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GREEN FUNCTIONS FOR MICROPOLAR ELASTICITY

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1. Introduction

Let us consider an infinite homogeneous isotropic and centro-symmetric elastic body. The action of body forces and couples produces in the body a field of displacements $\mathbf{u}(\mathbf{x}, t)$ and rotations $\boldsymbol{\omega}(\mathbf{x}, t)$. These fields vary with the position of the point $\mathbf{x}$ and the time t.

The strain of the body is characterized by two asymmetric tensors: the strain tensor γ_{ji} and the flexural-torsional tensor $\varkappa_{ji}$, where, [1–4]

$$\gamma_{ji} = u_{i,j} - \varepsilon_{kji}\omega_k, \quad \varkappa_{ji} = u_{i,j}. \tag{1.1}$$

The state of stress is determined by the stress tensor σ_{ji} and the couple-stress tensor μ_{ji}. The relations between the state of stress and the state of strain are expressed by the equations

$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij}, \tag{1.2}$$

$$\mu_{ji} = (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ij}, \quad i, j = 1, 2, 3. \tag{1.3}$$

The quantities μ, λ, α, β, γ, ε are material constants. On substituting (1.2) and (1.3) in the equations of motion

$$\sigma_{ji,j} + X_i - \varrho\ddot{u}_i = 0, \tag{1.4}$$

$$\varepsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i - J\ddot{\omega}_i = 0, \tag{1.5}$$

and expressing the quantities γ_{ji}, $\varkappa_{ji}$ in terms of the displacement u_i and the rotation ω_i, according to Eq. (1.1), we obtain a set of six differential equations which can be presented in the vectorial form:

$$(\mu+\alpha)\nabla^2\mathbf{u} + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\omega} + \mathbf{X} = \varrho\ddot{\mathbf{u}}, \tag{1.6}$$

$$(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega} + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\omega} - 4\alpha\boldsymbol{\omega} + 2\alpha\,\mathrm{rot}\,\mathbf{u} + \mathbf{Y} = J\ddot{\boldsymbol{\omega}}, \tag{1.7}$$

where $\mathbf{X}$ is the vector of body force, $\mathbf{Y}$ — the body couple vector, ϱ — the density, and J — the rotational inertia. The time derivative of the functions u_i, ω_i is denoted by a dot.

Our aim is to find a fundamental solution to the set of Eqs. (1.6) and (1.7), assuming that mass forces and moments vary in function of time in a harmonic manner.

We shall seek for the displacements $\mathbf{u} = \mathbf{U}^{(k)}(\mathbf{x}, \boldsymbol{\xi}, t)$ and the rotations $\boldsymbol{\omega} = \boldsymbol{\Omega}^{(k)}(\mathbf{x}, \boldsymbol{\xi}, t)$ produced by the action of a concentrated force at a point $\boldsymbol{\xi}$ directed parallel to the $\mathbf{x}_k$-

axis as well as the displacements $\mathbf{u} = \mathbf{V}^{(k)}(\mathbf{x}, \boldsymbol{\xi}, t)$ and rotations $\boldsymbol{\omega} = \mathbf{W}^{(k)}(\mathbf{x}, \boldsymbol{\xi}, t)$ produced by the action of a concentrated body couple in the point $\boldsymbol{\eta}$, directed parallel to the x_k-axis.

We shall obtain pairs of tensors $(U_j^{(k)}, \Omega_j^{(k)})$ and $(V_j^{(k)}, W_j^{(k)})$ which will be referred to, in a general manner, as Green functions for micropolar elasticity.

Our problem will be solved by two methods. The first consists in resolving the displacement vector $\mathbf{u}$ and the rotation vector $\boldsymbol{\omega}$ into potential and solenoidal parts. The other method of solution will involve the use of stress functions.

Let us present the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ in the form

$$\mathbf{u} = \operatorname{grad} \Phi + \operatorname{rot} \boldsymbol{\Psi}, \quad \operatorname{div} \boldsymbol{\Psi} = 0, \tag{1.8}$$

$$\boldsymbol{\omega} = \operatorname{grad} \Sigma + \operatorname{rot} \mathbf{H}, \quad \operatorname{div} \mathbf{H} = 0. \tag{1.9}$$

The same can be done for the body forces and body couples:

$$\mathbf{X} = \varrho(\operatorname{grad} \vartheta + \operatorname{rot} \boldsymbol{\chi}), \tag{1.10}$$

$$\mathbf{Y} = J(\operatorname{grad} \sigma + \operatorname{rot} \boldsymbol{\eta}). \tag{1.11}$$

On substituting (1.8) to (1.11) in Eqs. (1.6), (1.7), we reduce these equations to a system of four wave equations

$$\Box_1 \Phi + \varrho \vartheta = 0, \tag{1.12}$$

$$\Box_3 \Sigma + J\sigma = 0, \tag{1.13}$$

$$(\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \boldsymbol{\Psi} = 2\alpha J \operatorname{rot} \boldsymbol{\eta} - \varrho \Box_4 \boldsymbol{\chi}, \tag{1.14}$$

$$(\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \mathbf{H} = 2\alpha\varrho \operatorname{rot} \boldsymbol{\chi} - J \Box_2 \boldsymbol{\eta}. \tag{1.15}$$

The notations introduced are

$$\Box_1 = (\lambda + 2\mu)\nabla^2 - \varrho\partial_t^2, \qquad \Box_2 = (\mu + \alpha)\nabla^2 - \varrho\partial_t^2;$$

$$\Box_3 = (\beta + 2\gamma)\nabla^2 - 4\alpha - J\partial_t^2, \quad \Box_4 = (\gamma + \varepsilon)\nabla^2 - 4\alpha - J\partial_t^2,$$

$$\nabla^2 = \partial_i \partial_i, \quad \partial_t^2 = \partial^2/\partial t^2.$$

It is known that Eq. (1.12) represents a longitudinal wave and Eq. (1.13) a rotation wave. Equations (1.14) and (1.15) represent modified transversal waves. Let us observe that in the infinite elastic region the body force $\mathbf{X}' = \varrho \operatorname{grad} \vartheta$ produces only a longitudinal wave, and the body couple $\mathbf{Y}' = J \operatorname{grad} \sigma$ a rotation wave only.

Let us assume that the causes producing wave perturbations — that is, body forces and body couples — vary harmonically with time

$$\mathbf{X} = (\mathbf{x}, t) = \mathbf{X}^*(\mathbf{x}) e^{-i\omega t}, \quad \mathbf{Y} = (\mathbf{x}, t) = \mathbf{Y}^*(\mathbf{x}) e^{-i\omega t}. \tag{1.16}$$

The results of these actions — that is, the displacement $\mathbf{u}$ and the rotation $\boldsymbol{\omega}$ — will also vary harmonically in time. Denoting by an asterisk the amplitudes of these functions, we reduce Eqs. (1.12)–(1.15) to the following set of equations:

$$(\nabla^2 + \sigma_1^2)\Phi^* = -\frac{1}{c_1^2}\vartheta^*, \tag{1.17}$$

$$(\nabla^2 + \sigma_3^2)\Sigma^* = -\frac{1}{c_3^2}\sigma^*, \tag{1.18}$$

(1.19) $$(\nabla^2+k_1^2)(\nabla^2+k_2^2)\mathbf{\Psi}^* = \frac{r}{c_4^2}\operatorname{rot}\boldsymbol{\eta}^* - \frac{1}{c_2^2}D_2\boldsymbol{\chi}^*,$$

(1.20) $$(\nabla^2+k_1^2)(\nabla^2+k_2^2)\mathbf{H}^* = \frac{p}{c_2^2}\operatorname{rot}\boldsymbol{\chi}^* - \frac{1}{c_4^2}D_1\boldsymbol{\eta}^*,$$

where

$$\sigma = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad \sigma_3 = \left(\frac{\omega^2-\omega_0^2}{c_3^2}\right)^{1/2}, \quad c_2 = \left(\frac{\beta+2\gamma}{J}\right)^{1/2}, \quad \omega_0^2 = \frac{4\alpha}{J};$$

$$r = \frac{2\alpha}{\varrho c_2^2}, \quad p = \frac{2\alpha}{Jc_4^2}, \qquad c_2 = \left(\frac{\mu+\alpha}{\varrho}\right)^{1/2}; \quad c_4 = \left(\frac{\gamma+\varepsilon}{J}\right)^{1/2}. \quad \sigma_2 = \frac{\omega}{c_2}$$

$$\sigma_4 = \frac{\omega}{c_4}, \quad D_1 = \nabla^2+\sigma_2^2, \qquad D_2 = \nabla^2+\sigma_4^2-2p,$$

and k_1^2, k_2^2 are the roots of the equation:

(1.21) $$k^4 - k^2[\sigma_2^2+\sigma_4^2+p(r-2)]+\sigma_2^2(\sigma_4^2-2p) = 0.$$

The discriminant of

$$k_{1,2}^2 = \frac{1}{2}\left[\sigma_2^2+\sigma_4^2+p(\mathrm{r}-2) \pm \sqrt{[\sigma_4^2-\sigma_2^2+p(r-2)]^2+4pr\sigma_2^2}\,\right]$$

is positive.

