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A RECIPROCITY THEOREM FOR COUPLED MECHANICAL AND THERMOELECTRIC FIELDS IN PIEZOELECTRIC CRYSTALS

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The equations governing small vibration of piezoelectric crystals, taking into consideration the coupling between deformation, temperature and electric field have been derived by R. D. MINDLIN, [1]. These constitutive equations and those of motion will be used in the present paper to derive a reciprocity theorem, and to discuss a number of conclusions following from this theorem.

The coupled problem under consideration consists in determining the stresses $\sigma_{ij}(x, t)$ and the strains $\varepsilon_{ij}(x, t)$ of the $C^{(1)}$ class, the displacements $u_i(x, t)$, the temperature $\theta(x, t)$ and the electric potential $\Phi(x, t)$, of the $C^{(2)}$ class, for $x \in B$, $t \geqslant 0$, that is in the region B, bounded by the surface S. By x we denote the geometric coordinate $x \equiv (x_1, x_2, x_3)$, and by t — the time.

In the region B and for $t > 0$ the following equations should be satisfied.

The equation of motion

$$\sigma_{ij,j} = \varrho \ddot{u}_i - X_i, \qquad i, j = 1, 2, 3, \tag{1}$$

the generalized heat equation

$$\varkappa_{ij} \theta_{,ij} = T_0 \dot{s}, \qquad i, j = 1, 2, 3, \tag{2}$$

and the equation of quasi-stationary electric field

$$\dot{D}_{i,i} = -\chi, \qquad i = 1, 2, 3. \tag{3}$$

X_i denotes components of the mass force, s — the entropy per unit volume, D_i — components of electric displacement, χ —intensity of electric charge, T_0 — the absolute temperature referred to the natural state in which the stress and the strain are zero, $\varkappa_{ij}$ are coefficients of heat conduction and ϱ the density.

These equations should be completed with boundary and initial conditions. The following quantities may be assigned at the surface S.

The displacements or loads

$$u_i = U_i(x,t), \tag{4a}$$

$$\sigma_{ij} n_j = R_i(x,t), \qquad x \in S, \qquad t > 0. \tag{4b}$$

The temperature or heat flow

$$\theta = \vartheta(x,t), \tag{5a}$$

$$-\varkappa_{ij} \theta_{,i} n_j = k(xt), \qquad x \in S, \qquad t > 0. \tag{5b}$$

The electric potential Φ or the electric surface load

$$\Phi = \varphi(x,t), \tag{6a}$$

$$D_i n_i = d(x,t), \qquad x \in S, \qquad t > 0. \tag{6b}$$

The initial conditions will be assumed to be homogeneous, it being understood that all the causes producing the coupled mechanical-thermal-electric field started to act at the moment $t = 0^+$

$$u_i(x,0) = 0, \quad \dot{u}_i(x,0) = 0, \quad \theta(x,0) = 0, \quad \Phi(x,0) = 0, \quad x \in B, \quad t = 0. \tag{7}$$

In addition, we have the constitutive equations which have the form [1, 2].

$$\sigma_{ij} = c^{E\theta}_{ijkl}\varepsilon_{kl} - e^{\theta}_{kij}E_k - \lambda^{E}_{ij}\theta, \qquad E_k = -\Phi_{,k}, \tag{8}$$

$$D_j = e^{\theta}_{jkl}\varepsilon_{kl} + \gamma^{\theta s}_{ij}E_i + p^{s}_{j}\theta, \tag{9}$$

$$s = \lambda^{E}_{kl}\varepsilon_{kl} + p^{s}_{i}E_i + a^{sE}\theta, \qquad x \in B+S, \qquad t \geqslant 0. \tag{10}$$

These equations must be completed by the stress-strain relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{11}$$

Equation (8) is the Hooke relation generalized to coupled problems, the constants $c^{E\theta}_{ijkl}$ playing the role of stiffness constants measured under isothermal conditions with constant electric field. The relation (9) enables us to express the components of the electric displacement vector and the relation (10) — the entropy, by means of the functions ε_{ij}, E_i and θ. The material constants in these equations are discussed in detail in Mason's monograph [2]. In further consideration we shall need Eqs. (1), (2), (3) expressed in functions u_i, Φ and θ. We shall obtain [1]

$$c^{E\theta}_{ijkl}u_{k,li} + e^{\theta}_{kij}\Phi_{,ki} - \lambda^{E}_{ij}\theta_{,i} + X_i = \varrho\ddot{u}_i, \tag{12}$$

$$e^{\theta}_{kij}u_{i,jk} - \gamma^{\theta s}_{ij}\Phi_{,ij} + p^{s}_{i}\theta_{,i} = -\chi, \tag{13}$$

$$\frac{1}{T_0}\varkappa_{ij}\theta_{,ij} - a^{sE}\dot{\theta} + p^{s}_{i}\dot{\Phi}_{,i} - \lambda^{E}_{ij}\dot{u}_{i,j} = -W, \tag{14}$$

where W denotes the quantity of heat generated by a source of heat per unit time and unit volume.

