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THERMOELASTIC WAVE-MOTIONS IN AN INFINITE BODY

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1. Introduction

Coupled thermoelasticity is a synthesis of the theory of elasticity and that of heat conduction. Within the framework of this theory, which is founded on the thermodynamics of irreversible processes, a number of general theorems have been proved. The most important of them include the variational theorems, [1, 2], and the reciprocity theorem [3]. In the present paper, we shall be concerned with the propagation of a longitudinal thermoelastic wave in an infinite body, our aim being to prove the generalized G. Kirchhoff theorem. The problem is to represent the potential of thermoelastic displacement and the temperature at an internal point of the region, by means of surface integrals involving the potential as well as the temperature and the normal derivatives of these quantities.

After the generalized Kirchhoff theorem has been derived, two particular cases are considered. The first is that of the theory of thermal stresses in which the coupling between the temperature and the displacement field has been disregarded. The other case concerns the passage from the coupled thermoelasticity to the classical dynamical theory of elasticity.

Finally, the generalized Kirchhoff equation is expressed in an approximate manner making use of the perturbation method.

2. Equations of Thermoelasticity

Let us consider a homogeneous isotropic elastic body, in the region B. The following linearized differential equations are valid [1]:

$$\mu u_{i,kk}+(\lambda+\mu)u_{k,ki}+X_i=\gamma\theta_{,i}+\varrho_0\ddot{u}_i, \tag{2.1}$$

$$\theta_{,kk}-\frac{1}{\varkappa}\dot{\theta}-\beta_0\dot{u}_{k,k}=-\frac{Q_0}{\varkappa}, \qquad i,j=1,2,3. \tag{2.2}$$

Equations (2.1) are the displacement equations; Eq. (2.2) is the generalized heat equation. The quantities $u_i(\mathbf{x},t)$ are the components of the displacement vector and $\theta(\mathbf{x},t)= T-T_0$ is the temperature increase. By T, we denote the absolute temperature at the point $\mathbf{x}$ and the time t; T_0 is the temperature of the natural state, in which the stresses and strains are zero. $X_i(\mathbf{x},t)$ are the components of the vector of mass forces. The function $Q(\mathbf{x}, t) = W/c_\varepsilon$ expresses the intensity of the heat sources. By W, we denote the quantity of heat produced per unit time and volume, and by c_ε—the specific heat with constant strain.

The quantities μ, λ are Lamé's constants, referred to the isothermal state, ϱ_0—is the density and $\gamma = 3K\alpha_t$, where $K = \lambda+(2/3)\mu$ is the modulus of compressibility and α_t is the coefficient of linear thermal dilatation.

Next $\varkappa = \lambda_0/c_\varepsilon$, where λ_0 is the coefficient of heat conduction and $\beta_0 = \gamma T_0/\lambda_0$. The dot above the symbol of a function denotes differentiation with respect to time.

Equations (2.1) and (2.2) should be completed by the Duhamel-Neumann equations expressing the dependency of the stress and strain tensor on the temperature

$$\sigma_{ij} = 2\mu\varepsilon_{ij}+(\lambda\varepsilon_{kk}-\gamma\theta)\,\delta_{ij}, \tag{2.3}$$

and the relations between the strain and the displacement are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}). \tag{2.4}$$

On resolving the displacement vector into the potential part and the solenoidal part

$$u_i = \Phi_{,i}+\epsilon_{ijk}\psi_{k,j}, \tag{2.5}$$

we reduce the set of Eqs. (2.1), (2.2) (with $X_i = 0$, $Q = 0$) to the simpler set of equations

$$\left(\nabla^2-\frac{1}{c_1^2}\partial_t^2\right)\Phi = m_0\theta, \tag{2.6}$$

$$\left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)\psi_i = 0, \tag{2.7}$$

$$\left(\nabla^2-\frac{1}{\varkappa}\partial_t\right)\theta-\beta_0\partial_t\nabla^2\Phi = 0, \tag{2.8}$$

where the following notations have been introduced

$$c_1 = \left(\frac{\lambda+2\mu}{\varrho_0}\right)^{1/2}, \quad c_2 = \left(\frac{\mu}{\varrho_0}\right)^{1/2}, \quad m_0 = \frac{\gamma}{c_1^2\varrho_0},$$

$$\beta_0 = \frac{T_0\gamma}{c_\varepsilon\varkappa}, \quad \partial_t = \frac{\partial}{\partial t}, \quad \nabla^2 = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}.$$

Equation (2.6) represents a longitudinal wave, (2.7)—a transversal wave, and (2.8) is the generalized heat equation.

In further considerations, it will be more convenient to introduce the following new variables:

$$\zeta_i = \frac{c_1}{\varkappa}x_i, \quad \tau = \frac{c_1^2}{\varkappa}t.$$

By using these variables, the wave Eqs. (2.6) to (2.8) assume a somewhat simpler form

$$(\nabla^2-\partial^2\tau)\,\Phi(\zeta, \tau) = m\theta(\zeta, \tau), \tag{2.9}$$

(2.10) $$(\nabla^2-\sigma^2\partial^2\tau)\psi_i(\boldsymbol{\zeta},\tau)=0,$$

(2.11) $$(\nabla^2-\partial\tau)\theta(\boldsymbol{\zeta},\tau)-\beta\partial_\tau\nabla^2\Phi(\boldsymbol{\zeta},\tau)=0,$$

where

$$\nabla^2=\frac{\partial}{\partial\zeta_i}\frac{\partial}{\partial\zeta_i},\qquad \partial_\tau=\frac{\partial}{\partial\tau},\qquad m=m_0\frac{\varkappa^2}{c_1^2},$$
$$\beta=\frac{\beta_0c_1^2}{\varkappa},\qquad \sigma^2=\frac{c_1^2}{c_2^2}.$$

3. The Integral Form of the Solution of the Equation of Potential of Thermoelastic Displacement

Let us consider the inner region B^+ bounded by the surface A. By B^- we shall denote the complement to the region B^+ in the infinite three-dimensional space E_3.

We shall consider the propagation of a longitudinal wave in the internal region B^+. The representation of the function Φ at the point $\boldsymbol{\eta}\in B^+$ in terms of the functions θ, $\partial\theta/\partial n$, Φ, $\partial\Phi/\partial n$ on a closed surface A is of particular interest.

