POLISH ACADEMY OF SCIENCES
INSTITUTE OF BASIC TECHNICAL PROBLEMS

PROCEEDINGS OF VIBRATION PROBLEMS

QUARTERLY

VOL. 5 WARSAW 1964 No. 3
PAŃSTWOWE WYDAWNICTWO NAUKOWE

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OF VIBRATION PROBLEMS
WARSAW, 3, 5 (1964)

MIXED BOUNDARY-VALUE PROBLEMS OF ELASTODYNAMICS

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1. Introduction

In the papers [1] (1955) and [2] (1962), a general method was presented of solving problems of elastostatics with mixed boundary conditions. In the period from 1955 to 1962, a number of dynamic problems were solved concerning, mainly, stationary vibrations of plates [3, 4, 5], and also stationary vibrations of elastic parallelepipeds and finite cylinders [6, 7, 8], the methods used basing as presented n [1] and [2].

In this paper will be presented the application of the method suggested for solving static problems to problems of dynamics–that is, of stationary and non-stationary vibrations of elastic bodies.

Let us consider a simply connected elastic body B bounded by the surface S, and let this surface consists of two smooth surfaces intersecting along the curve

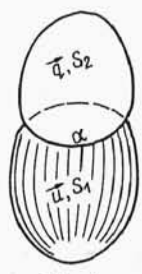

Fig. 1

a (Fig. 1). Inside the body, there act body forces $\mathbf{X}$, on the surface S_2 external forces $\mathbf{q}$, and on the surface S_1, the displacements $\mathbf{u}$ are given.

We assume that the external and body forces, which are functions of the time variable t, start acting at the instant $t = 0+$. By x we denote a point inside of the body with the coordinates x_1, x_2, x_3, while ξ is a point of the surface S, with the coordinates ξ_1, ξ_2, ξ_3.

The functions $u_i(x, t)$, representing the components of the displacement vector, have to satisfy the motion equations, in terms of displacements,

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + X_i = \varrho \ddot{u}_i, \quad i, j = 1, 2, 3 \tag{1.1}$$

the boundary conditions

$$(1.2)\qquad \begin{cases} u_i(\xi, t) = f_i(\xi, t) & \text{on the surface } S_1, \\ \sigma_{ij}(\xi, t)n_j = q_i(\xi, t) & \text{on the surface } S_2, \end{cases}$$

and the initial conditions

$$(1.3)\qquad u_i(x, 0) = 0, \quad \dot{u}_i(x, 0) = 0, \quad x \in B.$$

In the above equations μ and λ are the Lame constants, σ_{ij} the components of the stress tensor, while n_i denotes the components of the unit normal vector of the surface S. By a dot over a function we denote its derivative with respect to the time-variable.

The stresses σ_{ij} are connected with the deformations ε_{ij} by Hooke's law

$$(1.4)\qquad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\,\delta_{ij}\varepsilon_{kk}, \quad i, j, k = 1, 2, 3,$$

while the deformations are given, in terms of the displacements u_i, by the relations

$$(1.5)\qquad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

Thus, after solving Eq. (1.1) with the prescribed boundary conditions (1.2) and the initial conditions (1.3), and having determined the displacements u_i, we obtain the deformations ε_{ij} and stresses σ_{ij} from (1.5) and (1.4).

Let us observe that on the surface S_1 the displacements are known, while the support reactions–that is, the surface forces $\sigma_{ij}n_j = q_i$ are on this surface unknown. Similarly, on the surface S_2 we know the loading $q_i = \sigma_{ij}n_j$, while the function u_i is here unknown. Hence, in solving problems with mixed boundary conditions, we can assume as the unknown function either q_i on S_1 or u_i on S_2.

These two possibilities of choosing the unknown function lead to two variants of solving the problems. We shall discuss them successively in the further sections of this paper. We shall, however, first introduce the operator notation for the equation (1.1) and the boundary conditions (1.2). Equation (1.1) will then assume the form:

$$(1.1')\qquad D_{ij}[u_j(x, t)] + X_i(x, t) = 0 \quad x \in B,$$

where

$$D_{ij} = \mu\,\delta_{ij}\Box_2^2 + (\lambda+\mu)\,\partial_i\,\partial_j,$$

$$\Box_2^2 = \nabla^2 - \frac{1}{c_1^2}\,\partial_t^2, \quad c_2^2 = \frac{\mu}{\varrho};$$

while the boundary conditions will take the form:

$$(1.2')\qquad \begin{cases} u_i(\xi, t) = f_i(\xi, t) & \text{on the surface } S_1, \\ q_i(\xi, t) = L[u_i(\xi, t)] = \mu(u_{i,j} + u_{j,i})n_j + \lambda n_i \operatorname{div} \mathbf{u} & \text{on the surface } S_2. \end{cases}$$

2. The First Variant of Solution

Let us try to reduce the problem with mixed boundary conditions, as formulated in Sec. 1, to a matter of solving simpler problems, where the type of boundary conditions occurring on the two adjacent surfaces S_1 and S_2 is the same.

To this end we introduce what is called the "fundamental system" which is represented by an elastic body of the same shape as in Fig. 1, but free from loading on the surfaces S_1 and S_2. Since, however, this body will be subject to forces, it should be prevented from suffering displacements and rotations. We achieve this by assuming that at any arbitrary point O of this body the displacements and rotations vanisch. In the "fundamental system", chosen in this way, we now determine the Green tensor-field of displacements $G(x, x', t) = [G_{ik}(x, x', t)]$.

This field will be constructed as follows. We apply at the point $x' \in B$ of our "fundamental system" an instantaneous concentrated force, parallel to the axis x_k.