In the problem under consideration, the motion is assumed to be produced by mass forces X_i, a heat source Q, electric charges χ, and also the quantities appearing in the boundary conditions such as surface load R_i, temperature (heating) ϑ, potential Φ etc.

The resulting quantities are the displacements u_i, the temperature θ and the electric potential Φ. In order to obtain the reciprocity theorem, we shall consider two sets of causes and effects; the latter will be denoted by «primes». In further analysis, we shall use the equations (1) to (14) in a transformed shape by applying to them the one-sided Laplace transformation. We start out from the following identity obtained from Eq. (8):

$$(15)\qquad (\bar\sigma_{ij}+\lambda^E_{ij}\bar\theta+e^\theta_{kij}\bar E_k)\bar\varepsilon'_{ij}=(\bar\sigma'_{ij}+\lambda^E_{ij}\bar\theta'+e^\theta_{kij}\bar E'_k)\bar\varepsilon_{ij}=c^{E\theta}_{ijkl}\bar\varepsilon_{kl}\bar\varepsilon'_{ij}=c^{E\theta}_{ijkl}\bar\varepsilon'_{kl}\bar\varepsilon_{ij},$$

where

$$\bar\sigma_{ij}(x,p)=\mathcal{L}[\sigma_{ij}(x,t)]=\int_0^\infty \bar e^{pt}\sigma_{ij}(x,t)dt,\ etc.$$

On integrating (15) over the region B, we obtain the equation

$$(16)\qquad \int_B(\bar\sigma_{ij}\bar\varepsilon'_{ij}-\bar\sigma'_{ij}\bar\varepsilon_{ij})dV=-\int_B\lambda^E_{ij}(\bar\theta\bar\varepsilon'_{ij}-\bar\theta'\bar\varepsilon_{ij})dV-\int_B e^\theta_{kij}(\bar E_k\bar\varepsilon'_{ij}-\bar E'_k\bar\varepsilon_{ij})dV.$$

Bearing in mind that $\bar\sigma_{ij}\bar\varepsilon'_{ij}=\bar\sigma_{ij}(\bar\varepsilon'_{ij}+\bar\omega'_{ij})=\bar\sigma_{ij}\bar u'_{i,j}$ because $\bar\sigma_{ij}\bar\omega'_{ij}=0$ (in view of the skew-symmetry of the rotation tensor), the left-hand side of (16) can, taking into consideration (1) and the boundary conditions (4a), (4b), be represented in the form

$$(17)\qquad \int_B(\bar\sigma_{ij}\bar\varepsilon'_{ij}-\bar\sigma'_{ij}\bar\varepsilon_{ij})dV=\int_B(\bar\sigma_{ij}\bar u'_{i,j}-\bar\sigma'_{ij}\bar u_{i,j})dV$$

$$=\int_B[(\bar\sigma_{ij}\bar u'_i)_{,j}-(\bar\sigma'_{ij}\bar u_i)_{,j}]dV-\int_B(\bar\sigma_{ij,j}\bar u'_i-\bar\sigma'_{ij,j}\bar u_i)dV$$

$$=\int_S(\bar R_i\bar U'_i-\bar R'_i\bar U_i)dS+\int_B(\bar X_i\bar u'_i-\bar X'_i\bar u_i)dV.$$

By uniting (16) and (17), we obtain

$$(18)\qquad \int_B(\bar X_i\bar u'_i-\bar X'_i\bar u_i)dV+\int_S(\bar R_i\bar U'_i-\bar R'_i\bar U_i)dS$$

$$=-\int_B\lambda^E_{ij}(\bar\theta\bar\varepsilon'_{ij}-\bar\theta'\bar\varepsilon_{ij})dV+\int_B e^\theta_{kij}(\bar\Phi_{,k}\bar\varepsilon'_{ij}-\bar\Phi'_{,k}\bar\varepsilon_{ij})dV,\qquad E_k=-\Phi_{,i}.$$