Let the functions Φ,θ be regular in B^+, and have their first and second derivatives continuous in the closed region B^+UA. Let us assume next that the initial conditions for the functions Φ and θ are homogeneous. We consider therefore the set of equations

(3.1) $$(\nabla^2-\partial_\tau^2)\Phi=m\theta,\quad (\nabla^2-\partial_\tau)\theta-\beta\partial_\tau\nabla^2\Phi=0,$$

with no singularities in the region B^+UA. Eliminating from these equations the temperature, we obtain

(3.2) $$[(\nabla^2-\partial_\tau^2)(\nabla^2-\partial_\tau)-m\beta\partial_\tau\nabla^2]\Phi(\boldsymbol{\zeta},\tau)=0.$$

On performing on this equation the Laplace transformation and taking into consideration the homogeneity of the initial conditions, we have

(3.3) $$\Box_{\lambda_1}^2\Box_{\lambda_2}^2\tilde{\Phi}(\boldsymbol{\zeta},p)=0,$$

where

$$\Box_{\lambda_\alpha}^2=\nabla^2-\lambda_\alpha^2,\quad \alpha=1,2,\quad \tilde{\Phi}(\boldsymbol{\zeta},p)=\mathscr{L}(\Phi(\boldsymbol{\zeta},\tau))=\int_0^\infty\Phi(\boldsymbol{\zeta},\tau)e^{-p\tau}d\tau,$$

The quantities λ_1,λ_2 are the roots of the equation

$$\lambda^4-\lambda^2p[p+(1+\varepsilon)]+p^3=0,\quad \varepsilon=\beta m,$$

where

$$\left.\begin{matrix}\lambda_1^2\\ \lambda_2^2\end{matrix}\right\}=\frac{p}{2}\{p+1+\varepsilon\pm[p^2-2p(1-\varepsilon)+(1+\varepsilon)^2]^{1/2}\}.$$

Let us consider the auxiliary solution $G(\boldsymbol{\zeta},\boldsymbol{\eta},\tau)$, which is the solution of the equation

(3.4) $$[(\nabla^2-\partial_\tau^2)(\nabla^2-\partial_\tau)-\varepsilon\partial_\tau\nabla^2]G(\boldsymbol{\zeta},\eta,\tau)=-m\delta(\boldsymbol{\zeta}-\boldsymbol{\eta})\,\delta(\tau),\qquad \boldsymbol{\eta}\in B^+$$

in the infinite region.

The right-hand member of this equation represents the action of a concentrated instantaneous impulse at the point $\boldsymbol{\eta}$. It can easily be observed that this impulse is equivalent to the action of a concentrated instantaneous heat source at the point $\boldsymbol{\eta}$. It is assumed that the initial conditions $G(\boldsymbol{\zeta}, \boldsymbol{\eta}, 0)$, $\dot{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, 0)$ are homogeneous. On performing on (3.4) the Laplace transformation, we find

$$\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p) = -m\delta(\boldsymbol{\zeta}-\boldsymbol{\eta}). \tag{3.5}$$

The solution of this equation is already known [4, 5], and has the form:

$$\tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p) = \frac{m(e^{-\lambda_1\varrho}-e^{-\lambda_2\varrho})}{4\pi\varrho(\lambda_1^2-\lambda_2^2)}, \qquad \varrho^2 = (\zeta_i-\eta_i)(\zeta_i-\eta_i). \tag{3.6}$$

The function G represents a spherical wave with its centre at the point $\boldsymbol{\eta}$, propagating from this centre to infinity. Let us denote by $K(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)$ the temperature accompanying the potential $G(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau)$. On performing on the equation

$$(\nabla^2-\partial_\tau^2)G(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) = mK(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) \tag{3.7}$$

the Laplace transformation, we obtain

$$\tilde{K}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p) = \frac{1}{m}D_1\tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p), \qquad D_1 = \nabla^2-p^2.$$

Hence,

$$\tilde{K}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p) = \frac{(\lambda_1^2-p^2)e^{-\lambda_1\varrho}-(\lambda_2^2-p^2)e^{-\lambda_2\varrho}}{4\pi\varrho(\lambda_1^2-\lambda_2^2)}. \tag{3.8}$$

In order to obtain an identity determining the function $\tilde{\Phi}$ in the region B^+ for $\boldsymbol{\eta} \in B^+$ in terms of volume and surface integrals, we start out from the equation

$$\int_V (\tilde{G}\,\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{\Phi}-\tilde{\Phi}\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{G})dV = \int_V \tilde{G}\nabla^4\tilde{\Phi}-\tilde{\Phi}\nabla^4\tilde{G}- -(\lambda_1^2+\lambda_2^2)(\tilde{G}\nabla^2\tilde{\Phi}-\tilde{\Phi}\nabla^2\tilde{G}))dV. \tag{3.9}$$

On applying the formula for the bi-Laplacian

$$\int_V (\tilde{G}\nabla^4\tilde{\Phi}-\tilde{\Phi}\nabla^4\tilde{G})dV = \int_A \left(\nabla^2\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n}-\tilde{\Phi}\frac{\partial}{\partial n}\nabla^2\tilde{G}+\tilde{G}\frac{\partial}{\partial n}\nabla^2\tilde{\Phi}-\nabla^2\tilde{\Phi}\frac{\partial\tilde{G}}{\partial n}\right)dV \tag{3.10}$$

and the Green's transformation

$$\int_V (\tilde{G}\nabla^2\tilde{\Phi}-\tilde{\Phi}\nabla^2\tilde{G})dV = \int_A \left(\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n}-\tilde{\Phi}\frac{\partial\tilde{G}}{\partial n}\right)dA, \tag{3.11}$$

Eq. (3.9) takes the form

$$\int_V (\tilde{G}\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{\Phi}-\tilde{\Phi}\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{G})dV = \int_A \left(\tilde{G}\frac{\partial}{\partial n}\square^2\tilde{\Phi}-\tilde{\Phi}\frac{\partial}{\partial n}\square^2\tilde{G}\right)dA - \int_A \left(\nabla^2\tilde{\Phi}\frac{\partial\tilde{G}}{\partial n}-\nabla^2\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n}\right)dA, \qquad \square^2 = \nabla^2-(\lambda_1^2+\lambda_2^2). \tag{3.12}$$