Let us consider the homogeneous Eq. (1.19). Its solution can be presented (by virtue of the theorem of T. Boggio, [5]) as a sum of two partial solutions

(1.22) $$\mathbf{\Psi}^* = \mathbf{\Psi}'^* + \mathbf{\Psi}''^*,$$

satisfying Helmholtz vector equations

(1.23) $$(\nabla^2+k_1^2)\mathbf{\Psi}'^* = 0, \quad (\nabla^2+k_2^2)\mathbf{\Psi}''^* = 0.$$

Particular integrals of these equations are the functions $R^{-1}e^{\pm ik_\alpha R}\alpha = 1, 2$; $i = \sqrt{-1}$. The solutions $R^{-1}e^{ik_\alpha R}$ are, however, the only having a physical sense, because the expressions

$$\operatorname{Re}\left[e^{-i\omega t}\frac{e^{ik_\alpha R}}{R}\right] = \frac{1}{R}\cos\omega\left(t-\frac{R}{\vartheta_\alpha}\right), \quad \vartheta_\alpha = \frac{\omega}{k_\alpha}, \quad \alpha = 1, 2$$

are the only to express a wave moving from the place of perturbation to infinity. The solution of the homogeneous Eq. (1.14) will, therefore, take the form:

(1.24) $$\mathbf{\Psi}^* = \mathbf{A}\frac{e^{ik_1R}}{R}+\mathbf{B}\frac{e^{ik_2R}}{R}.$$

An analogous solution of the homogeneous Eq. (1.12) is presented by the function:

(1.25) $$\mathbf{H}^* = \mathbf{C}\frac{e^{ik_1R}}{R}+\mathbf{D}\frac{e^{ik_2R}}{R}.$$

In the waves $\mathbf{\Psi}$, $\mathbf{H}$, real phase velocities are the only that can occur. We must, therefore, have $k_1^2 > 0$, $k_2^2 > 0$. The first condition is satisfied for a positive determinant of Eq. (1.21). The other condition will be satisfied if $\sigma_4 > 2p$ or if $\omega^2 > 4\alpha/J$. This follows from the relation $k_1^2 k_2^2 = \sigma_2^2(\sigma_4^2 - 2p) > 0$. In the expressions (1.24), (1.25), there are two waves undergoing dispersion (because k_1 and k_2 are functions of the frequency ω). The rotation wave $\sum^*$ will exist if $\sigma_3^2 > 0$. This condition leads to the inequality $\omega^2 > 4\alpha/J$.

2. Body Force Effect

Let us consider the action of body forces. Let us observe that the lack of body couples ($\mathbf{Y} = 0$) results in $\sigma^* = 0$, $\boldsymbol{\eta}^* = 0$. No rotation wave will occur in the infinite elastic space ($\Sigma^* = 0$). The following set of equations remains to be solved:

$$(\nabla^2 + \sigma_1^2)\Phi^* = -\frac{1}{c_1^2}\vartheta^*, \tag{2.1}$$

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\boldsymbol{\Psi}^* = -\frac{1}{c_2^2} D_2 \boldsymbol{\chi}^*, \tag{2.2}$$

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\mathbf{H}^* = \frac{p}{c_2^2} \operatorname{rot} \boldsymbol{\chi}^*. \tag{2.3}$$

If the body forces $\mathbf{X}$ are distributed over a closed region B, the quantities ϑ^* and $\boldsymbol{\chi}^*$ will be determined from the following equations [6]:

$$\vartheta^*(\mathbf{x}) = -\frac{1}{4\pi\varrho} \int_B X_j^*(\boldsymbol{\xi}) \frac{\partial}{\partial x_j}\left(\frac{1}{R(\mathbf{x}, \boldsymbol{\xi})}\right) dV(\boldsymbol{\xi}), \quad j = 1, 2, 3, \tag{2.4}$$

$$\chi_i^*(\mathbf{x}) = -\frac{1}{4\pi\varrho} \int_B \varepsilon_{ijk} X_j^*(\boldsymbol{\xi}) \frac{\partial}{\partial x_k}\left(\frac{1}{R(\mathbf{x}, \boldsymbol{\xi})}\right) dV(\boldsymbol{\xi}) \quad i, j, k = 1, 2, 3. \tag{2.5}$$

By introducing in these equations the formula

$$X_j^*(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)\delta_{1j}, \quad j = 1, 2, 3,$$

expressing a concentrated force at the origin in the direction of the x_1-axis, we obtain:

$$\begin{gathered} \vartheta^* = -\frac{1}{4\pi\varrho}\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right), \quad \chi_1^* = 0, \quad \chi_2^* = \frac{1}{4\pi\varrho}\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right), \\ \chi_3^* = -\frac{1}{4\pi\varrho}\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right), \quad R = (x_1^2 + x_2^2 + x_3^2)^{1/2}. \end{gathered} \tag{2.6}$$

It remains to solve the equations:

$$(\nabla^2 + \sigma_1^2)\Phi^* = \frac{1}{4\pi\varrho c_1^2}\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right); \tag{2.7}$$

$$\begin{gathered} (\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\Psi_2^* = -\frac{1}{4\pi\varrho c_2^2}(\nabla^2 + \sigma_4^2 - 2p)\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right), \\ (\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\Psi_3^* = \frac{1}{4\pi\varrho c_2^2}(\nabla^2 + \sigma_4^2 - 2p)\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right); \end{gathered} \tag{2.8}$$

$$(\nabla^2+k_1^2)(\nabla^2+k_2^2)H_1^* = -\frac{p}{4\pi\varrho c_2^2}(\nabla^2-\sigma_1^2)\left(\frac{1}{R}\right),$$

(2.9) $$(\nabla^2+k_1^2)(\nabla^2+k_2^2)H_2^* = \frac{p}{4\pi\varrho c_2^2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right),$$

$$(\nabla^2+k_1^2)(\nabla^2+k_2^2)H_3^* = \frac{p}{4\pi\varrho c_2^2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right).$$

The solution of Eq. (2.7) is known from classical elastokinetics (dynamic theory of elasticity) [6]:

(2.10) $$\Phi^*(\mathbf{x}) = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_1}\left(\frac{e^{i\sigma_1 R}-1}{R}\right).$$

Equations (2.8), (2.9) will be solved by applying Fourier integral transformation of the exponential type. Thus, for instance, the solution of the equation for Ψ_2^* will be presented in the form of the integral:

(2.11) $$\Psi_2^{**}(\mathbf{x}) = \frac{1}{8\varrho c_2^2\pi^3}\frac{\partial}{\partial x_3}\int\int_{-\infty}^{\infty}\int \frac{(\boldsymbol{\alpha}^2-\sigma_4^2+2p)\,e^{-i\alpha_k x_k}d\alpha_1\,d\alpha_2\,d\alpha_3}{\boldsymbol{\alpha}^2(\boldsymbol{\alpha}^2-k_1^2)(\boldsymbol{\alpha}^2-k_2^2)},$$

where

$$\boldsymbol{\alpha}^2 = \alpha_1^2+\alpha_2^2+\alpha_3^2.$$

Bearing in mind that

$$\int\int_{-\infty}^{\infty}\int \frac{e^{-i\alpha_k x_k}d\alpha_1\,d\alpha_2\,d\alpha_3}{\alpha^2-k_j^2} = 2\pi^2\frac{e^{ik_jR}}{R},$$

the function Ψ_2^* of (2.11) can be presented in the form:

(2.12) $$\Psi_2^* = \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_3}\left(A_1\frac{e^{ik_1R}}{R}+A_2\frac{e^{ik_2R}}{R}+A_3\frac{1}{R}\right),$$

where

$$A_1 = \frac{\sigma_2^2-k_2^2}{k_1^2-k_2^2},\quad A_2 = \frac{\sigma_2^2-k_1^2}{k_2^2-k_1^2},\quad A = -1.$$

Solving Eq. (2.8) in an analogous manner, we have:

(2.13) $$\Psi_3^* = -\frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_2}\left(A_1\frac{e^{ik_1R}}{R}+A_2\frac{e^{ik_2R}}{R}+A_3\frac{1}{R}\right).$$

Application of Fourier integral transformations to the set of Eqs. (2.9) yields

(2.14) $$H_1^* = \frac{p}{4\pi\varrho c_2^2}\left[\frac{e^{ik_1R}-e^{ik_2R}}{R(k_1^2-k_2^2)}+\frac{\partial^2}{\partial x_1^2}\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+B_3\frac{1}{R}\right)\right],$$

(2.15) $$H_2^* = \frac{p}{4\pi\varrho c_2^2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+B_3\frac{1}{R}\right),$$

(2.16) $$H_3^* = \frac{p}{4\pi\varrho c_2^2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3}\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+B_3\frac{1}{R}\right),$$

where
$$B_1 = \frac{1}{k_1^2(k_1^2-k_2^2)}, \quad B_2 = \frac{1}{k_2^2(k_2^2-k_1^2)}, \quad B_3 = \frac{1}{k_1^2 k_2^2}.$$

The displacements $\mathbf{u}$ and the rotations $\boldsymbol{\omega}$ will be found from (1.8) and (1.9).

Since $\boldsymbol{\eta}^* = 0$, therefore:

$$(2.17) \qquad \begin{aligned} &u_1^* = \partial_1 \Phi^* + \partial_2 \Psi_3^* - \partial_3 \Psi_2^*, \quad u_2^* = \partial_2 \phi^* - \partial_1 \Psi_3^*, \quad u_3 = \partial_3 \phi^* + \partial_1 \Psi_2^* \\ &\omega_1^* = \partial_2 H_3^* - \partial_3 H_2^*, \quad \omega_2^* = \partial_3 H_1^* - \partial_1 H_3^*, \quad \omega_3^* = \partial_1 H_2^* - \partial_2 H_1^*. \end{aligned}$$

As a result, we shall obtain the following equations for the amplitudes $\mathbf{u}^*$ and $\boldsymbol{\omega}^*$:

$$(2.18) \qquad \begin{aligned} u_j^* = U_j^{*(1)} = &\frac{1}{4\pi\varrho\omega^2}\left(A_1 k_1^2 \frac{e^{ik_1R}}{R} + A_2 k_2^2 \frac{e^{ik_2R}}{R}\right)\delta_{1j} \\ &+ \frac{1}{4\pi\varrho\omega^2}\partial_1\partial_j\left(A_1\frac{e^{ik_1R}}{R} + A_2\frac{e^{ik_2R}}{R} + A_3\frac{e^{i\sigma_1R}}{R}\right), \quad j = 1, 2, 3 \end{aligned}$$

$$(2.19) \qquad \omega_j^* = \Omega_j^{*(1)} = \frac{p}{4\pi\varrho c_2^2(k_1^2-k_2^2)}\varepsilon_{1jk}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right), \quad j, k = 1, 2, 3.$$

We have obtained three components of the displacement vector $U_j^{*(1)}$ and three components of the vector of rotation $\Omega_j^{*(1)}$. We now displace the concentrated force from the origin to the point $\boldsymbol{\xi}$, and let it act parallel to the x_l-axis. Then, Eqs. (2.18), (2.19) become:

$$(2.20) \qquad \begin{aligned} u_j^* = U_j^{*(l)} = &\frac{1}{4\pi\varrho\omega^2}\left(A_1 k_1^2 \frac{e^{ik_1R}}{R} + A_2 k_2^2 \frac{e^{ik_2R}}{R}\right)\delta_{jl} \\ &+ \frac{1}{4\pi\varrho\omega^2}\partial_l\partial_j\left(A_1\frac{e^{ik_2R}}{R} + A_2\frac{e^{ik_2R}}{R} + A_3\frac{e^{i\sigma_1R}}{R}\right), \quad j, l = 1, 2, 3, \end{aligned}$$

and

$$(2.21) \qquad \omega_j^* = \Omega_j^{*(l)} = \frac{p}{4\pi\varrho c_2^2(k_1^2-k_2^2)}\varepsilon_{ljk}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right), \quad j, k, l = 1, 2, 3.$$

In Eqs. (2.20), (2.21), R has a different meaning. We have

$$R = [(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2]^{1/2}.$$

Thus, we have obtained the displacement tensor $U_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi})$ and the rotation tensor $\Omega_{j\,i}^{*(l)}(\mathbf{x}, \boldsymbol{\xi})$. These tensors constitute two symmetric matrices.

Let us introduce in Eqs. (2.20), (2.21) $\alpha = 0$, thus passing to classical elastokinetics [6]. We have:

$$(2.22) \qquad \begin{aligned} &U_j^{*(l)} = \frac{e^{i\tau R}}{4\pi\mu R}\delta_{jl} - \frac{1}{4\pi\varrho\omega^2}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(\frac{e^{i\sigma R}-e^{i\tau R}}{R}\right), \\ &\Omega_j^{*(l)} = 0, \quad j, l = 1, 2, 3, \end{aligned}$$

with the notations

$$\tau = \frac{\omega}{c_2^0}, \quad c_2^0 = \left(\frac{\mu}{\varrho}\right)^{1/2}, \quad \sigma = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}.$$

Let us return to Eqs. (2.18) and (2.19) and observe, that the concentrated force directed parallel to x_1 does not produce rotation ω_1^*. We have $\omega_1^* = \Omega_1^{*(1)} = 0$. This results in the fact that the components $\varkappa_{1j}$ ($j = 1, 2, 3$) of the curvature-twist tensor are zero. The components of the strain tensor γ_{ji} are different to zero.

Equations (2.20) and (2.21) express waves of three types. Waves connected with the values k_1, k_2 undergo dispersion.

3. Body Couples Effects

Let us consider the action of body couples. Since $\mathbf{X} = 0$, therefore also $\vartheta = 0$, $\boldsymbol{\chi} = 0$. No longitudinal wave ($\Phi^* = 0$) will occur in the infinite space. We must now solve the set of equations

$$(3.1) \qquad (\nabla^2+k_3^2)\sum{}^* = -\frac{1}{c_3^2}\sigma^*,$$

$$(3.2) \qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)\boldsymbol{\Psi}^* = \frac{r}{c_4^2}\operatorname{rot}\boldsymbol{\eta}^*,$$

$$(3.3) \qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)\mathbf{H}^* = -\frac{1}{c_4^2}D_1\boldsymbol{\eta}^*.$$

If the body couples $\mathbf{Y}$ are distributed over a closed region B, the quantities σ^* and $\boldsymbol{\eta}^*$ will be found from the equations:

$$(3.4) \qquad \sigma^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\int_B Y_j^*(\boldsymbol{\xi})\frac{\partial}{\partial x_j}\left(\frac{1}{R(\mathbf{x},\boldsymbol{\xi})}\right)dV(\boldsymbol{\xi}), \quad j = 1, 2, 3.$$

$$(3.5) \qquad \chi_i^*(\mathbf{x}) = -\frac{1}{4\pi\varrho}\int_B \varepsilon_{ijk}Y_j^*(\boldsymbol{\xi})\frac{\partial}{\partial x_k}\left(\frac{1}{R(\mathbf{x},\boldsymbol{\xi})}\right)dV(\boldsymbol{\xi}), \quad i, j, k = 1, 2, 3.$$