Owing to this loading in the "fundamental system", there occurs the displacement vector $\mathbf{G}^{(k)}$, with the components $G_{ik}(i = 1, 2, 3)$. Directing the instantaneous concentrated force, successively, parallel to the axes x_1, x_2, and x_3, and then assuming for k, successively, the indices $k = 1, 2, 3$, we obtain nine quantities $G_{ik}(i, k = 1, 2, 3)$ which form the symmetric $(G_{ik} = G_{ki})$ Green tensor of displacements. The Green functions G_{ik} must satisfy the equations of motion, in terms of displacements,

$$D_{ij}[G_{jk}(x, x', t)] + \delta(x-x')\delta(t)\delta_{ik} = 0, \quad i, j, k = 1, 2, 3, \quad x, x' \in B \tag{2.1}$$

with the initial conditions

$$G_{ik}(x, x', 0) = 0, \quad \dot{G}_{ik}(x, x', 0) = 0, \tag{2.2}$$

and the homogeneous boundary conditions

$$L[G_{ik}(\xi, x', t)] = 0 \text{ on the surfaces } S_1 \text{ and } S_2, \xi \in S. \tag{2.3}$$

The relations (2.1) represent three sets of equations (for $k = 1, 2, 3$). In (2.1) $\delta(x-x')\delta(t) = \delta(x_1-\xi_1)\,\delta(x_2-\xi_2)\,\delta(x_3-\xi_3)\,\delta(t)$ denotes the Dirac function expressing the instantaneous concentrated force applied at the point $x' \in B$, while by δ_{ik} we understand the Kronecker symbol. The quantity $\delta(x-x')\delta(t)\delta_{ik}$ indicates that the concentrated force is directed parallel to the axis x_k. In the boundary conditions, the expression $L(G_{ik})$ denotes the operation $\sigma_{ij}^{(k)}n_j$, where $\sigma_{ij}^{(k)}$ is the stress produced by the instantaneous concentrated force acting at the point x' and directed parallel to the axis x_k. The condition (2.3) shows that the surface of the body is free from loadings. In what follows, we assume that the Green functions G_{ik} are known—that is, they have been determined in the chosen "fundamental system".

Now we proceed to solving the second auxiliary problem in our fundamental system. Let the loading $q_i(x, t)$ act on the surface S_2 of the "fundamental system", and, inside of the body, the body forces $X_i(x, t)$, applied at the instant $t = 0^+$. All these forces will produce, in the elastic body, the displacements u_i^0 and stresses σ_{ij}^0.

The displacements have to satisfy the motion equations:

$$D_{ij}[u_j^0(x, t)] + X_i(x, t) = 0 \tag{2.4}$$

the homogeneous initial conditions

$$u_i^0(x, 0) = 0, \quad \dot{u}_i^0(x, 0) = 0,$$

and the boundary conditions

$$L[u_i^0(\xi,t)] = \sigma_{ij}^0 n_j = q_i(\xi,t) \quad \text{on the surface } S_2, \tag{2.5}$$
$$L[u_i^0(\xi,t)] = 0 \quad \text{on the surface } S_1.$$

We assume that the displacements u_i^0 have been determined in the chosen "fundamental system", and, in what follows, we shall consider these quantities as known functions.

In our further considerations, we shall use the reciprocal Betti theorem.

This theorem has, with respect to dynamic problems, the forms:

$$\iiint\limits_{(B)} (X_i u_i' - X_i' u_i)\, dV + \iint\limits_{(S)} (p_i u_i' - p_i' u_i)\, dS - \varrho \iiint\limits_{(B)} (\ddot{u}_i u_i' - \ddot{u}_i' u_i)\, dV = 0, \tag{2.6}$$

or

$$\iiint\limits_{(B)} [u_i D_{ij}(u_j') - u_i' D_{ij}(u_j)]\, dV + \iint\limits_{(S)} [u_i' L(u_i) - u_i L(u_i')]\, dS - \varrho \iiint\limits_{(B)} (\ddot{u}_i u_i' - \ddot{u}_i' u_i)\, dV = 0. \tag{2.7}$$

In the above equations, X_i, p_i denote body and surface forces, respectively, which produce in the body the displacements u_i, while X_i' and p_i' belong to the other system of forces producing the displacements u_i'.

Let us apply to the equation (2.6) the one-side Laplace transformation given by the formulae

$$\tilde{u}_i(x,p) = \int\limits_0^\infty u_i(x,t) e^{-pt}\, dt, \quad \tilde{p}(x,p) = \int\limits_0^\infty p_i(x,t) e^{-pt}\, dt, \tag{2.8}$$

where p is the parameter of transformation — that is, a number with the real part so chosen as to make the integral (2.8) convergent. After carrying out the Laplace transformation on the expression (2.6), we have

$$\iiint\limits_{(B)} (\tilde{X}_i \tilde{u}_i' - \tilde{X}_i' \tilde{u}_i)\, dV + \iint\limits_{(S)} (\tilde{p}_i \tilde{u}_i' - \tilde{p}_i' \tilde{u}_i)\, dS \tag{2.9}$$
$$- \varrho \iiint\limits_{(B)} \{[p^2 \tilde{u}_i - p u_i(x,0) - \dot{u}_i(x,0)] \tilde{u}_i' - [p^2 \tilde{u}_i' - p u_i'(x,0) - \dot{u}_i'(x,0)] \tilde{u}_i\}\, dV = 0.$$

However, according to our above assumptions concerning the application of loadings at the instant $t = 0^+$, and to the consequent initial conditions, we have to substitute in the expression (2.9) $u_i(x,0) = \dot{u}_i(x,0) = u_i'(x,0) = \dot{u}_i'(x,0) = 0$. Then Eq. (2.9) undergoes considerable simplification, and takes the forms.:

$$\iiint\limits_{(B)} (\tilde{X}_i \tilde{u}_i' - \tilde{X}_i' \tilde{u}_i)\, dV + \iint\limits_{(S)} (\tilde{p}_i \tilde{u}_i' - \tilde{p}_i' \tilde{u}_i)\, dS = 0, \tag{2.10}$$

or

$$\iiint\limits_{(B)} [\tilde{u}_i D_{ij}(\tilde{u}_j') - \tilde{u}_j D_{ij}(\tilde{u}_j')]\, dV + \iint\limits_{(S)} [\tilde{u}_i' L(\tilde{u}_i) - \tilde{u}_i L(\tilde{u}_i')]\, dS = 0. \tag{2.11}$$

Let us now apply the formula (2.11) to the displacements u_i as given by Eq. (1.1), and to the displacements G_{ik} satisfying the relations (2.1). Substituting into (2.11)

$$\tilde{u}'_i(x,p) = \tilde{G}_{ik}(x,x',p), \qquad D_{ij}[\tilde{G}_{jk}(x,x',p)] = -\delta(x-x')\delta_{ik}$$

and denoting by $L(u_i) = q_i$ the known loading acting on the surface S_2, and by $L(u_i) = R_i$ the unknown function of the support reactions on the surface S_1, we obtain:

$$(2.12)\qquad \iiint\limits_{(B)} [\tilde{G}_{ik}(x,x',p)\tilde{X}_i(x,p) - \delta(x-x')\delta_{ik}\tilde{u}_i(x,p)]\,dV(x) + \iint\limits_{(S_1)} \tilde{R}_i(\xi,p)\tilde{G}_{ik}(\xi,x',p)\,dS(\xi) + \iint\limits_{(S_2)} \tilde{q}_i(\xi,p)\tilde{G}_{ik}(\xi,x',p)\,dS(\xi).$$

