r, t)$. It can be stated that the formula $(4.21)_3$ describes displacements in a classical elastic medium, whereas Eqs. $(4.21)_1$, $(4.21)_2$ refer to a hypothetical medium in which there can exist only rotations but not displacements.

5. The Action of Loadings Harmonically Varying in Time

In the present section, we shall consider a particular case of loading the semi-space boundary—namely the loading varying harmonically in time. We shall discuss two different variants of such a loading. First, we shall assume that there exist only the stresses normal to the semi-space surface. Next, we shall investigate the second case in which there are assumed only the moment stresses on the semi-space surface and their vectors are directed along the positive axis z.

In the first case, we deal with the following boundary conditions

$$\sigma_{zz}(r, 0, t) = -f_1(r)e^{-i\omega t}, \quad \sigma_{zr}(r, 0, t) = 0, \quad \mu_{z\varphi}(r, 0, t) = 0. \tag{5.1}$$

The boundary conditions thus assumed produce the displacements u_r, u_z and rotation ω_φ in the elastic semi-space. Making use of (3.16) with $\tilde{f}_2 = \tilde{f}_3 = 0$, we arrive at the following expressions for the displacements and rotation sought:

$$\begin{aligned} u_r &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^{\infty} \eta^2(\alpha_{11} e^{-\sigma z} + \lambda_1 \alpha_{21} e^{-\lambda_1 z} + \lambda_2 \alpha_{31} e^{-\lambda_2 z}) \tilde{f}(\eta, \zeta) \mathscr{I}_1(\eta r) d\eta, \\ u_z &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^{\infty} \eta [\sigma\alpha_{11} e^{-\sigma z} - \eta^2(\alpha_{21} e^{-\lambda_1 z} + \alpha_{31} e^{-\lambda_2 z})] \tilde{f}_1(\eta, \zeta) \mathscr{I}_0(\eta r) d\eta, \\ \omega_z &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta t} d\zeta \int_0^{\infty} \eta^2 [\sigma\alpha_{11} e^{-\sigma z} - \eta^2(\alpha_{21} e^{-\lambda_1 z} + \alpha_{31} e^{-\lambda_2 z})] \tilde{f}_1(\eta, \zeta) \mathscr{I}_0(\eta r) d\eta. \end{aligned} \tag{5.2}$$

In these formulae

$$\tilde{f}_1(\eta, \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\zeta} dt \int_0^{\infty} f_1(r) e^{-i\omega t} r \mathscr{I}_0(\eta r) dr = \sqrt{2\pi}\, \delta(\zeta - \omega) \tilde{f}_1(\eta), \tag{5.3}$$

where

$$\tilde{f}_1(\eta) = \int_0^{\infty} f_1(r) r \mathscr{I}_0(\eta r) dr.$$

Use has been made here of $\int_{-\infty}^{\infty} e^{it(\zeta-\omega)} dt = 2\pi\delta(\zeta - \omega)$, where δ denotes the Dirac function.

Making use of (5.3) and integrating, we obtain from (5.2):

$$\begin{aligned} u_r &= -e^{-i\omega t} \int_0^{\infty} |(\alpha_{11} e^{-\sigma z} + \alpha_{21} \lambda_1 e^{-\lambda_1 z} + \alpha_{31} \lambda_2 e^{-\lambda_2 z})|_{\zeta=\omega} \tilde{f}_1(\eta) \eta^2 \mathscr{I}_1(\eta r) d\eta, \\ u_z &= -e^{-i\omega t} \int_0^{\infty} |(\sigma\alpha_{\cdot 1} e^{-\sigma z} - \eta^2 \alpha_{21} e^{-\lambda_1 z} - \eta^2 \alpha_{31} e^{-\lambda_2 z})|_{\zeta=\omega} \tilde{f}_1(\eta) \eta \mathscr{I}_0(\eta r) d\eta, \\ \omega_\varphi &= e^{-i\omega t} \int_0^{\infty} |(\alpha_{21} \varkappa_1 e^{-\lambda_1 z} + \alpha_{31} \varkappa_2 e^{-\lambda_2 z})|_{\zeta=\omega} \eta^2 \tilde{f}_1(\eta) \mathscr{I}_1(\eta r) d\eta. \end{aligned} \tag{5.4}$$

In $\alpha_{i1}(i = 1, 2, 3)$, $\varkappa_r$, λ_r $(r = 1, 2)$ and σ involved in (5.4), the ζ should be replaced by ω. Moreover, it should be noted that the expression

$$(5.5)\qquad \Delta|_{\zeta=\omega} = |[2\mu\sigma^2+\lambda(\sigma^2-\eta^2)](\lambda_2\varkappa_2 a_1-\lambda_1\varkappa_1 a_2)-4\mu^2\eta^2\sigma\lambda_1\lambda_2(\varkappa_2-\varkappa_1)|_{\zeta=\omega} = 0,$$

involved in α_{i1} can be regarded as the condition of the appearance of surface waves in the elastic semi-space. If we consider the set of homogeneous Eqs. (3.11) for monochromatic vibrations, the compliance conditions of this set would be the condition of zero value of a characteristic determinant of this set of equations — that is, the condition (5.5). This equation has been derived in [5, 6].