On introducing in these equations the expression

$$Y_j^*(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)\delta_{1j}, \quad j = 1, 2, 3,$$

— that is, a concentrated body couple acting at the origin in the x_1 direction — we obtain:

$$(3.6) \qquad \sigma^* = -\frac{1}{4\pi J}\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right), \quad \eta_1^* = 0, \quad \eta_2^* = \frac{1}{4\pi J}\frac{\partial}{\partial x_3}\left(\frac{1}{R}\right),$$

$$\eta_3^* = -\frac{1}{4\pi J}\frac{\partial}{\partial x_2}\left(\frac{1}{R}\right).$$

On solving Eqs. (3.1) to (3.3) in the same manner as was done in Sec. 2, we find:

$$(3.7) \qquad \sum{}^* = -\frac{1}{4\pi J c_3^2 k_3^2}\frac{\partial}{\partial x_1}\left(\frac{e^{ik_3R}-1}{R}\right),$$

$$(3.8) \qquad \Psi_j^* = \frac{r}{4\pi J c_4^2(k_1^2-k_2^2)}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right)\delta_{j1} + \frac{r}{4\pi J c_4^2}\partial_1\partial_j\left(B_1\frac{e^{ik_1R}}{R}+B_2\frac{e^{ik_2R}}{R}+\frac{B_3}{R}\right)$$

$$(3.9) \qquad H_j^* = \frac{1}{4\pi J c_4^2}\varepsilon_{1jk}\frac{\partial}{\partial x_k}\left(C_1\frac{e^{ik_1R}}{R}+C_2\frac{e^{ik_2R}}{R}+C_3\frac{1}{R}\right),$$

where
$$C_1 = \frac{k_1^2-\sigma_2^2}{k_1^2(k_1^2-k_2^2)}, \quad C_3 = \frac{k_2^2-\sigma_2^2}{k_2^2(k_2^2-k_1^2)}, \quad C_3 = -\frac{\sigma_2^2}{k_1^2 k_2^2}.$$

The displacements and rotations will be found from the equations

$$\text{(3.10)} \quad \begin{aligned} &u_1^* = \partial_2 \Psi_3^* - \partial_3 \Psi_2^*, \quad u_2^* = \partial_3 \Psi_1^* - \partial_1 \Psi_3^*, \quad u_3^* = \partial_1 \Psi_2^* - \partial_2 \Psi_1^*, \\ &\omega_1^* = \partial_1 \Sigma^* + \partial_2 H_3^* - \partial_3 H_2^*, \quad \omega_2^* = \partial_2 \Sigma^* - \partial_1 H_3^*, \quad \omega_3^* = \partial_3 \Sigma^* + \partial_1 H_2^*. \end{aligned}$$

On substituting (3.7)–(3.9) in (3.10), we obtain:

$$\text{(3.11)} \qquad u_j^* = V_j^{*(1)} = \frac{r}{4\pi J c_4^2 (k_1^2-k_2^2)} \epsilon_{1jk} \left(\frac{e^{ik_1R} - e^{ik_2R}}{R} \right),$$

$$\text{(3.12)} \qquad \omega_j^* = W_j^{*(1)} = \frac{1}{4\pi J c_4^2} \left(k_1^2 C_1 \frac{e^{ik_1R}}{R} + k_2^2 C_2 \frac{e^{ik_2R}}{R} \right) \delta_{1j}$$
$$+ \frac{1}{4\pi J c_4^2} \partial_1 \partial_j \left(C_1 \frac{e^{ik_1R}}{R} + C_2 \frac{e^{ik_2R}}{R} + C_3 \frac{e^{ik_3R}}{R} \right), \quad k, j = 1, 2, 3.$$

On moving the concentrated body couple to the point ξ, and directing the body couple vector parallel to the x_l-axis, we obtain the Green tensor of displacement $V_j^{*(l)}(\mathbf{x}, \xi)$ and the rotation tensor $W_j^{*(l)}(\mathbf{x}, \xi)$.

Thus, for example, we have

$$\text{(3.13)} \qquad V_j^{*(l)}(\mathbf{x}, \xi) = \frac{r}{4\pi J c_4^2 (k_1^2-k_2^2)} \epsilon_{ljk} \frac{\partial}{\partial x_k} \left(\frac{e^{ik_1R} - e^{ik_2R}}{R} \right), \quad l, j, k = 1, 2, 3.$$

where
$$R = [(x_i - \xi_i)(x_i - \xi_i)]^{1/2}$$

On returning to Eqs. (3.7)–(3.9), let us observe that the action of the concentrated body couple $Y_j^* = \delta(x_1)\delta(x_2)\delta(x_3)\delta_{1j}$ produces zero displacement in the direction of the axis x_1 ($V_1^{*(1)} = 0$), therefore also $\gamma_{11} = 0$. Since k_1, k_2, k_3 are functions of the frequency ω, all the types of waves occurring in the expressions (3.11) and (3.12) undergo dispersion.

4. Determination of the Green Functions by Means of Stress Functions

We shall now describe in brief the other method for finding the Green function. Use will be made of the stress functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ generalized by N. Sandru, [7]. These functions are connected with the displacements and rotations by the relations:

$$\text{(4.1)} \qquad \mathbf{u} = \Box_1 \Box_4 \boldsymbol{\varphi} - \operatorname{grad} \operatorname{div} \Gamma \boldsymbol{\varphi} - 2\alpha \operatorname{rot} \Box_3 \boldsymbol{\psi},$$

$$\text{(4.2)} \qquad \boldsymbol{\omega} = \Box_2 \Box_3 \boldsymbol{\psi} - \operatorname{grad} \operatorname{div} \Theta \boldsymbol{\psi} - 2\alpha \operatorname{rot} \Box_1 \boldsymbol{\varphi},$$

where
$$\Gamma = (\lambda+\mu-\alpha)\Box_4 - 4\alpha^2, \quad \Theta = (\beta+\gamma-\varepsilon)\Box_2 - 4\alpha^2.$$

On introducing (4.1) and (4.2) in the set of Eqs. (1.6), (1.7), we obtain:

$$\text{(4.3)} \qquad \Box_1 (\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \boldsymbol{\varphi} + \mathbf{X} = 0,$$

$$\text{(4.4)} \qquad \Box_3 (\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \boldsymbol{\psi} + \mathbf{Y} = 0.$$

The particular usefulness of these equations for the determination of Green function is evident. It suffices to find a particular integral of these equations and to determine the displacement $\mathbf{u}$ and the rotation $\boldsymbol{\omega}$ from Eqs. (4.1) and (4.2).

From Eqs. (4.3), (4.4), we find that $\boldsymbol{\varphi} = 0$ for no body forces and $\boldsymbol{\psi} = 0$ for no body couples. By considering harmonic body forces and couples, Eqs. (4.3) and (4.4) can be reduced to the form:

$$(4.5)\qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)(\nabla^2+\sigma_1^2)\boldsymbol{\varphi}^*+\varkappa\mathbf{X}^* = 0,$$

$$(4.6)\qquad (\nabla^2+k_1^2)(\nabla^2+k_2^2)(\nabla^2+k_3^2)\boldsymbol{\psi}^*+\sigma\mathbf{Y}^* = 0,$$

where

$$\varkappa = \frac{1}{(\lambda+2\mu)(\mu+\alpha)(\gamma+\varepsilon)}, \qquad \sigma = \frac{1}{(\beta+2\gamma)(\mu+\alpha)(\gamma+\varepsilon)}.$$

The symbols k_1^2, k_2^2, k_3^2, σ_1^2 have the same meaning as in Sec. 2. Let us observe that the solution of the homogeneous Eqs. (4.5) and (4.6) has the form:

$$(4.7)\qquad \boldsymbol{\varphi}^* = \mathbf{A}\frac{e^{ik_1R}}{R}+\mathbf{B}\frac{e^{ik_2R}}{R}+\mathbf{C}\frac{e^{i\sigma_1R}}{R},$$

$$(4.8)\qquad \boldsymbol{\psi}^* = \mathbf{D}\frac{e^{ik_1R}}{R}+\mathbf{E}\frac{e^{ik_2R}}{R}+\mathbf{F}\frac{e^{ik_3R}}{R}.$$

It is seen that the first two wave terms of (4.7) undergo dispersion. In Eq. (4.8) all the three wave terms are dispersed.