Making use now of Eqs. (3.3) and (3.5), we obtain the equation

$$\tilde{\Phi}(\boldsymbol{\eta}, p) = \frac{1}{m}\int_A \left[\tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p)\frac{\partial}{\partial n}\square^2\tilde{\Phi}(\boldsymbol{\zeta}, p)-\tilde{\Phi}(\boldsymbol{\zeta}, p)\frac{\partial}{\partial n}\square^2\tilde{G}(\boldsymbol{\zeta}, \boldsymbol{\eta}, p)\right]dA(\boldsymbol{\zeta}) \tag{3.13}$$

$$-\frac{1}{m}\int_A\left[\nabla^2\tilde{\Phi}(\boldsymbol{\zeta},p)\frac{\partial}{\partial n}\tilde{G}(\boldsymbol{\zeta},\boldsymbol{\eta},p)-\nabla^2\tilde{G}(\boldsymbol{\zeta},\boldsymbol{\eta},p)\frac{\partial\tilde{\Phi}(\boldsymbol{\zeta},p)}{\partial n}\right]dA(\boldsymbol{\zeta}),$$

which is valid for $\boldsymbol{\eta}\in B^+$. If $\boldsymbol{\eta}\in B^-$, we have $\tilde{\Phi}(\boldsymbol{\eta},p)=0$.

It is known that the temperature θ is related with the potential Φ by the equation

$$(\nabla^2-\partial_\tau)\Phi=m\theta.$$

On applying to this equation the Laplace transformation, we obtain:

$$\tilde{\theta}=\frac{1}{m}D_1\tilde{\Phi},\qquad D_1=\nabla^2-p^2. \tag{3.14}$$

Making use of this relation and introducing the operator $\Box_\lambda^2=\nabla^2-(\lambda_1^2+\lambda_2^2-p^2)$, we obtain the function $\tilde{\Phi}(\boldsymbol{\eta},p)$ in its final form

$$\tilde{\Phi}(\boldsymbol{\eta},p)=\int_A\left(\tilde{G}\frac{\partial\tilde{\theta}}{\partial n}-\tilde{\theta}\frac{\partial\tilde{G}}{\partial n}\right)dA+\frac{1}{m}\int_A\left(\Box_\lambda^2\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n}-\tilde{\Phi}\frac{\partial}{\partial n}\Box_\lambda^2\tilde{G}\right)dA(\boldsymbol{\zeta}),\quad \boldsymbol{\eta}\in B^+. \tag{3.15}$$

For $\boldsymbol{\eta}\in B^-$, we have $\tilde{\Phi}(\boldsymbol{\eta},p)\equiv 0$.

Equation (3.15) expresses the function $\tilde{\Phi}(\boldsymbol{\eta},p)$ in the region B^+ in terms of the functions $\partial\tilde{\Phi}/\partial n$, $\tilde{\Phi}$, $\partial\tilde{\theta}/\partial n$, $\tilde{\theta}$ on the surface A. On applying to (3.15) the inverse Laplace transformation, we obtain

$$\begin{aligned}\Phi(\boldsymbol{\eta},\tau)=\int_0^\tau d\tau'\Bigg\{\int_A\left(G(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')\frac{\partial\theta(\boldsymbol{\zeta},\tau')}{\partial n}-\theta(\boldsymbol{\zeta},\tau')\frac{\partial G(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')}{\partial n}\right)dA(\boldsymbol{\zeta})\\+\frac{1}{m}\int_A\Bigg(\hat{G}(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')\frac{\partial\Phi(\boldsymbol{\zeta},\tau')}{\partial n}\\-\Phi(\boldsymbol{\zeta},\tau')\frac{\partial}{\partial n}\hat{G}(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')\Bigg)dA(\boldsymbol{\zeta})\Bigg\},\quad \boldsymbol{\eta}\in B^+.\end{aligned} \tag{3.16}$$

The notation $\hat{G}=\mathscr{L}^{-1}(\Box_\lambda^2\tilde{G})$ expresses the inverse Laplace transformation of the function $\Box_\lambda^2\tilde{G}$. Equation (3.16) is a generalization of the familiar Kirchhoff equation of the classical dynamic theory of elasticity to coupled problems of the thermoelasticity. If the causes producing deformation vary in time in a harmonic manner, Eq. (3.16) constitutes an analogue of the Helmholtz theorem of the classical dynamic theory of elasticity [6].

4. Integral Form of the Solution of the Temperature Equation

To determine the temperature field $\theta(\boldsymbol{\zeta},\tau)$ connected with the potential $\Phi(\boldsymbol{\zeta},\tau)$, let us consider the singular solution of the equation

$$[(\nabla^2-\partial_\tau^2)(\nabla^2-\partial_\tau)-\varepsilon\partial_\tau\nabla^2]H(\boldsymbol{\zeta},\boldsymbol{\eta},\tau)=-(\nabla^2-\partial_\tau^2)\,\delta(\boldsymbol{\zeta}-\boldsymbol{\eta})\,\delta(\tau) \tag{4.1}$$

in the infinite elastic space. It is assumed here that the concentrated instantaneous perturbation acts at the point $\boldsymbol{\eta}$ and that the initial conditions of the function H are homogeneous.