We shall now consider the particular case in which $\alpha = 0$. Then, taking into account (3.19) and (3.20) with $\tilde{f}_2 = \tilde{f}_3 = 0$, we obtain from $(3.6)_2$:

$$(5.6)\qquad \begin{aligned} \Phi &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\zeta}d\zeta \int_0^{\infty} \frac{1}{\Delta_0}(\lambda_0^2+\eta^2)\eta^2 e^{-\sigma z}\tilde{f}_1(\eta,\zeta)\mathscr{J}_0(\eta r)\,d\eta,\\ \Psi &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\zeta}d\zeta \int_0^{\infty} \frac{2}{\Delta_0}\mu\sigma\eta^2 e^{-\lambda_0 z}\tilde{f}_1(\eta,\zeta)\mathscr{J}_0(\eta r)\,d\eta. \end{aligned}$$

Taking into account (5.3), we have the following expressions for the potentials:

$$(5.7)\qquad \begin{aligned} \Phi &= e^{-i\omega t}\int_0^{\infty} \frac{1}{D}\left(2\eta^2-\frac{\omega^2}{\hat{c}_2^2}\right)\eta^2 e^{-\left(\eta^2-\frac{\omega^2}{c_1^2}\right)^{1/2}z}\tilde{f}_1(\eta)\mathscr{J}_0(\eta r)\,d\eta,\\ \Psi &= 2\mu e^{-i\omega t}\int_0^{\infty} \frac{1}{D}\left(\eta^2-\frac{\omega^2}{c_1^2}\right)^{1/2}\left(\eta^2-\frac{\omega^2}{\hat{c}_2^2}\right)e^{-\left(\eta^2-\frac{\omega^2}{\hat{c}_2^2}\right)^{1/2}z}\tilde{f}_1(\eta)\mathscr{J}_0(\eta r)\,d\eta,\\ \Gamma &= 0, \end{aligned}$$

where

$$D = \eta\mu\left\{\left(2\eta^2-\frac{\omega^2}{\hat{c}_2^2}\right)-4\mu^2\eta^3\left(\eta^2-\frac{\omega^2}{c_1^2}\right)^{1/2}\left(\eta^2-\frac{\omega^2}{\hat{c}_2^2}\right)^{1/2}\right\}.$$

We shall now consider the second variant of loading—namely, the semi-space boundary is loaded with loadings varying harmonically in time and with their vectors being directed along the positive z-axis. The boundary conditions are of the form:

$$(5.8)\qquad \mu_{zz}(r,0,t) = -h_2(r,t),\quad \sigma_{z\varphi}(r,0,t) = 0,\quad \mu_{zr}(r,0,t) = 0.$$

The above boundary conditions produce, in the semi-space, the displacements u_φ and the rotations ω_r, ω_z which will be determined from (4.15):

$$(5.9)\qquad \begin{aligned} \omega_r &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int_0^{\infty}\eta^2(\beta_{12}e^{-\sigma_0 z}+\lambda_1\beta_{22}e^{-\lambda_1 z}+\lambda_2\beta_{32}e^{-\lambda_2 z})\tilde{h}_2(\eta,\zeta)\mathscr{J}_1(\eta r)\,d\eta,\\ \omega_z &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int_0^{\infty}\eta[\sigma_0\beta_{12}e^{-\sigma_0 z}-\eta^2(\beta_{zz}e^{-\lambda_1 z}+\beta_{32}e^{-\lambda_2 z})]\tilde{h}_2(\eta,\zeta)\mathscr{J}_0(\eta r)\,d\eta,\\ u_\varphi &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\zeta t}d\zeta\int_0^{\infty}\eta^2(\bar{\varkappa}_1\beta_{22}e^{-\lambda_1 z}+\bar{\varkappa}_2\beta_{32}e^{-\lambda_1 z})\tilde{h}_2(\eta,\zeta)\mathscr{J}_1(\eta r)\,d\eta. \end{aligned}$$

Here

(5.10) $$\tilde{h}_2(\eta, \zeta) = \sqrt{2\pi}\,\delta(\zeta-\omega)\tilde{h}_2(\eta),$$

and

$$\tilde{h}_2(\eta) = \int\limits_0^\infty h_2(r) r \mathscr{I}_0(\eta r)\, dr.$$

Finally, the rotations ω_r, ω_z and the displacement u_φ have the form

(5.11)
$$\omega_z = -e^{-i\omega t}\int\limits_0^\infty |(\beta_{12}e^{-\sigma_0 z}+\lambda_1\beta_{22}e^{-\lambda_1 z}+\lambda_2\beta_{32}e^{-\lambda_2 z})|_{\zeta=\omega}\eta^2\tilde{h}_2(\eta)\mathscr{I}_1(\eta r)\, d\eta,$$

$$\omega_z = -e^{-i\omega t}\int\limits_0^\infty |[\sigma_0\beta_{12}e^{-\sigma_0 z}-\eta^2(\beta_{22}e^{-\lambda_1 z}+\beta_{32}e^{-\lambda_2 z})]|_{\zeta=\omega}\eta\tilde{h}_2(\eta)\mathscr{I}_1(\eta r)\, d\eta,$$

$$u_\varphi = e^{-i\omega t}\int\limits_0^\infty |(\bar{\varkappa}_1\beta_{22}e^{-\lambda_1 z}+\bar{\varkappa}_2\beta_{32}e^{-\lambda_2 z})|_{\zeta=\omega}\eta^2\tilde{h}_2(\eta)\mathscr{I}_1(\eta r)\, d\eta.$$