Let us quote the equations for the amplitude of displacement and rotation:

$$(4.9)\qquad \mathbf{u}^* = (\lambda+2\mu)(\gamma+\varepsilon)(\nabla^2+k_3^2)(\nabla^2+\sigma_4^2-2p)\boldsymbol{\varphi}^*$$
$$-(\gamma+\varepsilon)(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}[(\nabla^2+\sigma_4^2-2p-\eta)]\boldsymbol{\varphi}^*-2\alpha(\beta+2\gamma)(\nabla^2+k_3^2)\operatorname{rot}\boldsymbol{\psi}^*,$$

$$(4.10)\qquad \boldsymbol{\omega}^* = (\mu+\alpha)(\beta+2\gamma)(\nabla^2+\sigma_2^2)(\nabla^2+k_3^2)\boldsymbol{\psi}^*$$
$$-(\beta+\gamma-\varepsilon)(\gamma+\varepsilon)\operatorname{grad}\operatorname{div}[\nabla^2+\sigma_2^2-2r-\zeta]\boldsymbol{\psi}^*-2\alpha(\lambda+2\mu)(\nabla^2+\sigma_1^2)\operatorname{rot}\boldsymbol{\varphi}^*,$$

where

$$\zeta = \frac{4\alpha^2}{(\beta+\gamma-\varepsilon)(\gamma+\varepsilon)}, \qquad \eta = \frac{4\alpha^2}{(\gamma+\varepsilon)(\lambda+\mu-\alpha)}.$$

Let us consider first the action of body forces. Since $\mathbf{Y}^* = 0$, therefore also $\boldsymbol{\psi}^* = 0$. It remains to consider Eq. (4.5) and to set $\boldsymbol{\psi}^* = 0$ in Eqs. (4.9), (4.10).

On applying to (4.5) the Fourier exponential transformation, and introducing the new notations

$$\mu_1 = k_1, \quad \mu_2 = k_2, \quad \mu_3 = \sigma_1,$$

we obtain, making use of the method used in [8]:

$$(4.11)\qquad \boldsymbol{\varphi}^* = \left(\frac{\mathbf{H}_1^*}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)}+\frac{\mathbf{H}_2^*}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_3^2)}+\frac{\mathbf{H}_3^*}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)}\right).$$

The vector functions $\mathbf{H}_1^*$, $\mathbf{H}_2^*$, $\mathbf{H}_3^*$ should satisfy the Helmholtz equations:

$$(4.12)\qquad (\nabla^2+\mu_1^2)\mathbf{H}_1^* = -\varkappa\mathbf{X}, \quad (\nabla^2+\mu_2^2)\mathbf{H}_2^* = -\varkappa\mathbf{X}^*, \quad (\nabla^2+\mu_3^2)\mathbf{H}_3^* = -\varkappa\mathbf{X}^*.$$

The solution of these equations is provided by the functions:

$$\mathbf{H}_j^*(\mathbf{x}) = \frac{\varkappa}{4\pi}\int_V \mathbf{X}^*(\boldsymbol{\xi})\frac{e^{ik_jR}}{R(\boldsymbol{\xi},\mathbf{x})}dV(\boldsymbol{\xi}), \quad j=1,2,3. \tag{4.13}$$

Therefore

$$\boldsymbol{\varphi}^*(\mathbf{x}) = \frac{\varkappa}{4\pi}\int_V \frac{\mathbf{X}^*(\boldsymbol{\xi})}{R(\boldsymbol{\xi},\mathbf{x})}\left(\sum_{\alpha=1}^{3}\frac{e^{i\mu_\alpha R}}{D_\alpha R}\right)dV(\boldsymbol{\xi}), \tag{4.14}$$

where

$$D_1 = \frac{1}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)}, \quad D_2 = \frac{1}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_3^2)}, \quad D_3 = \frac{1}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)}.$$

Let us assume that a concentrated force $\mathbf{X}(\mathbf{x},t) = e^{-i\omega t}(X^*,0,0)$, where $X^* = \delta(x_1)\delta(x_2)\delta(x_3)$, acts at the origin. Then, from Eq. (4.14), we shall obtain $\boldsymbol{\varphi}^* \equiv (\varphi_1^*,0,0)$, where

$$\varphi_1^* = \frac{\varkappa}{4\pi R_0}\sum_{\alpha=1}^{3}\frac{e^{i\mu_\alpha R}}{D_\alpha}, \quad R_0 = (x_1^2+x_2^2+x_3^2)^{1/2}. \tag{4.15}$$

On substituting φ_1^* of (4.15) in (4.9) and (4.10) and moving the concentrated force from the origin to the point $\boldsymbol{\xi}$ and directing it parallel to the x_l-axis, we obtain:

$$U_j^{*(l)}(\mathbf{x},\boldsymbol{\xi}) = \frac{\delta_{lj}}{4\pi R(\mu+\alpha)}\sum_{r=1}^{3}\frac{(\mu_3^2-\mu_r^2)(\sigma_4^2-2p-\mu_r^2)e^{i\mu_r R}}{D_r} - \frac{\lambda+\mu-\alpha}{4\pi(\lambda+2\mu)(\mu+\alpha)}\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_j}\left(\sum_{r=1}^{3}\frac{(\sigma_4^2-2p-\eta-\mu_r^2)e^{i\mu_r R}}{D_r R}\right), \tag{4.16}$$

$$\Omega_j^{*(l)} = \frac{-2\alpha\epsilon_{jkl}}{4\pi(\mu+\alpha)(\gamma+\varepsilon)}\frac{\partial}{\partial x_k}\left(\sum_{r=1}^{3}\frac{(\mu_3^2-\mu_r^2)}{D_r R}e^{i\mu_r R}\right). \tag{4.17}$$

It can be shown that these equations are identical with (2.20) and (2.21). For this, use must be made of the relations:

$$\mu_1^2+\mu_2^2 = \sigma_2^2+\sigma_4^2+p(r-2), \quad \mu_1^2\mu_2^2 = \sigma_2^2(\sigma_4^2-2p).$$

Let us assume that only body couples act in the body. Thus, $\mathbf{X}^* = 0$ and $\boldsymbol{\varphi}^* = 0$ The solution of Eq. (4.6) can be presented in the form:

$$\boldsymbol{\psi}^* = \sum_{\alpha=1}^{3}\frac{\boldsymbol{\Gamma}_\alpha^*}{F_\alpha}, \tag{4.18}$$

where

$$F_1 = \frac{1}{(k_1^2-k_2^2)(k_1^2-k_3^2)}, \quad F_2 = \frac{1}{(k_2^2-k_1^2)(k_2^2-k_3^2)}, \quad F_3 = \frac{1}{(k_3^2-k_1^2)(k_3^2-k_2^2)}.$$

The vector functions $\boldsymbol{\Gamma}_1^*$, $\boldsymbol{\Gamma}_2^*$, $\boldsymbol{\Gamma}_3^*$ must satisfy the Helmholtz equations:

$$(\nabla^2+k_1^2)\boldsymbol{\Gamma}_1^* = -\sigma\mathbf{Y}^*, \quad (\nabla^2+k_2^2)\boldsymbol{\Gamma}_2^* = -\sigma\mathbf{Y}^*, \quad (\nabla^2+k_3^2)\boldsymbol{\Gamma}_3^* = -\sigma\mathbf{Y}^*. \tag{4.19}$$