On applying to (4.1) the Laplace integral transformation, we obtain the equation

$$\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{H}(\boldsymbol{\zeta},\boldsymbol{\eta},p) = -D_1\,\delta(\boldsymbol{\zeta}-\boldsymbol{\eta}). \tag{4.2}$$

The solution of this equation is

$$\tilde{H}(\boldsymbol{\zeta},\boldsymbol{\eta},p) = \frac{D_1}{m}\,\tilde{G}(\boldsymbol{\zeta},\boldsymbol{\eta},p) = \frac{(\lambda_1^2-p^2)e^{-\varrho\lambda_1}-(\lambda_2^2-p^2)e^{-\varrho\lambda_2}}{4\pi\varrho(\lambda_1^2-\lambda_2^2)}. \tag{4.3}$$

Let us construct the following identity, analogous to (3.12)

$$\int\limits_V (\tilde{H}\,\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{\Phi}-\tilde{\Phi}\,\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{H})\,dV = \int\limits_A \left(\tilde{H}\frac{\partial}{\partial n}\,\square^2\tilde{\Phi}-\tilde{\Phi}\,\frac{\partial}{\partial n}\,\square^2\tilde{H}\right)dA - \int\limits_A \left(\nabla^2\tilde{\Phi}\,\frac{\partial\tilde{H}}{\partial n}-\nabla^2\tilde{H}\,\frac{\partial\tilde{\Phi}}{\partial n}\right)dA, \quad \square^2 = \nabla^2-(\lambda_1^2+\lambda_2^2). \tag{4.4}$$

Bearing in mind (3.3) and (4.2), we find:

$$D_1\,\tilde{\Phi}(\boldsymbol{\eta},p) = \int\limits_A \left(\tilde{H}\frac{\partial}{\partial n}\,\square^2\tilde{\Phi}-\tilde{\Phi}\,\frac{\partial}{\partial n}\,\square^2\tilde{H}\right)dA - \int\limits_A \left(\nabla^2\tilde{\Phi}\,\frac{\partial\tilde{H}}{\partial n}-\nabla^2\tilde{H}\,\frac{\partial\tilde{\Phi}}{\partial n}\right)dA. \tag{4.5}$$

Let us consider Eq. (3.14). After some simple transformations, we obtain the equation

$$m\tilde{\theta}(\boldsymbol{\eta},p) = m\int\limits_A \left(\tilde{H}\,\frac{\partial\tilde{\theta}}{\partial n}-\tilde{\theta}\,\frac{\partial\tilde{H}}{\partial n}\right)dA + \int\limits_A \left[\square^2_\lambda\tilde{H}\,\frac{\partial\tilde{\Phi}}{\partial n}-\tilde{\Phi}\,\frac{\partial}{\partial n}\,\square^2_\lambda\tilde{H}\right]dA, \qquad \boldsymbol{\eta}\in B^+, \tag{4.6}$$

the structure of which is similar to (3.15). The function $\tilde{\theta}(\boldsymbol{\eta},p)$ at the point $\boldsymbol{\eta}\in B^+$ is expressed, here also, in terms of surface integrals involving the functions $\tilde{\theta}$, $\partial\tilde{\theta}/\partial n$, $\tilde{\Phi}$, $\partial\tilde{\Phi}/\partial n$.

Equation (4.6) is valid for $\boldsymbol{\eta}\in B^+$. If $\boldsymbol{\eta}\in B^-$, we have $\tilde{\theta}(\boldsymbol{\eta},p)\equiv 0$ in the region B^+.

On applying to (4.6) the inverse integral transformation, we obtain

$$\theta(\boldsymbol{\eta},\tau) = \int\limits_0^\tau d\tau'\Bigg\{\int\limits_A \left[H(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')\frac{\partial\theta(\boldsymbol{\zeta},\tau')}{\partial n}-\theta(\boldsymbol{\zeta},\tau')\frac{\partial H(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')}{\partial n}\right)dA(\boldsymbol{\zeta}) + \frac{1}{m}\int\limits_A \left(\hat{H}(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')\frac{\partial\Phi(\boldsymbol{\zeta},\tau')}{\partial n}-\Phi(\boldsymbol{\zeta},\tau')\frac{\partial}{\partial n}\,\hat{H}(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')\right)dA(\boldsymbol{\zeta})\Bigg\}, \tag{4.7}$$

where the notation $\hat{H} = \mathscr{L}^{-1}(\square^2_\lambda H)$ has been introduced. Equation (4.7) can be treated as a generalization of the familiar Green's theorem of the theory of heat conduction to the problem of coupled thermoelasticity.

The equation for $\tilde{\theta}(\boldsymbol{\eta},p)$ can also be obtained in another way, making use of the function $\tilde{G}(\boldsymbol{\zeta},\boldsymbol{\eta},p)$. Let us perform on Eq. (3.15) the operation $\frac{1}{m}D_1$, it being borne in mind that $D_1\,\square^2_\lambda = \square^2_{\lambda_1}\square^2_{\lambda_2}+\varepsilon p^3$. Thus, bearing im mind Eqs. (3.8) and (3.14), we obtain:

$$(4.8)\qquad \tilde{\theta}(\boldsymbol{\eta}, p) = \int\limits_A \left(\tilde{K}\frac{\partial\tilde{\theta}}{\partial n} - \tilde{\theta}\frac{\partial \tilde{K}}{\partial n}\right) dA + \frac{\varepsilon p^3}{m^2}\int\limits_A \left(\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial\tilde{G}}{\partial n}\right) dA$$

for $\boldsymbol{\eta} \in B^+$ and $\tilde{\theta}(\boldsymbol{\eta}, p) \equiv 0$ for $\boldsymbol{\eta} \in B^-$.

5. The Non-coupled Problem of Thermoelasticity

In the analysis of non-coupled problems of thermoelasticity in the domain of the engineer's theory of thermal stresses, the set of Eqs. (2.6) to (2.8) is simplified by rejecting the term $\beta_0 \partial_\tau \nabla^2 \Phi$ in the equation of heat conduction (2.8). The rejection of this term influences in an insignificant manner the distribution of the temperature and the stress, but affects essentially the character of the wave motion. In the domain of coupled problems of thermoelasticity, the waves Φ, θ are damped and undergo dispersion. In the theory of thermal stresses, the function Φ is composed of a purely elastic wave and a diffusion wave, and the function θ has the character of a diffusion wave.

Within the framework of the theory of thermal stresses, the equations for the functions Φ and θ will be obtained from those obtained in Sec. 3 and Sec. 4, by substitution of $\varepsilon = 0$.