Bearing in mind the relation

$$\iiint\limits_{(B)} \delta(x-x')\delta_{ik}\tilde{u}_i(x,p)\,dV(x) = \tilde{u}_i(x',p),$$

we represent (2.12) in the form

$$(2.13)\qquad \tilde{u}_k(x',p) = \iiint\limits_{(B)} \tilde{X}_i(x,p)\tilde{G}_{ik}(x,x',p)\,dV(x) + \iint\limits_{(S_2)} \tilde{q}_i(\xi,p)\tilde{G}_{ik}(\xi,x',p)\,dS(\xi) + \iint\limits_{(S_1)} \tilde{R}_i(\xi,p)\tilde{G}_{ik}(\xi,x',p)\,dS(\xi).$$

Let us now employ the Betti theorem (2.11) to the functions u_1^0 and G_{ik}. Introducing into (2.11) the relations

$$\tilde{u}_i = \tilde{u}_i^0, \quad D_{ij}(\tilde{u}_j^0) = -\tilde{X}_i, \quad \tilde{u}'_i = \tilde{G}_{ik}, \quad D_{ij}(\tilde{G}_{ik}) = -\delta(x-x')\delta_{ik}$$

and $L(\tilde{u}_i^0) = \tilde{q}_i$ on the surface S_2, $L(\tilde{u}_i^0) = 0$ on S_1, and $L(\tilde{G}_{ik}) = 0$ on S_1 and S_2, we arrive at the following equation

$$(2.14)\qquad \tilde{u}_k^0(x',p) = \iiint\limits_{(B)} \tilde{X}_i(x,p)\tilde{G}_{ik}(x,x',p)\,dV(x) + \iint\limits_{(S_2)} \tilde{q}_i(\xi,p)\tilde{G}_{ik}(\xi,x',p)\,dS(\xi).$$

From comparison of (2.13) and (2.14) it follows the relation

$$(2.15)\qquad \tilde{u}_k(x',p) = \tilde{u}_k^0(x',p) + \iint\limits_{(S_1)} \tilde{R}_i(\xi,p)\tilde{G}_{ik}(\xi,x',p)\,dS(\xi).$$

We have obtained functional equations connecting the unknown functions of the displacements $\tilde{u}_k(x',p)$ with the unknown support reactions $\tilde{R}_i(\xi,p)$ on the surface S_1. Having applied to (2.15) the inverse Laplace transformation, we have

$$(2.16)\qquad u_k(x',t) = u_k^0(x',t) + \int_0^t d\tau \iint\limits_{(S_1)} R_i(\xi,\tau)G_{ik}(\xi,x',t-\tau)\,dS(\xi).$$

Now we can determine the unknown functions $\tilde{R}_i(\xi,p)$ using the boundary condition (1.2) on the surface S_1. On this surface S_1 we know the displacements $u_i(\xi,t) = f_i(\xi,t)$; thus we can find the Laplace transform $\tilde{f}_i(\xi,p)$. Passing in Eq. (2.15) with the point $x' \in B$ to the point ξ' on the surface S_1, we obtain the relation:

$$(2.17)\qquad \tilde{f}_k(\xi',p) = \tilde{u}_k^0(\xi',p) + \iint\limits_{(S_1)} \tilde{R}_i(\xi,p)\,G_{ik}(\xi,\xi',p)\,dS(\xi), \qquad i,k = 1,2,3.$$

In this equation, all quantities, except $\tilde{R}_i(\xi, p)$, are known. We have arrived at a system of integral equations of the first kind. After solving these equations for $\tilde{R}_i(\xi, p)$, we are able to determine the functions $\tilde{u}_k(x', p)$. The inverse Laplace transformation then yields the required functions $u_k(x', t)$. Finally, we determine from (1.4) and (1.5) the stresses σ_{ij} and deformations ε_{ij} in the elastic body with mixed boundary conditions.

The formulae (2.15)–(2.17) can be reduced to another form which is very convenient in application.

Let us introduce a new displacement tensor $U_{ik}(x, \xi', t)$. This tensor will be described in our "fundamental system" by the set of homogeneous differential equations:

$$D_{ij}[U_{jk}(x, \xi', t)] = 0 \quad x \in B, \quad \xi' \in S_1, \quad i, j, k = 1, 2, 3, \tag{2.18}$$

with the initial conditions

$$U_{ik}(x, \xi,' 0) = 0, \quad \dot{U}_{ik}(x, \xi', 0) = 0, \tag{2.19}$$

and non-homogeneous boundary conditions

$$\begin{aligned} &L[U_{ik}(\xi, \xi', t)] = \delta(\xi-\xi')\,\delta_{ik}\delta(t), \quad \xi, \xi' \in S_1 \text{ on the surface } S_1,\\ &L[U_{ik}(\xi, \xi', t)] = 0, \quad \xi, \xi' \in S_2 \quad \text{on the surface } S_2. \end{aligned} \tag{2.20}$$

It follows from (2.20) that at the point ξ' of the surface S_1 there acts an instantaneous concentrated force parallel to the axes $x_k (k = 1, 2, 3)$.

After solving Eq. (2.18) with the initial conditions (2.19) and boundary conditions (2.20), we obtain nine functions U_{ik}. However, owing to the symmetry of the displacement tensor ($U_{ik} = U_{ki}$), the number of independent functions reduces to six. Let us now connect the functions G_{ik} with U_{ik} using, to this end, the reciprocal theorem (2.11). Then we arrive at the relation

$$\iiint_{(B)} \tilde{U}_{jk}(x, \xi', p)\,\delta(x-x')\,\delta_{ik}\,dV(x) - \iint_{(S_1)} \tilde{G}_{jk}(\xi, x', p)\,\delta(\xi-\xi')\,\delta_{ik}\,dS(\xi) = 0,$$

whence it follows that

$$\tilde{U}_{ik}(x', \xi', p) = \tilde{G}_{ik}(\xi', x', p), \quad U_{ik}(x', \xi', t) = G_{ik}(\xi', x', t). \tag{2.21}$$

Inserting the above relation into (2.15), we have

$$\tilde{u}_k(x', p) = \tilde{u}_k^0(x', p) + \iint_{(S_1)} \tilde{R}_i(\xi, p)\tilde{U}_{ik}(x', \xi, p)\,dS(\xi). \tag{2.22}$$

The inverse Laplace transformation yields:

$$u_k(x', t) = u_k^0(x', t) + \int_0^t d\tau \iint_{(S_1)} R_i(\xi, \tau)\,U_{ik}(x', \xi, t-\tau)\,dS(\xi). \tag{2.23}$$