Equation (14) will now be made use of by multiplying it by θ', and integrating over the region B and subtracting from it the analogous equation with "primes", multiplied by θ and integrated over B. We obtain the equation

$$(19)\qquad \frac{1}{T_0}\int_B\varkappa_{ij}(\bar\theta_{,ij}\bar\theta'-\bar\theta'_{,ij}\bar\theta)dV=p\left\{\int_B\lambda^E_{ij}(\bar u_{i,j}\bar\theta'-\bar u'_{i,j}\bar\theta)dV\right.$$

$$\left.-\int_B p^s_i(\bar\Phi_{,i}\bar\theta'-\bar\Phi'_{,i}\bar\theta)dV\right\}-\int_B(\bar W\bar\theta'-\bar W'\bar\theta)dV.$$

The left-hand side of Eq. (19) yields, on applying the divergence theorem, the equation

$$(20)\qquad \frac{1}{T_0}\int_B\varkappa_{ij}(\bar\theta_{,ij}\bar\theta'-\bar\theta'_{,ij}\bar\theta)dV=\frac{1}{T_0}\int_S\varkappa_{ij}(\bar\theta_{,i}\bar\theta'-\bar\theta'_{,i}\bar\theta)n_jdS.$$

Bearing in mind the boundary conditions (5a), (5b) and uniting (19) with (20), we obtain

$$(21)\qquad \frac{1}{T_0}\int_S (\bar{k}\bar{\vartheta}'-\bar{k}'\bar{\vartheta})dS = p\left\{\int_B p_i^s(\bar{\Phi}_{,i}\bar{\theta}'-\bar{\Phi}_{,i}\bar{\theta})dV - \int_B \lambda_{ij}^E(\bar{u}_{i,j}\bar{\theta}'-\bar{u}'_{i,j}\bar{\theta})dV\right\} - \int_B (\bar{W}\bar{\theta}'-\bar{W}'\bar{\theta})dV.$$

Similarly, making use of (13), we obtain

$$(22)\qquad \int_B \gamma_{ij}^{\theta s}(\bar{\Phi}_{,ij}\bar{\Phi}'-\bar{\Phi}'_{,ij}\bar{\Phi})dV = \int_B e_{kij}^{\theta}(\bar{u}_{i,jk}\bar{\Phi}'-\bar{u}'_{i,jk}\bar{\Phi})dV + \int_B p_i^s(\bar{\theta}_{,i}\bar{\Phi}'-\bar{\theta}'_{,i}\bar{\Phi})dV + \int_B (\bar{\chi}\bar{\Phi}'-\bar{\chi}'\bar{\Phi})dV,$$

or, on applying the divergence theorem,

$$(23)\qquad \int_B \gamma_{ij}^{\theta s}(\bar{\Phi}_{,i}\bar{\Phi}'-\bar{\Phi}'_{,i}\bar{\Phi})n_j dV = \int_S e_{kij}^{\theta}(\bar{u}_{i,j}\bar{\Phi}'-\bar{u}'_{i,j}\bar{\Phi})n_k dS - \int_B e_{kij}^{\theta}(\bar{u}_{i,j}\bar{\Phi}'_{,k}-\bar{u}'_{i,j}\bar{\Phi}_{,k})dV + \int_S p_i^s(\bar{\theta}\bar{\Phi}'-\bar{\theta}'\bar{\Phi})n_i dS - \int_B p_i^s(\bar{\theta}\bar{\Phi}'_{,i}-\bar{\theta}'\bar{\Phi}_{,i})dV + \int_B (\bar{\chi}\bar{\Phi}'-\bar{\chi}'\bar{\Phi})dV.$$

Bearing in mind (9) and the boundary conditions (6a) and (6b), Eq. (23) may be given the form

$$(24)\qquad \int_S (\bar{d}\bar{\varphi}'-\bar{d}'\bar{\varphi})dS = \int_B e_{kij}^{\theta}(\bar{u}_{i,j}\bar{\Phi}_{,k}-\bar{u}'_{i,j}\bar{\Phi}_{,k})dV + \int_B p_i^s(\bar{\theta}\bar{\Phi}'_{,i}-\bar{\theta}'\bar{\Phi}_{,i})dV - \int_B (\bar{\chi}\bar{\Phi}'-\bar{\chi}'\bar{\Phi})dV,$$