In this case the Green's functions $\tilde{G}$, $\tilde{K}$, $\tilde{H}$ will take the form

$$(5.1)\qquad \tilde{G}|_{\varepsilon=0} = \frac{m}{4\pi\varrho}\,\frac{e^{-\varrho p} - e^{-\varrho p^{1/2}}}{p(p-1)}, \qquad \tilde{K}|_{\varepsilon=0} = \tilde{H}|_{\varepsilon=0} = \frac{e^{-\varrho p^{1/2}}}{4\pi\varrho}.$$

It can also be easily verified that

$$(5.2)\qquad \Box_\lambda^2 \tilde{G}|_{\varepsilon=0} = \frac{me^{-\varrho p}}{4\pi\varrho}, \qquad \Box_\lambda^2 \tilde{H}|_{\varepsilon=0} = 0.$$

Thus, Eq. (3.15) becomes

$$(5.3)\qquad \tilde{\Phi}(\boldsymbol{\eta}, p) = \int\limits_A \left[\tilde{G}|_{\varepsilon=0}\frac{\partial\tilde{\theta}}{\partial n} - \tilde{\theta}\frac{\partial}{\partial n}\tilde{G}|_{\varepsilon=0}\right] dA + \frac{1}{4\pi}\int\limits_A \left[\left(\frac{e^{-\varrho p}}{\varrho}\right)\frac{\partial\tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial}{\partial n}\left(\frac{e^{-\varrho p}}{\varrho}\right)\right] dA, \qquad \boldsymbol{\eta} \in B^+,$$

where θ is a known function, obtained from the heat equation for $\beta_0 = 0$.

Let us consider first the second of the surface integrals. On introducing the notations

$$[\tilde{\Phi}] = \tilde{\Phi}e^{-\varrho p}, \qquad \left[\frac{\partial\tilde{\Phi}}{\partial n}\right] = \frac{\partial\tilde{\Phi}}{\partial n}e^{-\varrho p},$$

the second of the integrals (5.3) can be transformed to obtain

$$(5.4)\qquad -\tilde{\Phi}_2(\boldsymbol{\eta}, p) = \frac{1}{4\pi}\int\limits_A \left\{ [\tilde{\Phi}]\frac{\partial}{\partial n}\left(\frac{1}{\varrho}\right) - \frac{1}{\varrho}\frac{\partial\varrho}{\partial n}[p\tilde{\Phi}] - \frac{1}{\varrho}\left[\frac{\partial\tilde{\Phi}}{\partial n}\right]\right\} dA, \qquad \boldsymbol{\eta} \in B^+.$$

Bearing in mind that

$$\mathscr{L}^{-1}\left(\frac{e^{-\varrho p}}{\varrho}\right)=\frac{1}{\varrho}\,\delta(\varrho-\tau),\quad \mathscr{L}^{-1}[\tilde{\Phi}]=\int\limits_0^\tau \Phi(\boldsymbol{\zeta},\tau-\tau')\delta(\varrho-\tau')d\tau'=\Phi(\boldsymbol{\zeta},\tau-\varrho)=[\Phi(\boldsymbol{\zeta},\tau)],$$

$$\mathscr{L}^{-1}[p\tilde{\Phi}]=\frac{\partial\Phi(\boldsymbol{\zeta},\tau-\varrho)}{\partial\tau}=\left[\frac{\partial\Phi(\boldsymbol{\zeta},\tau)}{\partial\tau}\right],$$

$$\mathscr{L}^{-1}\left[\frac{\partial\tilde{\Phi}}{\partial n}\right]=\int\limits_0^\tau \frac{\partial\Phi(\boldsymbol{\zeta},\tau-\tau')}{\partial n}\,\delta(\varrho-\tau')d\tau'=\frac{\partial\Phi(\boldsymbol{\zeta},\tau-\varrho)}{\partial n}=\left[\frac{\partial\Phi(\boldsymbol{\zeta},\tau)}{\partial n}\right],$$

we obtain, for $\Phi_2(\boldsymbol{\eta},\tau)$, the equation

$$\Phi_2(\boldsymbol{\eta},\tau)=-\frac{1}{4\pi}\int\limits_A\left\{[\Phi(\boldsymbol{\zeta},\tau)]\frac{\partial}{\partial n}\left(\frac{1}{\varrho}\right)-\frac{1}{\varrho}\frac{\partial\varrho}{\partial n}\left[\frac{\partial\Phi(\boldsymbol{\zeta},\tau)}{\partial\tau}\right]-\frac{1}{\varrho}\left[\frac{\partial\Phi(\boldsymbol{\zeta},\tau)}{\partial n}\right]\right\}dA(\boldsymbol{\zeta}),\qquad \boldsymbol{\eta}\in B^+. \tag{5.5}$$

The function $\Phi_2(\boldsymbol{\eta},\tau)$ is expressed in terms of the retarded potential $[\Phi(\boldsymbol{\zeta},\tau)]$ and its derivatives. Equation (5.5) represents the familiar form of the Kirchhoff equation.

Let us perform the inverse Laplace transformation of the function $\tilde{G}(\boldsymbol{\zeta},\boldsymbol{\eta},p)|_{\varepsilon=0}$. We obtain the function

$$G(\boldsymbol{\zeta},\boldsymbol{\eta},\tau)|_{\varepsilon=0}=\frac{m}{4\pi\varrho}\,(e^{\tau-\varrho}-1)H(\tau-\varrho)-\frac{m}{4\pi\varrho}\left[U(\varrho,\tau)-\operatorname{erfc}\frac{\varrho}{2\sqrt{\tau}}\right], \tag{5.6}$$

where

$$U=\frac{e^\tau}{2}\left[e^{-\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}}-\sqrt{\tau}\right)+e^{\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}}+\sqrt{\tau}\right)\right],$$

and

$$H(\tau-\varrho)=\begin{cases}0 & \text{for } \tau\leqslant\varrho\\ 1 & \text{for } \tau>\varrho,\end{cases}$$

is the Heaviside's function.

The first term of the expression (5.6) represents a spherical elastic wave, moving from the point $\boldsymbol{\eta}$ to infinity at the velocity $c_1=\left(\frac{(\lambda+2\mu)}{\varrho_0}\right)^{1/2}$. The second term has the character of a heat (diffusion) wave.

On performing on the first surface integral of Eq. (5.3) the inverse Laplace transformation, we obtain the equation

$$\Phi_1(\boldsymbol{\eta},\tau)=\int\limits_0^\tau d\tau'\int\limits_A\left\{G(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')|_{\varepsilon=0}\frac{\partial\theta(\boldsymbol{\zeta},\tau')}{\partial n}-\theta(\boldsymbol{\zeta},\tau')\frac{\partial}{\partial n}G(\boldsymbol{\zeta},\boldsymbol{\eta},\tau-\tau')|_{\varepsilon=0}\right\}dA, \tag{5.7}$$

in which the function $G|_{\varepsilon=0}$ is determined by Eq. (5.6). The potential of thermoelastic displacement Φ is composed, within the framework of the theory of thermal stresses, of two terms—a diffusion wave Φ_1 and a pure elastic wave Φ_2.