Passing with the point $x' \in B$ to the point $\xi' \in S_1$, and using the boundary condition (1.2) on S_1, we obtain the following integral equation

$$f_k(\xi', t) = u_k^0(\xi', t) + \int_0^t d\tau \iint_{(S_1)} R_i(\xi, \tau) U_{ik}(\xi', \xi, t-\tau)\,dS(\xi). \tag{2.24}$$

The functional relation (2.23) can be interpreted as follows. The displacement u_i at the point $x' \in B$ can be composed of two parts—of the displacement u_i^0 produced in the "fundamental system" by the body forces X_i and the loadings q_i applied on the surface S_2 (thus, in the system free from forces on the bounding surface S_1–that is for $R_i = 0$), and of the displacement produced by the unknown support reactions R_i, acting on S_1. The integral

$$\int_0^t d\tau \iint_{(S_1)} R_i(\xi, \tau) U_{ik}(\xi', \xi, t-\tau)\, dS(\xi),$$

expresses the superposition of the effects of this support reactions; the displacements produced by the reactions are integrated both over the surface S_1 and over the time, from the instant $\tau = 0$ to $\tau = t$.

Let us now consider the case of stationary vibrations, where the external loadings and the body forces are harmonic functions of the time variable:

$$X_i(x, t) = \mathrm{Re}[X_i^*(x)\, e^{i\omega t}], \quad q_i(x, t) = \mathrm{Re}[q^*(x)\, e^{i\omega t}]. \tag{2.25}$$

The quantities X_i^* and q^* are independent of the time, and let the frequency be different from the frequency of free vibrations. The displacements will also be periodic functions of the time variable, with the frequency ω

$$u_i(x, t) = \mathrm{Re}\,[u_i^*(x)\, e^{i\omega t}], \quad G_{ik}(x, x', t) = \mathrm{Re}\,[G_{ik}^*(x, x')\, e^{i\omega t}]. \tag{2.26}$$

According to the assumptions (2.25), the motion equations take the form:

$$\begin{aligned} &D_{ij}^*[u_j^*(x, \omega)] + X_i^*(x) = 0, \quad x \in B, \\ &D_{ij}^* = \mu\left(\nabla^2 + \frac{\omega^2}{c_2^2}\right)\delta_{ij} + (\lambda+\mu)\partial_i \partial_j. \end{aligned} \tag{2.27}$$

The appropriate boundary conditions will be written as follows

$$\begin{aligned} u_i^*(\xi) &= f_i^*(\xi), \quad \xi \in S_1, \\ L[u_i^*(\xi)] &= q_i^*(\xi), \quad \xi \in S_2. \end{aligned} \tag{2.28}$$

For the case of periodic vibrations, we write the reciprocal theorem (2.6) in the forms:

$$\iiint_{(B)} (X_i^* u_i'^* - X_i'^* u_i^*)\, dV + \iint_{(S)} (p_i^* u_i'^* - p_i'^* u_i^*)\, dS = 0, \tag{2.29}$$

or

$$\iiint_{(B)} [u_i^* D_{ij}^*(u_j'^*) - u_i'^* D_{ij}^*(u_j^*)]\, dV + \iint_{(S)} [u_i'^* L(u_i^*) - u_i^* L(u_i'^*)]\, dS = 0. \tag{2.29'}$$

Applying Eq. (2.29') first to the function u_i and G_{ik}, and next to u_i and G_{ik}, we obtain the following functional equation:

$$u_k^*(x', \omega) = u_k^{0*}(x', \omega) + \iint_{(S_1)} R_i^*(\xi)\, G_{ik}^*(\xi, x', \omega)\, dS(\xi). \tag{2.30}$$

Here u_u^{*0} denotes the amplitude of the displacements produced in the fundamental system by the body forces $X_i^* e^{i\omega t}$, and the surface loadings $q^* e^{i\omega t}$ acting on S_2. Introducing the Green functions U_{ik}^*, we can represent (2.30) in the form:

$$u_k^*(x', \omega) = u_k^{0*}(x', \omega) + \iint_{(S_1)} R_i^*(\xi) U_{ik}^*(x', \xi, \omega)\, dS(\xi). \tag{2.31}$$

Passing now with the point x' to the point ξ' on S_1, we arrive at the required system of integral equations:

$$f_k^*(\xi') = u_k^{*0}(\xi', \omega) + \iint_{(S_1)} R_i^*(\xi) U_{ik}^*(\xi', \xi, \omega)\, dS(\xi), \qquad i, k = 1, 2, 3 \tag{2.32}$$

whence we are able to determine the unknown functions $R_i^*(\xi)$.

Let us return to our fundamental functional Eq. (2.23). after passing with the point $x' \in B$ to the point ξ' on the surface S_1

$$u_k(\xi', t) = u_k^0(\xi', t) + \int_0^t d\tau \iint_{(S_1)} R_i(\xi, \tau) G_{ik}(\xi, \xi', t-\tau)\, dS(\xi). \tag{2.33}$$

If the surface S_1 is fixed elastically—that is, rests on a Winklerian foundation, then we have:

$$u_k(\xi', t) = -\frac{1}{\varkappa_k} R_k(\xi', t). \tag{2.34}$$

Here $\varkappa_k$ $(k = 1,2,3)$ are the moduli of the elastic foundation, and thus constant coefficients.

Inserting (2.34) into (2.33), we obtain a system of integral equations of the second kind

$$R_k(\xi', t) + \varkappa_k \int_0^t d\tau \iint_{(S_1)} R_i(\xi, \tau) G_{ik}(\xi, \xi', t-\tau)\, dS + \varkappa_k u_k^0(\xi', t) = 0 \qquad i, k = 1, 2, 3. \tag{2.35}$$

In this case, applying to Eqs. (2.35) the Laplace transformation, we arrive at a system of Fredholm equations of the second kind. This system can be solved by the iteration method, owing to the weak singularity of the kernel G_{ik}. For $\varkappa_k \to \infty$, we obtain a body which is rigidly fixed on its surface S_1, while for $\varkappa_k \to 0$, this surface is free from loading, and $R_k = 0$.

Let us observe that the surface S_1 may be constructed so as to carry over only reactions normal to this surface. Then the additional linear relations, connecting the components R_i, enable us to reduce the system of three integral Eq. (2.24) to one integral equation, since, in this case, only one support reaction, normal to the surface, is unknown.

Let us, finally, observe that our considerations include also such cases in which the surfaces S_1 reduce to curves.