The expression $\Delta|_{\zeta=\omega} = 0$ involved in the functions β_{iz} $(i = 1, 2, 3)$ is a characteristic determinant of the set of Eqs. (4.11). Under the assumption of homogeneous boundary conditions, the expression $\Delta|_{\zeta=\omega} = 0$ is the condition of the noncontradictory character of the homogeneous set of Eqs. (4.11). This equation is identical with the characteristic equation for the Love's waves appearing in a micropolar elastic semi-space [5, 6].

Finally, we shall consider additionally the case $\alpha = 0$. Then, for the boundary conditions (5.8), and taking into account (4.18), we obtain the following expressions for the potentials:

(5.12)
$$\Theta = e^{-i\omega t}\int\limits_0^\infty \frac{1}{D_0}\left[2\gamma\eta^2-(2\gamma+\beta)\frac{\omega^2}{c_3^2}\right]\eta e^{-\left(\eta^2-\frac{\omega^2}{c_4^2}\right)z}\tilde{h}_2(\eta)\mathscr{I}_0(\eta r)\, d\eta,$$

$$\Xi = e^{-i\omega t}\int\limits_0^\infty \frac{2}{D_0}\gamma\eta^3\left(\eta^2-\frac{\omega^2}{c_4^2}\right)e^{-\left(\eta^2-\frac{\omega^2}{c_3^2}\right)^{1/2}z}\tilde{h}_2(\eta)\mathscr{I}_0(\eta r)\, d\eta,$$

$$\Lambda = 0,$$

where

$$D_0 = -\left[4\eta^3\left(\eta^2-\frac{\omega^2}{c_4^2}\right)^{1/2}\left(\eta^2-\frac{\omega^2}{c_3^2}\right)^{1/2}-\left(2\eta^2-\frac{\omega^2}{\hat{c}_4^2}\right)\right]\eta\gamma^2, \quad \hat{c}_4^2 = \frac{\gamma}{J}.$$

By virtue of (4.2) and (4.3), we can determine the rotation vector $\boldsymbol{\omega} = (\omega_r, 0, \omega_z)$ in such a hypothetical medium that only moment rotations and stresses can occur.

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Streszczenie

OSIOWO-SYMETRYCZNE ZAGADNIENIE LAMBA W MIKROPOLARNEJ PÓŁPRZESTRZENI SPRĘŻYSTEJ

W pracy rozważono zagadnienie propagacji fal w półprzestrzeni sprężystej mikropolarnej izotropowej i centrosymetrycznej, wywołanych przez obciążenia zmienne w czasie i rozłożone osiowo-symetrycznie na powierzchni półprzestrzeni. Rozważono przypadek szczególny, w którym obciążenia zewnętrzne, wektor przemieszczeń $\mathbf{u}$ i wektor obrotów $\boldsymbol{\omega}$ są zależne jedynie od współrzędnych r, z (w układzie współrzędnych cylindrycznych r, φ, z). Rozwiązanie problemu sprowadza się do rozwiązania dwóch niezależnych od siebie układów równań. Pierwszy z nich wywołuje przemieszczenia $\mathbf{u} = (u_r, 0, u_z)$ i obroty $\boldsymbol{\omega} = (0, \omega_\varphi, 0)$, drugi zaś wywołuje przemieszczenia $\mathbf{u} = (0, u_\varphi, 0)$ i obroty $\boldsymbol{\omega} = (\omega_r, 0, \omega_z)$. Do rozwiązania zastosowano transformację całkową Fouriera-Hankela.

Резюме

ОСЕСИММЕТРИЧНАЯ ЗАДАЧА ЛАМБА В МИКРОПОЛЯРНОМ УПРУГОМ ПОЛУПРОСТРАНСТВЕ

В работе рассмотрена проблема распространения волн в микрополярном, изотропном и центросимметричном, упругом полупространстве, вызванных переменными во времени и осесимметрично распределенными на поверхности полупространства нагрузками. Рассмотрен частный случай, в котором внешние нагрузки, вектор перемещений $\mathbf{u}$ и вектор вращений $\boldsymbol{\omega}$ зависят только от координат r, z (в системе цилиндрических координат r, φ, z). Решение проблемы сводится к решению двух независимых от себя систем уравнений. Первая из них вызывает перемещения $\mathbf{u} = (u_r, 0, u_z)$ и вращения $\boldsymbol{\omega} = (0, \omega_\varphi, 0)$, вторая же вызывает перемещения $\mathbf{u} = (0, u_\varphi, 0)$ и вращения $\boldsymbol{\omega} = (\omega_r, 0, \omega_z)$. Для решения применено интегральное преобразование Фурье-Ханкеля.

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