On eliminating from Eqs. (18), (21) and (24) the repeated integrals, we arrive at the following equation involving all the causes and effects

$$(25)\qquad \int_B (\bar{X}_i\bar{u}'_i-\bar{X}'_i\bar{u}_i)dV + \int_S (\bar{R}_i\bar{U}'_i-\bar{R}'_i\bar{U}_i)dS + \frac{1}{p}\int_B \bar{W}\bar{\theta}'-\bar{W}'\bar{\theta})dV + \frac{1}{pT_0}\int_S (\bar{k}\bar{\vartheta}'-\bar{k}'\bar{\vartheta})dS + \int_B (\bar{\chi}\bar{\Phi}'-\bar{\chi}'\bar{\Phi})dV + \int_S (\bar{d}\bar{\varphi}'-\bar{d}'\bar{\varphi})dS = 0.$$

This equation expresses the reciprocity theorem in its transformed shape. On performing the inverse Laplace transformation we obtain the reciprocity theorem in the following final form

$$(26)\qquad \int_0^t d\tau \int_B \left[X_i(x,t-\tau)\frac{\partial u'_i(x_1\tau)}{\partial\tau} - X'_i(x,t-\tau)\frac{\partial u_i(x,\tau)}{d\tau}\right]dV(x)$$

(2.6) [cont.]
$$+\int_0^t d\tau \int_S \left[R_i(x,t-\tau)\frac{\partial U_i'(x,\tau)}{\partial \tau} - R_i'(x,t-\tau)\frac{\partial U_i(x,\tau)}{\partial \tau}\right] dS(x)$$

$$+\int_0^t d\tau \int_B \left[W(x,t-\tau)\theta'(x,\tau) - W'(x,t-\tau)\theta(x,\tau)\right] dV(x)$$

$$+\frac{1}{T_0}\int_0^t d\tau \int_S \left[k(x,t-\tau)\vartheta'(x,\tau) - k'(x,t-\tau)\vartheta(x,\tau)\right] dS(x)$$

$$+\int_0^t d\tau \int_B \left[\chi(x,t-\tau)\frac{\partial \Phi'(x,\tau)}{\partial \tau} - \chi'(x,t-\tau)\frac{\partial \Phi(x,\tau)}{\partial \tau}\right] dV(x)$$

$$+\int_0^t d\tau \int_S \left[d(x,t-\tau)\frac{\partial \varphi'(x,\tau)}{\partial \tau} - d'(x,t-\tau)\frac{\partial \varphi(x,\tau)}{\partial \tau}\right] dS(x) = 0.$$

The reciprocity theorem has a particularly simple form in the case of an infinite body, because in this case the surface integrals vanish.

Below are obtained equations for the functions u_i, θ and Φ inside the body if boundary values are known together with their derivatives.

As a set with "primes", let us assume an instantaneous concentrated force at a point ξ of the infinite body, directed in the x_j direction.

This force $X_i' = \delta(x-\xi)\delta(t)\delta_{ij}$, will produce in the infinite body a displacement $u_i' = G_i^{(j)}(x,\xi,t)$, $i,j = 1,2,3$, a temperature $\theta' = C^{(j)}(x,\xi,t)$, and a potential $\Psi^{(j)}(x,\xi,t)$. The above Green's functions $G_i^{(j)}$, $C^{(j)}$, $\Psi^{(j)}$ will be obtained by solving the set of Eqs. (12)–(14), assuming that $X_i' = \delta(x-\xi)\,\delta(t)\delta_{ij}$, $\chi' = 0$, $W' = 0$. These functions will, in what follows, be assumed to be known. On substituting $u_i' = G_i^{(j)}$, $\theta' = C^{(j)}$ and $\Phi' = \Psi_{(j)}$ and X_i in Eq. (25), we find

(27)
$$\bar{u}_j(\xi,p) = \int_B \bar{X}_i(x,p)\bar{G}_i^{(j)}(x,\xi,p)\,dV(x) + \int_S [\bar{R}_i(x,p)\bar{G}_i^{(j)}(x,\xi,p) - \bar{R}_i^{(j)}(x,\xi,p)\bar{U}_i(x,p)]dS(x)$$

$$+\frac{1}{p}\int_B \bar{W}(x,p)\bar{C}^{(j)}(x,\xi,p)dV(x) + \frac{1}{T_0 p}\int_S [\bar{k}(x,p)\bar{C}^{(j)}(x,\xi,p) - \bar{k}^{(j)}(x,\xi,p)\bar{\vartheta}(x,p)]dS(x)$$

$$+\int_B \bar{\chi}(x,p)\bar{\Psi}^{(j)}(x,\xi,p)dV(x) + \int_S [\bar{d}(x,p)\bar{\Psi}^{(j)}(x,\xi,p) - \bar{d}^{(j)}(x,\xi,p)\bar{\varphi}(x,p)]dS(x).$$