We can generalize our method in order to solve the problem of an elastic body having more then one support surface. Such a body is represented in Fig. 2. The bounding surface S consists of three smooth surfaces intersecting along the curves α and β. Let there act, inside of this body, body forces $\mathbf{X}$, and on its surface the external

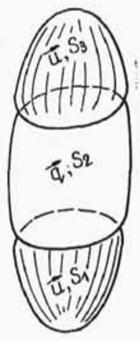

Fig. 2

loadings $\mathbf{q}$. We shall assume, for the sake of simplicity, that the elastic body considered is rigidly fixed on its surfaces S_1 and S_3. The displacements u_i must satisfy the differential equation

$$D_{ij}(u_j)+X_i = 0 \tag{2.36}$$

with the boundary conditions

$$u_i = 0 \text{ on } S_1 \text{ and } S_3, \quad L(u_i) = q_i \text{ on } S_2, \tag{2.37}$$

and the initial conditions

$$u_i = 0, \quad \dot{u}_i = 0, \quad \text{for} \quad t = 0. \tag{2.38}$$

As the "fundamental system" we assume, in this case, the elastic body which is rigidly fixed on the surface S_3 only, and free from loadings on the surfaces S_1 and S_2. In this fundamental system, we determine the Green functions G_{ik}. These last are described by the equations:

$$D_{ij}(G_{jk})+\delta(x-x')\delta(t)\delta_{ik} = 0 \tag{2.39}$$

with the conditions

$$\begin{aligned} &G_{ik} = 0 \text{ on } S_3, \quad L(G_{ik}) = 0 \text{ on } S_1 \text{ and } S_2, \\ &G_{ik} = 0, \quad \dot{G}_{ik} = 0 \quad \text{for} \quad t = 0. \end{aligned} \tag{2.40}$$

In a similar way, we determine in the "fundamental system" the displacements u_i^0, by solving the differential equations

$$D_{ij}(u_j^0)+X_i = 0 \tag{2.41}$$

with the conditions

$$\begin{aligned} &u_i^0 = 0 \text{ on } S_3, \quad L(u_i^0) = 0 \text{ on } S_1, \quad L(u_i^0) = q_i \text{ on } S_2, \\ &u_i^0 = 0, \quad \dot{u}_i^0 = 0, \quad \text{for } t = 0. \end{aligned} \tag{2.42}$$

Applying to the functions u_i and G_{ik}, and to the functions u_i^0 and G_{ik} the reciprocal theorem (2.11), we obtain the following functional equation:

$$\tilde{u}_k(x', p) = \tilde{u}_k^0(x', p) + \iint_{(S_1)} \tilde{R}_i(\xi, p)\tilde{G}_{ik}(\xi, x', p)\,dS(\xi), \tag{2.43}$$

where we have denoted by $R_i(\xi, t)$ the unknown support reactions on the surface S_1.

Passing from the point $x' \in B$ to the point $\xi' \in S_1$, we arrive at a system of integral equations of the first kind

$$0 = \tilde{u}_k^0(\xi', p) + \iint_{(S_1)} \tilde{R}_i(\xi, p)\tilde{G}_{ik}(\xi, \xi', p)\,dS(\xi), \qquad i, k = 1, 2. 3, \tag{2.44}$$

whence we can already determine the unknown functions $\tilde{R}_i(\xi, p)$. In the example presented above, we assumed a new fundamental system in the form of a body fixed at S_3, and free from loadings on S_1 and S_2. However, we can also assume as a fundamental system the elastic body which is prevented from motion at an arbitrary point 0 (we assume that at this point the displacements and rotations vanish) and free from loadings on the surfaces S_1, S_2 and S_3.

Then the Green functions G_{ik} should satisfy the equation

$$D_{ij}(G_{jk}) + \delta(x - x')\delta_{ik}\delta(t) = 0, \tag{2.45}$$

with the boundary conditions

$$L(G_{ik}) = 0 \quad \text{on } S_1, S_2 \text{ and } S_3, \tag{2.46}$$

and the initial conditions

$$G_{ik} = 0, \quad \dot{G}_{ik} = 0, \quad \text{for } t = 0. \tag{2.47}$$

The function u_i^0 has to satisfy the differential equation

$$D_{ij}(u_j^0) + X_i = 0, \tag{2.48}$$

with the boundary conditions

$$L(u_i^0) = 0 \text{ on } S_1 \text{ and } S_3, \; L(u_i^0) = q_i \text{ on } S_2, \tag{2.49}$$

and initial conditions

$$u_i^0 = 0, \quad \dot{u}_i^0 = 0 \quad \text{for } t = 0. \tag{2.50}$$

If we now apply the reciprocal theorem (2.11) to the function u_i, which is described by Eq. (2.39) and the conditions (2.40), and to the function G_{ik}, and, finally, to the functions u_i^0 and G_{ik}, we obtain the following functional equation:

$$\tilde{u}_k(x', p) = \tilde{u}_k^0(x', p) + \iint_{(S_1)} \tilde{R}_1(\xi_1, p)\tilde{G}_{ik}(\xi_1, x', p)\,dS(\xi_1) + \iint_{(S_3)} \tilde{R}_3(\xi_3, p)\tilde{G}_{ik}(\xi_3, x', p)\,dS(\xi_3), \tag{2.51}$$

where $R_1(\xi_1, t)$ and $R_3(\xi_3, t)$ are the unknown surface forces acting on S_1 and S_3. We shall find these last unknown functions from the condition of vanishing displacements on S_1 and S_3. Now, passing with the point $x' \in B$ to the point $\xi_1' \in S_1$, and with the point $x' \in B$ to the point $\xi_3' \in S_3$, we obtain a system of two integral equations:

$$0 = \tilde{u}_k^0(\xi_1', p) + \iint_{(S_1)} \tilde{R}_1(\xi_1, p)\, \tilde{G}_{ik}(\xi_1, \xi_1', p)\, dS(\xi_1) + \iint_{(S_3)} \tilde{R}_3(\xi_3, p)\, \tilde{G}_{ik}(\xi_3, \xi_1', p)\, dS(\xi_3),$$

(2.52)

$$0 = \tilde{u}_k^0(\xi_3', p) + \iint_{(S_1)} \tilde{R}_1(\xi_1, p)\, \tilde{G}_{ik}(\xi_1, \xi_3', p)\, dS(\xi_1) + \iint_{(S_3)} \tilde{R}_3(\xi_3, p)\, \tilde{G}_{ik}(\xi_3, \xi_3', p)\, dS(\xi_3) \quad i, k = 1, 2, 3,$$

$$\xi_1, \xi_1' \in S_1, \quad \xi_3, \xi_3' \in S_3.$$

After solving the system of Eqs. (2.52), and determining the functions R_1 and R_3, we find the displacement $u_k(x', t)$ from the functional Eq. (2.51). It is clear that we can extend our considerations to the case in which on the surfaces S_2, S_4, and S_6, ... are prescribed the loadings, while on the surfaces S_1, S_3, and S_5, ... are given the displacements.