Let us consider, in addition, the temperature field in the case of $\varepsilon = 0$. From Eq. (4.6) or (4.8), we obtain

$$\tilde{\theta}(\boldsymbol{\eta}, p) = \frac{1}{4\pi} \int_A \left\{ \frac{e^{-\varrho p^{1/2}}}{\varrho} \frac{\partial \tilde{\theta}}{\partial n} - \tilde{\theta} \frac{\partial}{\partial n} \left(\frac{e^{-\varrho p^{1/2}}}{\varrho} \right) \right\} dA, \quad \boldsymbol{\eta} \in B^+. \tag{5.8}$$

The performance on (5.8) of the inverse Laplace transformation yields

$$\theta(\boldsymbol{\eta}, \tau) = \frac{1}{4\pi} \int_0^\tau d\tau' \int_A \left\{ F(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau-\tau') \frac{\partial \theta(\boldsymbol{\zeta}, \tau')}{\partial n} - \theta(\boldsymbol{\zeta}, \tau') \frac{\partial}{\partial n} F(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau-\tau') \right\} dA(\boldsymbol{\zeta}), \quad \boldsymbol{\eta} \in B^+ \tag{5.9}$$

where

$$F(\boldsymbol{\zeta}, \boldsymbol{\eta}, \tau) = \frac{1}{8(\pi\tau)^{3/2}} \exp\left(\frac{-\varrho^2}{4\tau} \right).$$

Equations (5.9) enables determination of the temperature θ at the point $\boldsymbol{\eta}$ and at the moment τ, if the functions θ and $\partial\theta/\partial n$ are known at the surface A.

Let us observe that the quantities μ, λ involved in the equations of the present section (through the variables $\boldsymbol{\zeta}$ and τ) concern the isothermal state.

6. The Passage to the Classical Dynamic Theory of Elasticity

In the considerations of the present section, it will be more convenient to use the coordinates $\mathbf{x}$, t.

Let us consider the equation of the potential of thermoelastic displacement Φ, having a regular solution in B^+

$$\left(\nabla^2 - \frac{1}{c_1^2} \partial_t^2 \right) \Phi(\mathbf{x}, t) - m_0 \theta(\mathbf{x}, t) = 0, \tag{6.1}$$

where

$$c_1 = \left(\frac{\lambda_T + 2\mu_T}{\varrho_0} \right)^{1/2}, \qquad m_0 = \left(\frac{3\lambda_T + 2\mu_T}{c_1^2 \varrho_0} \right) \alpha_t,$$

and the quantities μ_T, λ_T are measured under isothermal conditions. In order to solve Eq. (6.1), let us consider a singular solution of the equation

$$\left(\nabla^2 - \frac{1}{c_1^2} \partial_t^2 \right) G^*(\mathbf{x}, \mathbf{x}', t) - m_0 K^*(\mathbf{x}, \mathbf{x}', t) = -\frac{1}{c_1^2} \delta(\mathbf{x}-\mathbf{x}')\delta(t) \tag{6.2}$$

in the infinite space. It is assumed that the initial conditions of the functions G^* and K^* are homogeneous. By $\theta(\mathbf{x}, t)$ we have denoted the temperature connected with the potential Φ and by K^* — the temperature connected with the function G^*. Let us perform on Eqs. (6.1) and (6.2) the Laplace transformation. On combining Eqs. (6.1) and (6.2), we find

$$\int_V (\tilde{G}^*\nabla^2\tilde{\Phi}-\tilde{\Phi}\nabla^2\tilde{G}^*)\,dV-m_0\int_V (\tilde{\theta}\tilde{G}^*-\tilde{K}^*\tilde{\Phi})dV=\frac{1}{c_1^2}\tilde{\Phi}(\mathbf{x}',p). \tag{6.3}$$

The classical dynamic theory of elasticity is, of course, based on the assumption that the heat exchange between elements of the body is very slow. Thermodynamic processes are assumed in the dynamic theory of elasticity to be adiabatic. This assumption implies the following relation between the temperature and the dilatation

$$\theta(\mathbf{x},t)=-\eta_T\varkappa u_{i,i}=-\eta_T\varkappa\nabla^2\Phi,\qquad \eta_T=\frac{\gamma_T T_0}{c_\varepsilon\varkappa}. \tag{6.4}$$

We have also the following analogous equation:

$$K^*(\mathbf{x},\mathbf{x}',t)=-\eta_T\varkappa\nabla^2 G^*(\mathbf{x},\mathbf{x}',t). \tag{6.5}$$

The relations (6.4) and (6.5) replace, in the classical dynamic theory of elasticity, the heat equation. On introducing (6.4) and (6.5) in (6.3), we obtain

$$\Phi(\mathbf{x}',p)=c^2\int_V (\tilde{G}^*\nabla^2\tilde{\Phi}-\tilde{\Phi}\nabla^2\tilde{G}^*)\,dV,\qquad c^2=c_1^2(1+\eta_T m_0\varkappa). \tag{6.6}$$

On substituting (6.5) in (6.2) and performing the Laplace transformation, we find:

$$\left(\nabla^2-\frac{p^2}{c^2}\right)\tilde{G}^*=-\frac{1}{c^2}\,\delta(\mathbf{x}-\mathbf{x}'). \tag{6.7}$$

A solution of this equation is the function

$$\tilde{G}^*(\mathbf{x},\mathbf{x}',p)=\frac{1}{4\pi c^2 r}\,e^{-\frac{rp}{c}}. \tag{6.8}$$

Making use of Green's transformation, we can reduce Eq. (6.6) to the form

$$\tilde{\Phi}(\mathbf{x}',p)=c^2\int_A\left(\tilde{G}^*\frac{\partial\tilde{\Phi}}{\partial n}-\tilde{\Phi}\frac{\partial\tilde{G}^*}{\partial n}\right)dA(\mathbf{x}),\qquad \mathbf{x}'\in B^+. \tag{6.9}$$