In this equation $R_i^{(j)}$ is the value of $\sigma_{ik}^{(j)}n_k$ on the surface S, the stress $\sigma_{ij}^{(j)}$ being expressed by (8) in terms of the Green's functions $G_i^{(j)}$, $C^{(j)}$ and $\Psi^{(j)}$. Similarly we have $\bar{k}^{(j)} = -\varkappa_{ik}C_{,i}^{(j)}n_k$ and $d^{(j)} = D_i^{(j)}n_i$, where $D_i^{(j)}$ is expressed by (9) in terms of the functions $G_i^{(j)}$, $C^{(j)}$, $\Psi^{(j)}$.

Equation (27) enables us to determine the displacements at the point $\xi \in B$ with a known distribution of mass forces, heat equations and electric charges and with known values of R_i, U_i, k, ϑ, d and φ at the surface of the body. Equation (27) comprises a number of special cases. Thus, with no electric coupling, this equation becomes that of the thermoelastic problem. If there is no electric and no thermal coupling, the body being merely loaded by the mass forces X_i and the loads R_i, Eq. (27) becomes the Somigliano equation generalized to the dynamic problem.

Let us assume in turn that the set with "primes" is a product of the action of an instantaneous concentrated source of heat in an infinite body. Under the action of this source of heat, there occur displacements $u'_i = V_i(x, \xi, t)$, temperature $\theta' = K(x, \xi, t)$, and potential $\Phi' = \Omega(x, \xi, t)$.

The above functions will be obtained by solving the set of equations (12)–(14) by assuming that $X'_i = 0$, $\chi' = 0$, $W' = \delta(x-\xi)\delta(t)$. Assuming the Green's functions thus obtained to be known, we obtain, by substituting them in the reciprocity Eq. (25), the relation

(28)

$$\begin{aligned}\bar{\theta}(\xi,p) = {} & p\int_B \bar{X}_i(x,p)\bar{V}_i(x,\xi,p)dV(x) + p\int_S [\bar{R}_i(x,p)\bar{V}_i(x,\xi,p) - \bar{R}_i^{(w)}(x,\xi,p)\bar{U}_i(x,p)]dS(x)\\ & + \int_B \bar{W}(x,p)\bar{K}(x,\xi,p)dV(x) + \frac{1}{T_0}\int_S [\bar{k}(x,p)\bar{K}(x,\xi,p) - \bar{k}^{(w)}(x,\xi,p)\bar{\vartheta}(x,p)]dS(x)\\ & + p\int_B \bar{\chi}(x,p)\bar{\Omega}(x,\xi,p)dV(x) + p\int_S [\bar{d}(x,p)\bar{\Omega}(x,\xi,p) - \bar{d}^{(w)}(x,\xi,p)\bar{\varphi}(x,p)]dS(x),\end{aligned}$$

where $\bar{R}_i^{(w)}$ is the expression $\bar{\sigma}_{ij}^{(w)}n_j$, where $\sigma_{ij}^{(w)}$ is obtained from Eq. (8) by substituting $u'_i = V_i$, $\theta' = K$, $\Phi' = \Omega$. Next, $k^{(w)} = -\varkappa_{ik}K_{,i}n_k$ and $d^{(w)} = D_i^{(w)}n_i$, where $D_i^{(w)}$ is given by (9) and expressed in terms of the functions V_i, K, Ω.

Equation (28) enables us to determine the temperature inside the body for a known distribution of the mass forces, heat sources and electric charges, and for prescribed values of the functions R_i, U_i, k, ϑ, d and φ at the surface.

Finally, let us assume as a set with "primes" the result of action of an instantaneous concentrated electric charge at the point ξ of the infinite body. This charge $\chi' = \delta(x-\xi)\delta(t)$ produces the displacements $u'_i = W_i(x, \xi, t)$, temperature $\theta' = L(x, \xi, t)$ and potential $\Phi' = \Gamma(x, \xi, t)$.