3. The Second Variant of the Solution

Let us return to the solution of the problem as presented in Fig. 1. The displacements are given on the surface S_1, while the external forces are prescribed on S_2. The displacements, produced by the loading $\mathbf{q} \in S_2$ and body forces $\mathbf{X} \in B$, are determined by the equations (1.1), the boundary conditions (1.2) and the initial conditions (1.3).

Let us assume as the unknown functions of the problem the displacements $\boldsymbol{u}$ of the surface S_2. Having thus chosen the unknown functions, we have to assume a different fundamental system. We choose as the fundamental system the elastic body entirely fixed on S_1 and S_2. In this fundamental system, we define the Green function as follows. The functions G_{ik} should satisfy the differential equation

$$D_{ij}[G_{jk}(x, x', t)] + \delta(x - x')\,\delta(t)\,\delta_{ik} = 0, \quad x, x' \in B, \tag{3.1}$$

the boundary conditions

$$G_{ik}(\xi, x', t) = 0 \text{ on the surfaces } S_1 \text{ and } S_2, \tag{3.2}$$

and the initial conditions

$$G_{ik}(x, x', 0) = 0, \quad \dot{G}_{ik}(x, x', 0). \tag{3.3}$$

We also determine in our fundamental system the displacement $u_i^0(x, t)$ as the solution of the differential equation

$$D_{ij}[u_j^0(x, t)] + X_i(x, t) = 0 \tag{3.4}$$

with the boundary conditions

$$u_i^0(\xi, t) = 0 \text{ on the surfaces } S_1 \text{ and } S_2, \tag{3.5}$$

and initial conditions

$$u_i^0(x, 0) = 0, \quad \dot{u}_i^0(x, 0) = 0. \tag{3.6}$$

Assuming that the functions G_{ik} and u_i^0 have been determined in our fundamental system, we shall, in what follows, consider these quantities as known functions.

Let us now apply to the functions u_i (satisfying Eq. (1.1) and the conditions (1.2) and (1.3)), and to the functions G_{ik} the Betti reciprocal theorem (2.11).

Thus we obtain

$$(3.7)\qquad \iiint\limits_{(B)} [\tilde{G}_{ik}(x,x',p)\tilde{X}_i(x,p)-\delta(x-x')\delta_{ik}\tilde{u}_i(x,p)]\,dV(x) - \iint\limits_{(S_2)} \tilde{U}_i(\xi,p)\,L[\tilde{G}_{ik}(\xi,x',p)]\,dS(\xi)=0,$$

or

$$(3.7')\qquad \tilde{u}_k(x',p)=\iiint\limits_{(B)} [\tilde{G}_{ik}(x,x',p)\tilde{X}_i(x,p)\,dV(x) - \iint\limits_{(S_2)} \tilde{U}_i(\xi,p)\,L[\tilde{G}_{ik}(\xi,x',p)]\,dS(\xi),$$

where we have denoted by $\tilde{U}_i(\xi,t)$ the unknown displacement function on the surface S_2.

Let us again apply the Betti theorem to the function G_{ik} and u_i^0. This leads to the relation:

$$(3.8)\qquad \tilde{u}_k^0(x',p)=\iiint\limits_{(B)} \tilde{G}_{ik}(x,x',p)\tilde{X}_i(x,p)\,dV(x).$$

Hence we can represent Eq. (3.7′) in the form:

$$(3.9)\qquad \tilde{u}_k(x',p)=\tilde{u}_k^0(x',p)-\iint\limits_{(S_2)} \tilde{U}_i(\xi,p)\,L[\tilde{G}_{ik}(\xi,x',p)]\,dS(\xi).$$

We have obtained a functional equation with the unknown functions of displacements $u_i(x',t)$ of the interior points of the body, and the unknown displacements $U_i(\xi,t)$ of the surface S_2.

In order to determine the unknown functions $\tilde{U}_i(\xi,p)$, we perform on the expression (3.9) the operation $L'(\ldots)$. In this way we obtain:

$$(3.9')\qquad L'[\tilde{u}_k(x',p)]=L'[\tilde{u}_k^0(x',p)]-\iint\limits_{(S_2)} \tilde{U}_i(\xi,p)\,L'L[G_{ik}(\xi,x',p)]\,dS(\xi).$$

The dash at the operator L denotes that the operation refers to the point x'.

Now, we pass with the point $x'\in B$ to the point $\xi'\in S_2$, and we use the boundary condition which states that on S_2 there is given the loading $\mathbf{q}$. From (3.9), after passing with the point x' to the point ξ' on the boundary S_2, we arrive at the following system of integral equations of the first kind:

$$(3.10)\qquad \tilde{q}_k(\xi',p)=\tilde{q}_k^0(\xi',p)-\iint\limits_{(S_2)} \tilde{U}_i(\xi,p)\,L'L[\tilde{G}_{ik}(\xi,\xi',p)]\,dS(\xi).$$

In this equations, we know the transforms $\tilde{q}_k$, $\tilde{q}_k^0=L(\tilde{u}_k^0)$ and the functions G_{ik}. Having determined the unknown functions $\tilde{U}_i(\xi,p)$, we return to Eq. (3.9), whence we are able to obtain the transform of the displacement $\tilde{u}_k$ at the point x' of the elastic body.

Instead of the functions G_{ik}, we can here introduce a new Green tensor—namely, the functions K_{ik}, which satisfy the homogeneous differential equations

(3.11) $$D_{ij}[K_{jk}(x, \xi', t)] = 0$$

the boundary conditions

(3.12) $$K_{ik}(\xi, \xi', t) = 0 \quad \text{on} \quad S_1, \qquad K_{ik}(\xi, \xi', t) = \delta(\xi - \xi')\delta(t)\delta_{ik} \quad \text{on} \quad S_2,$$

and the initial conditions

(3.13) $$K_{ik}(x, \xi', 0) = 0, \quad \dot{K}_{ik}(x, \xi', 0) = 0.$$

Now, applying the Betti reciprocal theorem to the functions G_{ij} and K_{ij}, we obtain the following equation

$$\iiint_{(B)} \delta(x-x')\delta_{ik}\tilde{K}_{ij}(x, \xi', p)\,dV(x) + \iint_{(S_2)} \delta(\xi-\xi')\delta_{ik} L[G_{ij}(\xi, x', p)]\,dS(\xi) = 0,$$

whence follows the relation

(3.14) $$\tilde{K}_{jk}(x', \xi', p) = -L[\tilde{G}_{ik}(\xi', x', p)].$$

It is readily seen that Eq. (3.9) can be represented in the form:

(3.15) $$\tilde{u}_k(x', p) = \tilde{u}_k^0(x', p) + \iint_{(S_2)} \tilde{U}_i(\xi, p)\tilde{K}_{ik}(x', \xi, p)\,dS(\xi),$$

and the integral Eqs. (3.10) will take the form of the system:

(3.16) $$\tilde{q}_k(\xi', p) = \tilde{q}_k^0(\xi', p) + \iint_{(S_2)} \tilde{U}_i(\xi, p) L'[\tilde{K}_{ik}(\xi', \xi, p)]\,dS(\xi).$$

Let us also examine the case in which the surface S_2 leans on an elastic Winklerian foundation—that is, the case in which the relation holds

(3.17) $$q_k(\xi', t) = -\frac{1}{\varkappa_k} U_k(\xi', t), \quad k = 1, 2, 3.$$

In this last case, we obtain a system of integral equations of the second kind

(3.18) $$\tilde{U}_k(\xi', p) + \varkappa_k \iint_{(S_2)} \tilde{U}_i(\xi', p) L'[\tilde{K}_{ik}(\xi', \xi, p)]\,dS(\xi) + \varkappa_k \tilde{q}_k^0(\xi', p) = 0, \qquad k = 1, 2, 3.$$

The method presented above can be extended to the case in which on the surfaces $S_1, S_3, S_5, \ldots$ are given the displacements, while on the surfaces $S_2, S_4, S_6, \ldots$ are prescribed the loadings.

The second variant of the solution is obviously less important for practical application than the first. This results, first of all, from the difficulties connected with determining the Green functions G_{ik} and the displacements u_i^0 in the chosen fundamental system, which is represented by an elastic body rigidly fixed on the entire bounding surface. Moreover, the kernels of the integral equations $L'(K_{ik})$ exhibit strong singularities.

4. Solution for an Elastic Body with a Crack

Let us consider a simply connected body with a crack (Fig. 3). We denote the region of the body by B, and the bounding surface by S. The upper and lower surfaces of the crack we denote by S_2' and S_2'' and the remaining part of the surface by S_1. Let the body be subject to body forces and to external loadings $\mathbf{q}$ acting on the

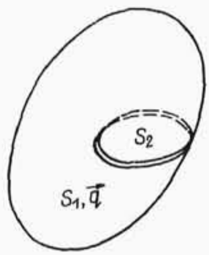

Fig. 3

surface S_1. These forces produce in the body the displacements $\mathbf{u}$, which are described by the differential equation:

(4.1) $$D_{ij}[u_j(x,t)]+X_i(x,t) = 0, \quad i,j = 1,2,3, \quad x \in B.$$

On the surface S_1, let there be prescribed the boundary conditions:

(4.2) $$L[u_i(\xi,t)] = q_i(\xi,t), \quad \xi \in S_1$$

and on the surfaces S_2' and S_2'' the conditions:

(4.3) $$L[u_i(\xi,t)] = 0, \quad \xi \in S_2', S_2''.$$

Moreover, we assume that the initial conditions are of homogeneous type:

(4.4) $$u_i(x,0) = 0, \quad \dot{u}_i(x,0) = 0.$$

As the fundamental system, we choose a body without a crack, and free from loadings on the surface S_1. Having prevented an arbitrary point O of the body from being displaced, we shall determine in our fundamental system the Green displacement functions. These last functions, G_{ik}, should satisfy the differential equation:

(4.5) $$D_{ij}[G_{jk}(x,x',t)]+\delta(x-x')\delta(t)\delta_{ik} = 0,$$

and the homogeneous boundary and initial conditions:

(4.6) $$L[G_{jk}(\xi,x',t)] = 0 \quad \text{on } S_1,$$
$$G_{ik}(\xi,x',0) = 0, \quad \dot{G}_{ik}(\xi,x',0) = 0.$$

Finally, we determine in the fundamental system the displacements $u_i^0(x,t)$ which are obtained from solving the differential equations:

(4.7) $$D_{ij}[u_j^0(x,t)]+X_i(x,t) = 0,$$

with the boundary conditions

(4.8) $$L[u_j^0(\xi,t)] = q_i(\xi,t), \quad \xi \in S_1,$$

and initial conditions

(4.9) $$u_i^0(\xi,0) = 0, \quad \dot{u}_i^0(\xi,0) = 0.$$

Let us now use with respect to the functions u_i and G_{ik} the Betti reciprocal theorem in the form (2.11). Then we obtain:

$$(4.10)\quad \iiint_{(B)} [\tilde{G}_{ik}(x, x', p)\tilde{X}_i(x, p) - \delta(x-x')\delta_{ik}\tilde{u}_i(x, p)]\, dV(x) - \iint_{(S_1)} \tilde{q}_i(\xi, p)\tilde{G}_{ik}(\xi, x', p)\, dS(\xi) + \iint_{(S_2'+S_2'')} \tilde{U}_i(\xi, p) L[\tilde{G}_{ik}(\xi, x', p)]\, dS(\xi) = 0,$$

or

$$(4.11)\quad \tilde{u}_k(x', p) = \tilde{u}_k^0(x', p) + \iint_{(S_2)} [\tilde{U}_i^+(\xi, p) + \tilde{U}_i^-(\xi, p)] L[G_{ik}(\xi, x', p)]\, dS(\xi).$$

Here, we denote by $\tilde{U}_i^+$ and $\tilde{U}_i^-$ the transforms of the displacements on the surfaces S_2' and S_2'', respectively. If we apply the Betti theorem to the functions G_{ik} and u_i^0, we arrive at the relation:

$$\tilde{u}_k^0(x', p) = \iiint_{(B)} \tilde{X}_i(x, p)\tilde{G}_{ik}(x, x', p)\, dV(x) + \iint_{(S_1)} \tilde{q}_i(\xi, p)\tilde{G}_{ik}(\xi, x', p)\, dS(\xi).$$

In the functional Eq. (4.11), the unknown quantities are represented by the transforms $\tilde{U}_i = \tilde{U}_i^+ + \tilde{U}_i^-$ and the displacements $\tilde{u}_k$. The unknown function $\tilde{U}_k$ will be obtained by performing on the equation (4.11) the operation $L'(\ldots)$, and passing with the point x' to the point ξ' on S_2. Using the condition (4.3), we arrive at the following system of integral equations:

$$(4.12)\quad L'[\tilde{u}_k(\xi', p)] = 0 = L'[\tilde{u}_k^0(\xi', p)] + \iint_{(S_2)} \tilde{U}_i(\xi, p) L'L[\tilde{G}_{ik}(\xi', \xi, p)]\, dS(\xi).$$

In the case in which within the crack there are prescribed the loadings $\tilde{q}_k(\xi', p) = L'(\tilde{u}_k(\xi, p))$, the Eq. (4.12) assumes the form:

$$(4.13)\quad \tilde{q}_k(\xi', p) = \tilde{q}_k^0(\xi', p) + \iint_{(S_2)} \tilde{U}_i(\xi, p) L'L[G_{ik}(\xi', \xi, p)]\, dS(\xi).$$

Having solved Eqs. (4.12) or (4.13), we can already determine the displacement u_i from (4.11).