Bearing in mind (6.8), we have

$$\Phi(\mathbf{x}',p)=\frac{1}{4\pi}\int_A\left[\left(\frac{e^{-\frac{pr}{c}}}{r}\right)\frac{\partial\tilde{\Phi}}{\partial n}-\tilde{\Phi}\frac{\partial}{\partial n}\left(\frac{e^{-\frac{rp}{c}}}{r}\right)\right]dA(\mathbf{x}), \tag{6.10}$$

where c is the velocity of the longitudinal wave with adiabatic constants μ, λ, and, [7],

$$c^2=c_1^2+\frac{\eta_T\varkappa\gamma_T}{\varrho_0}.$$

The inverse Laplace transformation performed on the expression (6.10) yields the Kirchhoff equation in its classical form

$$\Phi(\mathbf{x}',t)=-\frac{1}{4\pi}\int_A\left\{\Phi(\mathbf{x},t)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{cr}\frac{\partial r}{\partial n}\left[\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\right]-\frac{1}{r}\left[\frac{\partial\Phi(\mathbf{x},t)}{\partial n}\right]\right\}dA(\mathbf{x}),\qquad \mathbf{x}'\in B^+. \tag{6.11}$$

In this equation $[\Phi(\mathbf{x},t)]=\Phi(\mathbf{x},t-r/c)$ is a retarded potential.

7. An Approximate Form of the Potentials Φ, θ for the Coupled Problem

The determination of the potentials Φ, θ from Eqs. (3.16) and (4.7) encounters serious difficulties due to the necessity of performing the inverse Laplace transformation on the functions $\tilde{G}$, $\tilde{H}$ and their derivatives. This difficulty will be evaded by making use of the perturbation method, in which the small quantity is $\varepsilon = \beta m$ (1), which characterizes the coupling between the temperature field and the strain field. Let us assume, therefore, that

$$(7.1) \qquad \begin{aligned} \Phi &= \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \dots, \\ G &= G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \dots, \end{aligned}$$

and introduce them in Eq. (3.16). The function $\hat{G}$ occurring in this equation can be expressed in terms of the function G as follows

$$\hat{G} = (\nabla^2 - (1+\varepsilon)\partial_\tau)G.$$

Confining ourselves to the first two terms of the series (7.1), we find

$$(7.2) \qquad \Phi(\boldsymbol{\eta}, \tau) = \int_0^\tau d\tau' \left\{ \iint_A \left(G_0 \frac{\partial\theta}{\partial n} - \theta \frac{\partial G_0}{\partial n} \right) dA + \frac{1}{m} \int_A \left[(NG_0) \frac{\partial \Phi_0}{\partial n} - \Phi_0 \frac{\partial}{\partial n} (NG_0) \right] dA \right\} + \varepsilon \int_0^\tau d\tau' \left\{ \iint_A \left(G_1 \frac{\partial\theta}{\partial n} - \theta \frac{\partial G_1}{\partial n} \right) dA + \frac{1}{m} \int_A \left[(NG_1 - \partial_{\tau'} G_0) \frac{\partial \Phi_1}{\partial n} - \Phi_1 \frac{\partial}{\partial n} (NG_1 - \partial_{\tau'} G_0) \right] dA \right\},$$

where N is the operator $N = \nabla^2 - \partial_\tau$.

For $\varepsilon = 0$, Eq. (7.2) becomes the sum of Eqs. (5.4) and (5.7) of the problem of the theory of thermal stresses. On substituting (7.1) and

$$(7.3) \qquad H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \dots$$

in Eq. (4.7) and confining ourselves to two terms of the series (7.1) and (7.3), we find

$$(7.4) \qquad \theta(\boldsymbol{\eta}, \tau) = \int_0^\tau d\tau' \int_A \left(H_0 \frac{\partial\theta}{\partial n} - \theta \frac{\partial H_0}{\partial n} \right) dA + \varepsilon \int_0^\tau d\tau' \left\{ \int_A \left(H_1 \frac{\partial\theta}{\partial n} - \theta \frac{\partial H_1}{\partial n} \right) dA + \frac{1}{m} \int_A \left[(NH_1 - \partial_{\tau'} H_0) \frac{\partial \Phi_0}{\partial n} - \Phi_0 \frac{\partial}{\partial n} (NH_1 - \partial_{\tau'} H_0) \right] dA \right\}.$$

It remains to determine the functions G_0, G_1 and H_0, H_1. We expand the quantity $\lambda_1^2(\varepsilon, p)$, $\lambda_2^2(\varepsilon, p)$ in Maclaurin's series in ε. By confining ourselves to two expansion terms, we have

$$(7.5) \qquad \lambda_1^2 \approx p^2 + \frac{p^2}{p-1}\varepsilon, \qquad \lambda_2^2 \approx p - \frac{p}{p-1}\varepsilon,$$

(1) This quantity is equal to $\varepsilon = 0.0168$ for copper, $\varepsilon = 0.0356$ for aluminium and $\varepsilon = 0.00237$ for steel.

$$\lambda_1 \approx p+\frac{p}{2(p-1)}\,\varepsilon, \qquad \lambda_2 \approx p^{1/2}\left(1-\frac{\varepsilon}{2(p-1)}\right),$$

$$\frac{1}{\lambda_1^2-\lambda_2^2} \approx \frac{1}{p(p-1)}\left(1-\frac{\varepsilon(p+1)}{(p-1)^2)}\right).$$

On substituting λ_1, λ_2 in the expressions $e^{-\lambda_1\varrho}$, $e^{-\lambda_2\varrho}$, we find

$$\text{(7.6)} \qquad \begin{aligned} e^{-\lambda\varrho_1} &\approx \left(1-\frac{\varepsilon\varrho}{2}\,\frac{p}{p-1}\right)e^{-\varrho p}, \\ e^{-\varrho\lambda_2} &\approx \left(1+\frac{\varepsilon\varrho}{2}\,\frac{p^{1/2}}{p-1}\right)e^{-\varrho p^{1/2}}. \end{aligned}$$