These Green's functions will be obtained by solving the set of equations (12) to (14) with $X'_i = 0$, $W' = 0$, $\chi' = \delta(x-\xi)\delta(t)$. On substituting χ', W_i, Γ in the reciprocity Eq. (25), we find the following expression for the electric potential

$$\begin{aligned}(29)\quad \bar{\Phi}(\xi,p) = {} & \int_B \bar{X}_i(x,p)\bar{W}_i(x,\xi,p)dV(x) + \int_S [\bar{R}_i(x,p)\bar{W}_i(x,\xi,p) - \bar{R}_i^{(\chi)}(x,\xi,p)\bar{U}_i(x,p)]dS(x)\\ & + \frac{1}{p}\int_B \bar{W}(x,p)\bar{L}(x,\xi,p)dV(x) + \frac{1}{T_0 p}\int_S [\bar{k}(x,p)\bar{L}(x,\xi,p) - \bar{k}^{(\chi)}(x,\xi,p)\bar{\vartheta}(x,p)]dS(x)\\ & + \int_B \bar{\chi}(x,p)\bar{\Gamma}(x,\xi,p)dV(x) + \int_S [\bar{d}(x,p)\bar{\Gamma}(x,\xi,p) - \bar{d}^{(\chi)}(x,\xi,p)\bar{\varphi}(x,p)]dS(x).\end{aligned}$$

From this equation the electric potential Φ can be determined at the point $\xi \in B$ for a known distribution of the functions X_i, W, χ and for values of the functions R_i, U_i, ϑ, k, d, and φ prescribed at the surface S. Equations (27) to (29) express the Somigliano theorems generalized to the coupled dynamic problem. Let us observe also that Eqs. (27)–(29) undergo considerable simplification owing to the relations

$$(30)\qquad p\bar{V}_i(\xi,x,p) = \bar{C}^{(i)}(x,\xi,p) \qquad \bar{W}_i(\xi,x,p) = \bar{\Psi}^{(i)}(x,\xi,p),$$
$$\bar{L}(\xi,x,p) = p\bar{\Omega}(x,\xi,p),$$

which follows from the application of the reciprocity theorem for an infinite region assumed to be acted on, by turns, by the instantaneous concentrated agents X_i and W', then by X_i and χ' and, finally, W and χ'.

Let us build up the functions $G_i^{(j)}$, $C^{(j)}$, $\Psi^{(j)}$; V_i, K, Ω; W_i, L, Γ in a bounded region B, with appropriate boundary conditions. Let therefore an instantaneous concentrated force X_i' acts at the point ξ of the region B. The effect of its action is described by the functions $G_i^{(j)}$, $C^{(j)}$, $\Psi^{(j)}$, which are selected in such a way that the quantities $G_i^{(j)}$, $C^{(j)}$, $\Psi^{(j)}$ are zero at S. Next, the functions V_i, K, Ω describing the effect of an instantaneous concentrated source of heat are calculated in B with the boundary conditions $V_i = 0$, $K = 0$, $\Omega = 0$ at the surface S. Finally, we determine the functions W_i, L, Γ due to the action of an instantaneous concentrated electric charge χ' at a point $\xi \in B$, assuming that $W_i = 0$, $L = 0$, $\Gamma = 0$ at the surface S. With Green's functions constructed in this way, Eqs (27)–(29) become

$$(31)\qquad \bar{u}_j(\xi,p) = \int_B \bar{X}_i(x,p)\bar{G}_i^{(j)}(x,\xi,p)dV(x) - \int_S \bar{R}_i^{(j)}(x,\xi,p)\bar{U}_i(x,p)dS(x)$$
$$+\frac{1}{p}\int_B \bar{W}(x,p)\bar{C}^{(j)}(x,\xi,p)dV(x) - \frac{1}{T_0 p}\int_S \bar{k}^{(j)}(x,\xi,p)\bar{\vartheta}(x,p)dS(x)$$
$$+\int_B \bar{\chi}(x,p)\bar{\Psi}^{(j)}(x,\xi,p)dV(x) - \int_S \bar{d}^{(j)}(x,\xi,p)\bar{\varphi}(x,p)dS(x),$$