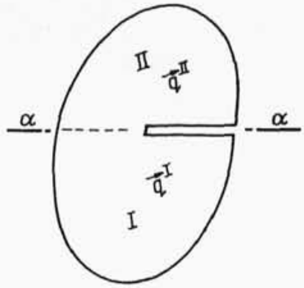

Fig. 4

Another way of obtaining the solution can also be demonstrated. Let us, thus, imagine the elastic body to be divided by a cut $a-a$, passing through the crack, into two parts I and II (Fig. 4), and let us choose the interaction between the parts

I and II as the unknown function. Here we denote the unknown interaction forces by $R_i(\xi, t)$ and by $u_i^{\mathrm{I}}(x_{\mathrm{I}}, t)$ the displacement of the point $x_{\mathrm{I}} \in B_{\mathrm{I}}$, which is produced by the external loading q_k^{I} acting on the external surface S_{I}, by the body forces X_i^{I}, and by the forces R_i acting on the surface S_α. Similarly, we denote by $u_i^{\mathrm{II}}(x_{\mathrm{II}}, t)$ the displacement of the point $x_{\mathrm{II}} \in B_{\mathrm{II}}$, which is produced by the external forces q_k^{II} and reactions R_i. According to the formula (2.20), we have:

$$\tilde{u}_k^{\mathrm{I}}(x_{\mathrm{I}}', p) = \tilde{q}_k^{\mathrm{I}0}(x_I', p) + \iint_{(S_\alpha)} \tilde{R}_i(\xi_\alpha, p)\, \tilde{U}_{ik}^{\mathrm{I}}(x_I', \xi_\alpha, p)\, dS(\xi_\alpha). \tag{4.14}$$

$$x_I' \in B_I, \quad \xi_\alpha \in S_\alpha,$$

$$\tilde{u}_k^{\mathrm{II}}(x_{\mathrm{II}}', p) = \tilde{u}_k^{\mathrm{II}0}(x_{\mathrm{II}}', p) + \iint_{(S_\alpha)} \tilde{R}_i(\xi_\alpha, p)\tilde{U}_{ik}^{\mathrm{II}}(x_{\mathrm{II}}', \xi_\alpha, p)\, dS(\xi_\alpha), \tag{4.15}$$

$$x_{\mathrm{II}}' \in B_{\mathrm{II}}, \quad \xi_\alpha \in S_\alpha, \quad i, k = 1, 2, 3.$$

In the two last-quoted formulae, U_{ik}^{I} and U_{ik}^{II} are the Green functions for the regions B_{I} and B_{II}, which have been discussed in Sec. 2 in a detailed form.

Now, we pass with the point $x_{\mathrm{I}}' \in B_{\mathrm{I}}$, and with the point $x_{\mathrm{II}}' \in B_{\mathrm{II}}$ to the point $\xi_\alpha' \in S_\alpha$ on the surface of the cut $\alpha - \alpha$. Owing to the continuity of the elastic body on the surface S_α, the following condition should be satisfied:

$$\tilde{u}_k^{\mathrm{I}}(\xi_\alpha', p) = \tilde{u}_k^{\mathrm{II}0}(\xi_\alpha', p), \tag{4.16}$$

or

$$\iint_{(S_\alpha)} \tilde{R}_i(\xi_\alpha, p)[\tilde{U}_{ik}^{\mathrm{I}}(\xi_\alpha', \xi_\alpha, p) - \tilde{U}_{ik}^{\mathrm{II}}(\xi_\alpha', \xi_\alpha, p)]\, dS(\xi_\alpha) + \tilde{u}_k^{\mathrm{I}0}(\xi_\alpha', p) - \tilde{u}_k^{\mathrm{II}0}(\xi_\alpha', p) = 0, \quad i, k = 1, 2, 3. \tag{4.17}$$

We have obtained a system of integral equations of the first kind, and its solution yields the functions $\tilde{R}_i(\xi_\alpha, p)$, which being substituted into the relations (4.14) and (4.15), enable us to find the required functions $\tilde{u}_k^{\mathrm{I}}$ and $\tilde{u}_k^{\mathrm{II}}$.

The above presented procedure can be extended to cases in which there exist more than one crack in the elastic body considered. For r cracks we obtain a system of $3r$ integral equations.

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Streszczenie

ZAGADNIENIA ELASTODYNAMIKI Z MIESZANYMI WARUNKAMI BRZEGOWYMI

W poprzednich pracach [1, 2] autor przedstawił metodę rozwiązania zagadnień elastostatyki z mieszanymi warunkami brzegowymi. W niniejszej pracy przedstawiono analogiczną metodę dla dynamicznych zagadnień teorii sprężystości. Korzystając z funkcji przemieszczeniowych Greena oraz z twierdzenia Bettiego o wzajemności sprowadzono zagadnienie do rozwiązania układu równań całkowych pierwszego rodzaju.

W zależności od wyboru t.zw. układu podstawowego podano dwa warianty rozwiązania. Podano wreszcie drogę postępowania dla zagadnienia szczelin w ciałach jednospójnych.

Резюме

УПРУГОДИНАМИЧЕСКИЕ ЗАДАЧИ СО СМЕШАННЫМИ КРАЕВЫМИ УСЛОВИЯМИ

В предыдущих работах [1] и [2] автор представил метод решения задач упругостатики со смешанными краевыми условиями. В настоящей же работе приводится аналогичный метод для динамических задач теории упругости. Используя функции Грина в перемещениях и теорему Бетти о взаимности, задача сводится к решению системы интегральных уравнений первого рода.

В зависимости от выбора так называемой основной системы даются два варианта решения. В заключение предлагается метод решения eадаг, касающихся щелей в односвязных телах.

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Received March 9, 1964.