By analogy with the solution (4.14), we have:

$$\boldsymbol{\psi}^*(\mathbf{x}) = \frac{\sigma}{4\pi} \int\limits_V \frac{\mathbf{Y}^*(\boldsymbol{\xi})}{R(\boldsymbol{\xi}, \mathbf{x})} \left(\sum_{\alpha=1}^{3} \frac{e^{ik_\alpha R}}{F_\alpha} \right). \tag{4.20}$$

Let us consider the action of a concentrated couple at the origin parallel to the x_1-axis. On introducing in (4.20) the expression $Y_j^* = \delta(x_1)\delta(x_2)\delta(x_3)\delta_{1j}$, we obtain:

$$\boldsymbol{\psi}^* \equiv (\psi_1^*, 0, 0), \quad \psi_1^* = \frac{\sigma}{4\pi R_0} \sum_{\alpha=1}^{3} \frac{e^{ik_\alpha R_0}}{F_\alpha}. \tag{4.21}$$

On introducing $\boldsymbol{\psi}^*$ of (4.21) and $\boldsymbol{\varphi}^* = 0$ in (4.9) and (4.10), moving the concentrated body couple to the point $\boldsymbol{\xi}$ and directing it parallel to the x_l-axis, we shall obtain the following expressions for displacements and rotations:

$$u_j^* = V_j^{*(l)} = -\frac{2\alpha\varepsilon_{kjl}}{4\pi(\mu+\alpha)(\gamma+\varepsilon)} \frac{\partial}{\partial x_k} \left(\sum_{s=1}^{3} \frac{e^{ik_sR}(k_3^2-k_s^2)}{F_s R} \right), \tag{4.22}$$

$$\omega_j^* = W_j^{*(l)} = \frac{\delta_{jl}}{4\pi(\gamma+\varepsilon)} \sum_{s=1}^{3} \frac{(\sigma_2^2-k_s^2)(k_3^2-k_s^2)e^{ik_sR}}{F_s R} \tag{4.23}$$

$$-\frac{(\beta+\gamma-\varepsilon)}{4\pi(\beta+2\gamma)(\mu+\alpha)} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \left(\sum_{s=1}^{3} \frac{(\sigma_2^2-2r-\xi-k_s^2)e^{ik_sR}}{RF_s} \right), \quad j, l = 1, 2, 3.$$

It can easily be seen, after some minor rearrangements, that these last equations are identical with those obtained in Sec. 2.

5. Two-dimensional Problems

Let a body force $X_j = \delta(x_1)\delta(x_2)\delta_{1j}e^{i\omega t}$ act in the infinite elastic body in the direction of the x_1-axis and let these forces be uniformly distributed along the x_3-axis. In this case, the displacements and the rotations are independent of the variable x_3, and we are concerned with the two-dimensional problem.

The relevant equations for the two-dimensional problem will be obtained from the equations of the foregoing sections by means of the principle of superposition.

Let us start out from the Green function for displacement $U_j^{*(1)}$, Eq. (2.18), assuming that the concentrated force acts at the point $(0, 0, \xi_3)$ in the direction of the x_1-axis. On integrating the function $U_j^{*(1)}$ along the x_3-axis from $-\infty$ to ∞, we shall find the corresponding equations for the displacement in the two-dimensional problem.

Let us observe that

$$\int\limits_{-\infty}^{\infty} \frac{\exp\left[ik_\alpha\sqrt{r^2+\xi_3^2}\right]}{\sqrt{r^2+\xi_3^2}} d\xi_3 = 2K_0(-ik_\alpha r), \quad \alpha = 1, 2, \tag{5.1}$$

$$r = (x_1^2+x_2^2)^{1/2},$$

where $K_0(z)$ is the modified Bessel function of the third kind. On the other hand, we have

$$2K_0(-ik_\alpha r) = \pi i H_0^{(1)}(k_\alpha r), \quad \alpha = 1, 2, \tag{5.2}$$

where $H_0^{(1)}(k_\alpha r)$ is a Hankel function.

If now we integrate the displacements $U_j^{*(1)}$ of (2.18) along the x_3-axis, we obtain, bearing in mind (5.1) and (5.2):

$$U_j^{*(1)}(x_1, x_2; 0, 0) = \frac{i}{2\varrho\omega^2}\big[\big(A_1 k_1^2 H_0^{(1)}(k_1 r) + A_2 k_2^2 H_0^{(1)}(k_2 r)\big)\delta_{1j} + \partial_1\partial_j\big(A_1 H_0^{(1)}(k_1 r) + A_2 H_0^{(1)}(k_2 r) + A_3 H_0^{(1)}(\sigma_1 r)\big)\big], \quad j = 1, 2. \tag{5.3}$$

In a similar manner, we shall also determine the rotation $\Omega_j^{*(1)}(x_1, x_2; 0, 0)$. In Eq. (2.19), we obtain:

$$\Omega_1^{*(1)} = \Omega_2^{*(1)} = 0, \quad \Omega_3^{*(1)} = -\frac{p_i \epsilon_{1jk}}{2\varrho c_2^2(k_1^2 - k_2^2)}\frac{\partial}{\partial x_k}\big(H_0^{(1)}(k_1 r) - H_0^{(1)}(k_2 r)\big). \tag{5.4}$$

We must now direct the linear force parallel to $x_l (l = 1, 2)$, and move it to the point $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Then

$$U_j^{*(l)}(x_1, x_2, \xi_1, \xi_2) = \frac{i}{2\varrho\omega^2}\big[\big(A_1 k_1^2 H_0^{(1)}(k_1 r) + A_2 k_2^2 H_0^{(1)}(k_2 r)\big)\delta_{jl} + \partial_l\partial_j\big(A_1 H_0^{(1)}(k_1 r) + A_2 H_0^{(1)}(k_2 r) + A_3 H_0^{(1)}(\sigma_1 r)\big)\big], \quad j, l = 1, 2. \tag{5.5}$$

The Green functions can also be determined starting from Eqs. (4.5) and (4.6), and treating them as concerning a two-dimensional problem.

The solution of (4.5) will be assumed in the form (4.11). However, the functions $\mathbf{H}_1^*$, $\mathbf{H}_2^*$, $\mathbf{H}_3^*$ should satisfy the two-dimensional Helmholtz equations

$$\begin{aligned} &(\partial_1^2 + \partial_2^2 + \mu_1^2)\mathbf{H}_1^* = -\varkappa \mathbf{X}^*, \quad (\partial_1^2 + \partial_2^2 + \mu_2^2)\mathbf{H}_2^* = -\varkappa \mathbf{X}^*, \\ &(\partial_1^2 + \partial_2^2 + \mu_3^2)\mathbf{H}_3^* = -\varkappa \mathbf{X}^*, \quad \mathbf{H}_1^* = (\overset{1}{H}_1^*, \overset{2}{H}_2^*), \quad \mathbf{X}^* = (X_1^*, X_2^*). \end{aligned} \tag{5.6}$$

For the concentrated force $X_j^* = \delta(x_1)\delta(x_2)\delta_{1j}$ acting in the x_1-direction, we obtain, from Eqs. (5.6):

$$\begin{aligned} &\overset{1}{H}_1^* = \frac{i\varkappa}{4} H_0^{(1)}(\mu_1 r), \quad \overset{2}{H}_1^* = \overset{3}{H}_1^* = 0, \\ &\overset{1}{H}_2^* = \frac{i\varkappa}{4} H_0^{(1)}(\mu_2 r), \quad \overset{2}{H}_2^* = \overset{3}{H}_2^* = 0, \\ &\overset{1}{H}_3^* = \frac{i\varkappa}{4} H_0^{(1)}(\mu_3 r), \quad \overset{2}{H}_3^* = \overset{3}{H}_3^* = 0. \end{aligned} \tag{5.7}$$

On substituting the above in (4.14), we find $\boldsymbol{\varphi}^* = (\varphi_1^*, 0, 0)$, where

$$\varphi_1^* = \frac{\varkappa i}{4}\sum_{\alpha=1}^{3}\frac{1}{D_\alpha} H_0^{(1)}(\mu_\alpha r). \tag{5.8}$$

On substituting (5.8) in (4.9) and (4.10) (in which all the derivatives with respect to x_3 should be rejected), we shall obtain Eqs. (5.3) and (5.4).