Finally, on substituting (7.5), (7.6) in the expression (3.6), we find

$$\tilde{G}_0 = \tilde{G}|_{\varepsilon=0} = \frac{m}{4\pi\varrho}\,\frac{e^{-\varrho p}-e^{-\varrho p^{1/2}}}{p(p-1)},$$

$$\tilde{G}_1 = -\frac{1}{4\pi\varrho}\,\frac{m}{p(p-1)^2}\left[\left(\frac{p+1}{p-1}+\frac{\varrho p}{2}\right)e^{-\varrho p}-\left(\frac{p+1}{p-1}+\frac{\varrho p^{1/2}}{2}\right)e^{-\varrho p^{1/2}}\right].$$

The quantity $\tilde{G}_0 = \tilde{G}|_{\varepsilon=0}$ is expressed by Eq. (5.6). For the function $G_1(\varrho, \tau)$, we obtain the following equation [8]

$$\text{(7.7)} \qquad \begin{aligned} G_1(\varrho, \tau) = -\frac{m}{4\pi\varrho}\Bigg\{\Bigg[\left((\tau-\varrho)^2+(\tau-\varrho)\left(\frac{\varrho}{2}-1\right)+1\right)e^{\tau-\varrho}-1\Bigg]H(\tau-\varrho)-\alpha(\varrho, \tau)\frac{\varrho}{2} \\ -\left[\frac{1}{2}\left(\tau^2+\frac{\varrho^2}{4}\right)U+\frac{1}{2}\left(\frac{\varrho}{4}-\tau\varrho\right)V-\frac{\varrho}{4}\left(\frac{\tau}{\pi}\right)^{1/2}\exp\left(-\frac{\varrho^2}{4\tau}\right)\right] \\ -\left[\frac{\tau^2}{2}-\tau+\frac{\varrho^2}{8}+1\right]U+\left(\frac{5}{8}\,\varrho-\frac{\tau\varrho}{2}\right)V+\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}}\right)-\frac{\varrho}{4}\left(\frac{\tau}{\pi}\right)^{1/2}\exp\left(\frac{-\varrho^2}{4\pi}\right)\Bigg\}, \end{aligned}$$

where the following functions have been introduced

$$V = \frac{e^{\tau}}{2}\left[e^{-\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}}-\sqrt{\tau}\right)-e^{\varrho}\operatorname{erfc}\left(\frac{\varrho}{2\sqrt{\tau}}+\sqrt{\tau}\right)\right],$$

$$\alpha(\varrho, \tau) = \int_0^{\tau}\left[\left(\tau_0+\frac{1}{2}\right)V(\varrho, \tau_0)-\frac{\varrho}{2}\,U(\varrho, \tau_0)+\left(\frac{\tau_0}{\pi}\right)^{1/2}\exp\left(\frac{-\varrho^2}{4\tau_0}\right)\right]d\tau_0.$$

The function $U(\varrho, \tau)$ was given in Sec. 5. In the function $G_1(\varrho, \tau)$ we can discern, similarly to the case of the function G_0, two types of waves—an elastic wave and a diffusion wave. If we introduce (7.5) and (7.6) into the expression (4.3) for the function $\tilde{H}(\varrho, p)$, we obtain

$$\text{(7.8)} \qquad \begin{aligned} \tilde{H}_0(\varrho, p) &= \frac{1}{4\pi\varrho}\,e^{-\varrho p^{1/2}}, \\ \tilde{H}_1(\varrho, p) &= \frac{p}{4\pi\varrho(p-1)^2}\left\{e^{-\varrho p}+\left[\frac{\varrho}{2}\,p^{-1/2}\,(p-1)-1\right]e^{-\varrho p^{1/2}}\right\}. \end{aligned}$$

On performing on these functions the inverse Laplace transformation, we find

$$H_0(\varrho,\tau)=\frac{1}{8(\pi\tau)^{3/2}}\exp\left(\frac{-\varrho^2}{4\tau}\right),$$

$$(7.9)\qquad H_1(\varrho,\tau)=\frac{1}{4\pi\varrho}\left\{(\tau-\varrho+1)e^{\tau-\varrho}H(\tau-\varrho)-\left[(\tau+1)U(\varrho,\tau)-\frac{\varrho}{2}V(\varrho,\tau)\right]\right.$$
$$\left.+\frac{\varrho}{2}\left[V+\frac{1}{(\pi\tau)^{1/2}}\exp\left(\frac{-\varrho^2}{4\tau}\right)\right]\right\}.$$

The function H_0 has a diffusion character. In the function $H_1(\varrho,\tau)$ are contained terms having the character of an elastic wave and a diffusion wave.

For practical purposes, the strain and the stress in machines and structures can be determined by rejecting the coupling between the temperature field and the strain field, and by assuming $\varepsilon=0$ in Eqs. (7.2) and (7.4). In this way we arrive at the equations discussed in detail in Sec. 5.

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Streszczenie

FALE TERMOSPRĘŻYSTE W CIELE NIESKOŃCZONYM

W pracy rozpatruje się propagację fali podłużnej w nieograniczonym termosprężystym ośrodku. Potencjał termosprężystego przemieszczenia oraz temperaturę w pewnym punkcie wyrażono w postaci całek powierzchniowych w których występują potencjał, temperatura oraz ich pochodne w kierunku normalnej. Uzyskany wynik stanowi rozszerzenie znanego z klasycznej teorii termoprężystości twierdzenia Kirchhoffa na sprzężoną termosprężystość. Rozpatrzono dwa przypadki szczególne tego ogólnego twierdzenia, przy czym pierwszy z nich dotyczy niesprzężonego zagadnienia termosprężystego, a drugi klasycznej elastokinetyki.

Резюме

ТЕРМОУПРУГИЕ ВОЛНЫ В БЕСКОНЕЧНОМ ТЕЛЕ

В работе рассматирвается распространение продольных упругих волн в бесконечной среде. Потенциал термоупругого перемещения Φ и температура θ в некоторой точке $\eta \in B^+$ выражается в виде поверхностных интегралов, в которых существуют: потенциал, температура и их производные по направлению нормали. Полученный результат является расширением известной теоремы Кирхгоффа из классической динамической теории упругости на случай сопряженной термоупругости. Рассматриваются два частных случая этой общей теоремы, причем первый из них касается несопряженной задачи термоупругости, а второй классической динамической упругости.

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