$$(32)\qquad \bar{\theta}(\xi,p) = p\int_B \bar{X}_i(x,p)\bar{V}_i(x,\xi,p)dV(x) - p\int_S \bar{R}_i^{(w)}(x,\xi,p)\bar{U}_i(x,p)dS(x)$$
$$+\int_B \bar{W}(x,p)\bar{K}(x,\xi,p)dV(x) - \frac{1}{T_0}\int_S \bar{k}^{(w)}(x,\xi,p)\bar{\vartheta}(x,p)dS(x)$$
$$+p\int_S \bar{\chi}(x,p)\bar{\Omega}(x,\xi,p)dV(x) - p\int_S \bar{d}^{(w)}(x,\xi,p)\bar{\varphi}(x,p)dS(x),$$

$$(33)\qquad \bar{\Phi}(\xi,p) = \int_B \bar{X}_i(x,p)\bar{W}_i(x,\xi,p)dV(x) - \int_S \bar{R}^{(\chi)}(x,\xi,p)\bar{U}_i(x,p)dS(x)$$
$$+\frac{1}{p}\int_B \bar{W}(x,p)\bar{L}(x,\xi,p)\,dV(x) - \frac{1}{T_0 p}\int_S \bar{k}^{(\chi)}(x,\xi,p)\bar{\vartheta}(x,p)dS(x)$$
$$+\int_B \bar{\chi}(x,p)\bar{\Gamma}(x,\xi,p)dV(x) - \int_S \bar{d}^{(\chi)}(x,\xi,p)\bar{\varphi}(x,p)dS(x).$$

From these equations, the quantities u_j, θ, Φ can be determined inside the body for values of the displacements U_i, temperature ϑ, and potential φ prescribed at the boundary. Equations (31)–(33) constitute a generalization of the known Green's theorem of electrostatics. Let us observe that the relations (30) remain valid in the present case also.

Let us select now the set with ,,primes'' in such a way that the quantities becoming zero at the surface S are the load, the heat flow and the surface electric load. Let the quantities acting by turns at the point $\xi \in B$ be an instantaneous concentrated force, an instantaneous concentrated source of heat, and instantaneous concentrated charge. In this way, new Green's functions are obtained. On substituting these in Eqs. (27) to (29), we obtain the relations

$$\begin{aligned}(34)\qquad \bar{u}_j(\xi,p) &= \int_B \bar{X}_i(x,p)\bar{G}_i^{(j)}(x,\xi,p)dV(x)+\int_S \bar{R}_i(x,p)\bar{G}_i^{(j)}(x,\xi,p)dS(x)\\ &+\frac{1}{p}\int_B \bar{W}(x,p)\bar{C}^{(j)}(x,\xi,p)dV(x)+\frac{1}{T_0 p}\int_S \bar{k}(x,p)\bar{C}^{(j)}(x,\xi,p)dS(x)\\ &+\int_B \bar{\chi}(x,p)\bar{\Psi}^{(j)}(x,\xi,p)dV(x)+\int_S \bar{d}(x,p)\bar{\Psi}^{(j)}(x,\xi,p)dS(x),\end{aligned}$$

$$\begin{aligned}(35)\qquad \bar{\theta}(\xi,p) &= p\int_B \bar{X}_i(x,p)\bar{V}_i(x,\xi,p)dV(x)+p\int_S \bar{R}_i(x,p)\bar{V}_i(x,\xi,p)dS(x)\\ &+\int_B \bar{W}(x,p)\bar{K}(x,\xi,p)dV(x)+\frac{1}{T_0}\int_S \bar{k}(x,p)\bar{K}(x,\xi,p)dS(x)\\ &+p\int_B \bar{\chi}(x,p)\bar{\Omega}(x,\xi,p)dV(x)+p\int_S \bar{d}(x,p)\bar{\Omega}(x,\xi,p)dS(x),\end{aligned}$$

$$\begin{aligned}(36)\qquad \bar{\Phi}(\xi,p) &= \int_B \bar{X}_i(x,p)\bar{W}_i(x,\xi,p)dV(x)+\int_S \bar{R}_i(x,p)\bar{W}_i(x,\xi,p)dS(x)\\ &+\frac{1}{p}\int_B \bar{W}(x,p)\bar{L}(x,\xi,p)dV(x)+\frac{1}{T_0 p}\int_S \bar{k}(x,p)\bar{L}(x,\xi,p)dS(x)\\ &+\int_B \bar{\chi}(x,p)\bar{\Gamma}(x,\xi,p)dV(x)+\int_S \bar{d}(x,p)\bar{\Gamma}(x,\xi,p)dS(x).\end{aligned}$$

It is seen that Eqs. (34) to (36) may be used to find the functions u_j, θ, Φ at the point $\xi \in B$ for prescribed load R_i, heat flow k and electric surface load d. Equations (30) also remain valid. In the particular elasto-kinetic case, where the coupling with the electric and temperature field is not taken into consideration, there remains only the first Eq. (34) with the first two terms on the right-hand side.