6. Singular Solution of Higher Order

Let us consider first the spatial problem. Let a concentrated force of intensity $\frac{P}{\Delta\xi_1}e^{-i\omega t}$ act at the point $\left(\xi_1+\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)$ parallel to the x_1-axis, and let a force of the same intensity act at the point $\left(\xi_1-\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)$ in the direction of the negative x_1-axis. Then, the amplitude of the displacement u_j^* produced by these forces will be:

$$(6.1)\qquad u_j^* = \frac{P}{\Delta\xi_1}\left[U_j^{*(1)}\left(x_1, x_2, x_3; \xi_1-\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right) - U_j^{*(1)}\left(x_1, x_2, x_3; \xi_1+\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)\right].$$

By letting $\Delta\xi_1 \to 0$, we shall obtain the displacement $\hat{U}_j^{*(1)}$ for what is referred to as a double force without moment:

$$(6.2)\qquad \hat{U}_j^{*(1)} = -P\frac{\partial}{\partial\xi_1}U_j^{*(1)}(\mathbf{x}, \boldsymbol{\xi}).$$

Similarly, for the double force, we obtain the following rotation function

$$(6.3)\qquad \hat{\Omega}_j^{*(1)} = -P\frac{\partial}{\partial\xi_1}\Omega_j^{*(1)}(\mathbf{x}, \boldsymbol{\xi}).$$

Generally, if a double force without moment acts at the point $\boldsymbol{\xi}$ in the direction of the x_l-axis, the corresponding singularities are given by the equations:

$$(6.4)\qquad \hat{U}_j^{*(l)} = -P\frac{\partial}{\partial\xi_l}U_j^{*(l)}(\mathbf{x}, \boldsymbol{\xi}),$$

$$(6.5)\qquad \hat{\Omega}_j^{*(l)} = -P\frac{\partial}{\partial\xi_l}\Omega^{*(l)}(\mathbf{x}, \boldsymbol{\xi}),$$

with the functions $U_j^{*(l)}$, $\Omega_j^{*(l)}$ as expressed by Eqs. (2.20) and (2.21).

Let now three double forces of intensity $Pe^{-i\omega t}$ act in the direction of the x_1, x_2 and x_3-axis.

It is known that such a set of forces constitutes what is termed *centre of compression* or *nucleus of dilatation*. Let us denote by $\bar{U}_{j_i}^*$ the displacement components, and by $\bar{\Omega}_j$ the rotation components. Making use of the results obtained for double forces, we shall obtain, by superposition, the expression:

$$(6.6)\qquad \bar{U}_j^* = -\left(\frac{\partial}{\partial\xi_1}U_j^{*(1)} + \frac{\partial}{\partial\xi_2}U_j^{*(2)} + \frac{\partial}{\partial\xi_3}U_j^{*(3)}\right) = \frac{1}{4\pi\varrho c_1^2}\frac{e^{i\sigma R}}{R},$$

$$(6.7)\qquad \bar{\Omega}_j^* = 0.$$

It can easily be shown that a compression centre produces only longitudinal waves.

Let a force $\frac{M}{\Delta\xi_1}e^{-i\omega t}$ act at the point $\left(\xi_1+\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)$ in the direction of the positive x_2-axis and let the same force act at the point $\left(\xi_1-\frac{\Delta\xi_1}{2}\xi_2, \xi_3\right)$ in the opposite direction. Then,

$$u_j^* = \frac{M}{\Delta\xi_1}\left[U_j^{*(2)}\left(x_1, x_2, x_3; \xi_1-\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right) - U_j^{*(2)}\left(x_1, x_2, x_3; \xi_1+\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)\right].$$

By making $\Delta\xi_1$ tends to zero, we find the displacement u_j^*, corresponding to the double force with moment

$$u_j^* = -M\frac{\partial}{\partial\xi_1}U_j^{*(2)}. \tag{6.8}$$

Let now a force $\frac{Me^{-i\omega t}}{\Delta\xi_2}$ act at the point $\left(\xi_1, \xi_2+\frac{\Delta\xi_2}{2}, \xi_3\right)$ in the direction of the negative x_1-axis and let a force of the same intensity act at the point $\left(\xi_1, \xi_2-\frac{\Delta\xi_2}{2}, \xi_3\right)$ in the direction of the x_1-axis. As a result, we obtain:

$$u_j^* = M\frac{\partial}{\partial\xi_2}U_j^{*(1)}. \tag{6.9}$$

The sum of these two double forces with moment will produce the displacements:

$$u_j^* = -M\left(\frac{\partial U_j^{*(2)}}{\partial\xi_1}-\frac{\partial U_j^{*(1)}}{\partial\xi_2}\right). \tag{6.10}$$

Similarly, we obtain:

$$\omega_j^* = -M\left(\frac{\partial \Omega_j^{*(2)}}{\partial\xi_1}-\frac{\partial \Omega_j^{*(1)}}{\partial\xi_2}\right). \tag{6.11}$$

Making use of Eqs. (2.20) and (2.21), we obtain

$$\mathbf{u}^* \equiv \frac{M}{4\pi\varrho\omega^2}\left(\frac{\partial F}{\partial\xi_2}, -\frac{\partial F}{\partial\xi_1}, 0\right), \tag{6.12}$$

where

$$F = A_1k_1^2\frac{e^{ik_1R}}{R}+A_2k_2^2\frac{e^{ik_2R}}{R},$$

and

$$\omega_j^* = -\frac{M}{4\pi\varrho c_2^2(k_1^2-k_2^2)}\left(\varepsilon_{2jk}\frac{\partial}{\partial\xi_1}-\varepsilon_{1jk}\frac{\partial}{\partial\xi_2}\right)\frac{\partial}{\partial x_3}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right), \tag{6.13}$$

or

$$\omega_j^* = \frac{M}{4\pi\varrho c_2^2(k_1^2-k_2^2)}\left[\partial_{j3}\Gamma^*(\mathbf{x})+\frac{\partial}{\partial\xi_j}\frac{\partial}{\partial x_3}\Gamma^*(\mathbf{x})\right], \tag{6.13'}$$

where

$$\Gamma^*(\mathbf{x}) = \frac{e^{ik_1R}-e^{ik_2R}}{R}.$$

Let now a concentrated couple of intensity $\frac{\mathfrak{M}}{\Delta\xi_1}e^{-i\omega t}$ act at the point $\left(\xi_1+\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)$ in the direction of the x_1-axis, and let a concentrated couple of the same intensity act at the point $\left(\xi_1-\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)$ in the direction of the negative x_1-axis.

The amplitude u_j^* resulting from these two body couples is:

$$u_j^* = \frac{\mathfrak{M}}{\Delta\xi_1}\left[V_j^{*(1)}\left(x_1, x_2, x_3; \xi_1-\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)-V_j^{*(1)}\left(x_1, x_2, x_3; \xi_1+\frac{\Delta\xi_1}{2}, \xi_2, \xi_3\right)\right]. \tag{6.14}$$

If $\Delta\xi_1$ tends to zero, we obtain the displacement $\hat{V}_j^{*(1)}$ for the double couple:

$$\hat{V}_j^{*(1)} = -\mathfrak{M}\frac{\partial}{\partial\xi_1}V_j^{*(1)}. \tag{6.15}$$

Similarly, for the rotation function, we have

$$\hat{W}_j^{*(1)} = -\mathfrak{M}\frac{\partial}{\partial\xi_1}W_j^{*(1)}, \tag{6.16}$$

the functions $V_j^{*(1)}$ and $W_j^{*(1)}$ being given by (3.11) and (3.12). If now three double couples of intensity $\mathfrak{M}e^{-i\omega t}$ act in the directions of the x_1, x_2 and x_3-axis, then, by superposition, we find that

$$\overline{V}_j^* = -\mathfrak{M}\left(\frac{\partial}{\partial\xi_1}V_j^{*(1)}+\frac{\partial}{\partial\xi_2}V_j^{*(2)}+\frac{\partial}{\partial\xi_3}V_j^{*(3)}\right) = 0, \tag{6.17}$$

$$\overline{W}_j^* = \frac{\mathfrak{M}}{4\pi Jc_3^2k_3^2}\frac{\partial}{\partial\xi_j}\left(\frac{e^{ik_3R}}{R}\right), \quad j = 1, 2, 3. \tag{6.18}$$

The action of the three double couples can be treated as that of *centre of torsion*. It is of interest to observe that there is no displacement field and the function $\overline{W}_1^*$ satisfied the homogeneous Eq. (1.18).

Let us now consider Eqs. (6.12) and (6.13). In the classical theory of elasticity, Eq. (6.12) is treated as a vector of displacement produced by the action of a concentrated moment acting at the origin and directed along the negative x-axis. On confronting this equation with (3.11), which takes now the form:

$$u_j^* = -\frac{Mr}{4\pi Jc_4^2(k_1^2-k_2^2)}\varepsilon_{3jk}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_1R}-e^{ik_2R}}{R}\right), \tag{6.19}$$

it is seen that the results are not in agreement. This results from the fact that in the micropolar theory of elasticity a concentrated body couple is a fundamental load, similarly to concentrated forces. The above problem has been analysed in detail by P. P. Teodorescu [9] in the static case.

Our considerations are also valid for the two-dimensional problem. Let us consider the case of a linear centre of compression. Let us make use of Eq. (6.6) which takes a somewhat different form:

$$\overline{U}_j^* = -\left(\frac{\partial}{\partial\xi_1}U_j^{*(1)}+\frac{\partial}{\partial\xi_2}U_j^{*(2)}\right), \quad \overline{\Omega}_j^* = 0, \tag{6.20}$$

where the displacement vector is taken from Eq. (5.5). As a result, we find:

$$U_j^*(x_1, x_2; \xi_1, \xi_2) = \frac{1}{4\pi\varrho c_1^2}H_0^{(1)}(\sigma r), \tag{6.21}$$

where

$$r = [(x_1-\xi_1)^2+(x_2-\xi_2)^2]^{1/2}.$$

7. Conclusions Following from the Reciprocity Theorem

One of the fundamental theorems of the theory of elasticity is the theorem of reciprocity of works. For a body with micropolar elasticity, and if causes and effects vary with time in a harmonic manner, we have [10]:

$$\int_V (X_i^* u_i'^* + Y_i^* \omega_i'^*) dV = \int_V (X_i'^* u_i^* + Y_i'^* \omega_i^*) dV. \tag{7.1}$$

In the form (7.1), the reciprocity theorem concerns, of course, an infinite body.

Let us consider loads of two types

a. Let a concentrated force $X_j^* = \delta(\mathbf{x}-\boldsymbol{\xi})\delta_{jr}$ act at a point $\boldsymbol{\xi}$, thus producing a displacement $U_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi})$ field and a rotation field $\Omega_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi})$. Let now a concentrated force $X_j^{*'} = \delta(\mathbf{x}-\boldsymbol{\eta})\delta_{jl}$ act at a point $\boldsymbol{\eta}$, parallel to the x_l-axis. This force will produce in the body a displacement $U_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta})$ and a rotation $\Omega_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta})$. From the reciprocity theorem (7.1), we have

$$\int_V \delta(\mathbf{x}-\boldsymbol{\xi})\delta_{jr} U_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta}) dV(\mathbf{x}) = \int_V \delta(\mathbf{x}-\boldsymbol{\eta})\delta_{jl} U_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi}) dV(\mathbf{x}).$$

Hence

$$V_r^{*(l)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = U_l^{*(r)}(\boldsymbol{\eta}, \boldsymbol{\xi}). \tag{7.2}$$

b. Let a concentrated body couple $Y_j^{-*} = \delta(\mathbf{x}-\boldsymbol{\xi})\delta_{jr}$ act at the point $\boldsymbol{\xi}$ and a body couple $Y_j^{*'} = \delta(\mathbf{x}-\boldsymbol{\eta})\delta_{jl}$ at the point $\boldsymbol{\eta}$. The body couple Y_j^* is connected with a field $V_j^{*(r)}$ and $W_j^{*(r)}$ and the body couple $Y_j^{*'}$ with a field $V_j^{*(l)}$ and $W_j^{*(l)}$. From Eq. (7.1), we obtain

$$V_r^{*(l)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = V_l^{*(r)}(\boldsymbol{\eta}, \boldsymbol{\xi}). \tag{7.3}$$

It can easily be seen, from (2.20) and (3.12), that the equations (7.2) and (7.3) are satisfied.

c. Let a concentrated force $X_j^* = \delta(\mathbf{x}-\boldsymbol{\xi})\delta_{jr}$ act at the point $\boldsymbol{\xi}$, thus producing a field $U_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi})$ and $\Omega_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi})$. Let now a concentrated body couple $Y_j'^* = \delta(\mathbf{x}-\boldsymbol{\eta})\delta_{jl}$ act at the point $\boldsymbol{\eta}$, in the direction of the x_l-axis, thus producing a displacement field $V_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta})$ and a rotation field $W_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta})$.

From the reciprocity theorem (7.1), we have

$$\int_V \delta(\mathbf{x}-\boldsymbol{\xi})\delta_{jr} V_j^{*(l)}(\mathbf{x}, \boldsymbol{\eta}) dV(\mathbf{x}) = \int_V \delta(\mathbf{x}-\boldsymbol{\eta})\delta_{jl} \Omega_j^{*(r)}(\mathbf{x}, \boldsymbol{\xi}) dV(\mathbf{x}).$$

Hence

$$V_r^{*(l)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \Omega_l^{*(r)}(\boldsymbol{\eta}, \boldsymbol{\xi}). \tag{7.4}$$

Making use of (2.21) and (3.13), we have:

$$\Omega_l^{*(r)}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{p}{4\pi\varrho c_2^2(k_1^2-k_2^2)} \varepsilon_{rlk} \left| \frac{\partial}{\partial x_k} \left(\frac{e^{ik_1R} - e^{ik_2R}}{R(\mathbf{x}, \boldsymbol{\xi})} \right) \right|_{\mathbf{x}=\boldsymbol{\eta}},$$

$$V_r^{*(l)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{r}{4\pi J c_4^2(k_1^2-k_2^2)} \varepsilon_{lrk} \left| \frac{\partial}{\partial x_k} \left(\frac{e^{ik_1R} - e^{ik_2R}}{R(\mathbf{x}, \boldsymbol{\eta})} \right) \right|_{\mathbf{x}=\boldsymbol{\xi}}.$$

It is evident, bearing in mind that $r = 2\alpha/\varrho c_2^2$, $p = 2\alpha/J c_4^2$, that the relation (7.4) is satisfied.

Thus, making use of the reciprocity theorem, we have obtained additionally a proof of correctness of the equations obtained in Secs. 2 and 3.

The relations (7.2)–(7.4) can be treated as a generalization of the familiar reciprocity theorem of J. C. Maxwell known from the classical dynamic theory of elasticity (elastokinetics).

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S t r e s z c z e n i e

FUNKCJE GREENA MIKROPOLARNEJ SPRĘŻYSTOŚCI

W pracy podano podstawowe rozwiązania równań różniczkowych mikropolarnej sprężystości (micropolar elasticity). Podano funkcje Greena (tensory przemieszczenia i obrotu) dla siły skupionej i momentu skupionego, działający w nieskończonym ośrodku sprężystym tak dla zagadnienia trój- jak i dwuwymiarowego. Omówiono wreszcie osobliwe rozwiązanie wyższych rzędów.

Р е з ю м е

ФУНКЦИИ ГРИНА МИКРОПОЛЯРНОЙ УПРУГОСТИ

В работе даются фундаментальные решения дифференциальных уравнений микрополярной упругости (micropolar elasticity) Даются функции Грина (тензоры перемещений и вращений) для сосредоточенной силы и сосредоточенного момента, действующих в бесконечной упругой среде, так для трехмерной, как и двухмерной проблемы. Наконец обсуждены особые решения высших порядков.

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