We have described only three particular cases of determining Green's functions. If, for instance, the Green's functions are constructed in such a way that the functions becoming zero at S are the temperature, the electric potential and the load, we shall obtain from Eqs. (27) to (29) the solution of the problem of determining the functions u_j, θ, Φ at $\xi \in B$ for assigned values of load R_i, temperature ϑ, and potential φ at the surface S.

The Somigliano and Green's equations described here may be transformed to the case of loads harmonically variable in time. Then, in place of the parameter p of the Laplace transformation, we should substitute $i\omega$, $i=\sqrt{-1}$, where ω is the frequency of vibration and the transforms of the functions should be replaced by their amplitudes.

The equations given here comprise a number of particular cases. For bodies showing no piezoelectric effect we are concerned with a coupled thermoelastic problem. If there are no heat sources and the surface is not heated — if the body is in a thermally adiabatic state, there remains only the mechanical-electrical coupling.

The stationary case deserves mentioning. In this case, the time-derivatives in (12) to (14) vanish, and the heat Eq. (14) becomes independent of the remaining equations. By performing the same transformations as before, we obtain the following reciprocity equations

$$\int_B (X_i u_i' - X_i' u_i)dV + \int_S (R_i U_i' - R_i' U_i)dS + \int_B (\chi\Phi' - \chi'\Phi)dV$$
$$+ \int_S (d\varphi' - d'\varphi)dS + \int_B \lambda_{ij}^E(\theta\varepsilon_{ij}' - \theta'\varepsilon_{ij})dV + \int_B p_i^s(\theta'\Phi_{,i} - \theta\Phi'_{,i})dV = 0, \tag{37}$$

$$\int_B (W\theta' - W'\theta)dV + \frac{1}{T_0}\int_S (k\dot{\vartheta}' - k'\dot{\vartheta}')dS = 0, \tag{38}$$

The functions θ, θ' in Eq. (37) are treated as functions obtained from the uncoupled heat equation. In (38), we recognize the reciprocity theorem for the stationary heat equation. Generalized equations of Somigliano and Green may also be given in this stationary case, similarly to the dynamic case.

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Streszczenie

TWIERDZENIE O WZAJEMNOŚCI DLA SPRZĘŻONYCH PÓL MECHANO-TERMO-ELEKTRYCZNYCH WYSTĘPUJĄCYCH W KRYSZTAŁACH PIEZOELEKTRYCZNYCH

Przedmiotem pracy jest wyprowadzenie twierdzenia o wzajemności dla ośrodka anizotropowego piezoelektrycznego, przy uwzględnieniu sprzężeń między polem deformacji, temperatury i polem elektrycznym. Twierdzenie to słuszne dla zagadnień dynamicznych zastosowane zostało do wyznaczenia przemieszczeń u_i, temperatury θ oraz potencjału elektrycznego Φ wewnątrz ciała, przy danych warunkach brzegowych. W rezultacie otrzymano rozszerzone na ośrodek anizotropowy piezoelektryczny oraz na zagadnienia dynamiczne znane z elastostatyki twierdzenie Somigliana i Greena. Podano wreszcie twierdzenie o wzajemności dla zagadnienia stacjonarnego.

Резюме

ТЕОРЕМА О ВЗАИМНОСТИ ДЛЯ СОПРЯЖЕННЫХ МЕХАНО-ТЕРМО-ЭЛЕКТРИЧЕСКИХ ПОЛЕЙ, ПОЯВЛЯЮЩИХСЯ В ПЬЕЗОЭЛЕКТРИЧЕСКИХ КРИСТАЛЛАХ

В работе выводится теорема о взаимности для пьезоэлектрической анизотропной среды, при учёте сопряжения между полем деформации, температуры и электрическими полями. Эта теорема, справедливая для динамических задач применяется при определении перемещений u_i, температуры θ и электрического потенциала Φ внутри тела, при заданных краевых условиях. В результате получено расширенную на анизотропную среду и на динамические задачи, известную из упругостатики теорему Сомильяна и Грина. В заключение приводится теорема о взаимности для стационарной задачи.

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