PROBLEMS OF HYDRODYNAMICS AND CONTINUUM MECHANICS

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1. THERMOELASTICITY EQUATIONS

Consider an isotropic homogeneous elastic body occupying a region B. The following linearized differential equations hold in this instance [1]:

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} + X_i = \gamma\Theta_{,i} + \rho_0\ddot{u}_i, \tag{1.1}$$

$$\Theta_{,kk} - \frac{1}{\varkappa}\dot{\Theta} - \eta_0\dot{u}_{k,k} = \frac{Q}{\varkappa}, \qquad i, j, k = 1, 2, 3. \tag{1.2}$$

The first of these are the equations for the displacements and the second is the generalized heat equation. The meaning of the notation is: the $u_i(x, t)$ are the displacement components, $\Theta(x, t) = T - T_0$ is the temperature increment, $T(x, t)$ is the absolute temperature, T_0 is the temperature corresponding to a natural state of zero stress and zero strain, the $X_i(x, t)$ are the components of body forces, $Q(x, t) = W/c_\varepsilon$ is the intensity of thermal sources, W is the amount of heat liberated in a unit volume per unit time, c_ε is the specific heat for fixed deformation, μ and λ are the Lamé constants referred to the natural state, ρ_0 is the density, $K = \lambda + \frac{2}{3}\mu$ is the bulk rigidity modulus, α_t is the coefficient of linear thermal expansion, λ_0 is the thermal conductivity coefficient, $\gamma \equiv 3K\alpha_t$, $\varkappa \equiv \lambda_0/c_\varepsilon$ and $\eta_0 \equiv \gamma T_0/\lambda_0$.

One must still add to (1.1), (1.2) the Duhamel–Neumann law

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma\Theta)\delta_{ij} \tag{1.3}$$

relating the stress tensor to the temperature and strain tensor and the strain-displacement relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.4}$$

Representing the displacements as a sum of potential and solenoidal parts

$$u_i = \Phi_{,i} + \epsilon_{ijk}\Psi_{k,j}, \tag{1.5}$$

we can reduce the system of equations (1.1) and (1.2) (for $X_i = Q = 0$) to the following simple form:

$$\left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right)\Phi = m_0\Theta, \tag{1.6}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\Psi_i = 0, \tag{1.7}$$

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\Theta - \eta_0\partial_t\nabla^2\Phi = 0. \tag{1.8}$$

Here we have introduced the notation

$$c_1 \equiv \left(\frac{\lambda + 2\mu}{\rho_0}\right)^{1/2}, \qquad c_2 \equiv \left(\frac{\mu}{\rho_0}\right)^{1/2}, \qquad m_0 \equiv \frac{\gamma}{c_1^2 \rho_0},$$

$$\eta_0 \equiv \frac{\gamma T_0}{c_\varepsilon \varkappa}, \qquad \partial_t \equiv \frac{\partial}{\partial t}, \qquad \nabla^2 \equiv \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}.$$

Equations (1.6) and (1.7) describe respectively longitudinal and transverse waves while equation (1.8) is the generalized heat equation.

It will be convenient to use new variables

$$\xi_i \equiv \frac{c_1}{\varkappa} x_i, \qquad \tau \equiv \frac{c_1^2}{\varkappa} t$$

in the subsequent considerations. The wave equations (1.6)–(1.8) can then be written in a slightly simpler form:

$$(\nabla^2 - \partial_\tau^2)\Phi(\xi, \tau) = m\Theta(\xi, \tau), \tag{1.9}$$

$$(\nabla^2 - \sigma^2 \partial_\tau^2)\Psi_i(\xi, \tau) = 0, \tag{1.10}$$

$$(\nabla^2 - \partial_\tau)\Theta(\xi, \tau) - \eta' \partial_\tau \nabla^2 \Phi(\xi, \tau) = 0, \tag{1.11}$$

where

$$\nabla^2 \equiv \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_i}, \qquad \partial_\tau \equiv \frac{\partial}{\partial \tau}, \qquad m \equiv m_0 \frac{\varkappa^2}{c_1^2}, \qquad \eta' \equiv \eta_0 c_1^2.$$

2. INTEGRAL REPRESENTATION OF THE SOLUTION TO THE EQUATION FOR THE THERMOELASTIC STRAIN POTENTIAL

Consider an internal region B^+ bounded by a surface A. Let B^- denote the complement of B^+ with respect to all of three-dimensional space. We shall study the propagation of a longitudinal wave inside the region B^+. We are especially interested in an expression for the values of the function Φ at the point η in terms of the distribution of Θ, $\partial\Theta/\partial n$, Φ, and $\partial\Phi/\partial n$ on the closed surface A.

Suppose that the functions Φ and Θ are regular in B^+ and have first and second derivatives in the closure of B^+ and that they satisfy homogeneous initial conditions.

In other words we are considering a system of equations

$$(\nabla^2 - \partial_\tau^2)\Phi = m\Theta, \qquad (\nabla^2 - \partial_\tau)\Theta - \eta' \partial_\tau \nabla^2 \Phi = 0, \tag{2.1}$$

with no singularities in the region $B^+ \cup A$. Eliminating the temperature from these equations, we obtain

$$[(\nabla^2 - \partial_\tau^2)(\nabla^2 - \partial_\tau) - \varepsilon\, \partial_\tau \nabla^2]\Phi(\xi, \tau) = 0. \tag{2.2}$$

Taking Laplace transforms of both sides and using the homogeneity of the initial conditions, we find

$$\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{\Phi}(\xi, p) = 0, \tag{2.3}$$

where

$$\square^2_{\lambda_\alpha} \equiv \nabla^2 - \lambda_\alpha^2, \qquad \alpha = 1, 2, \qquad \tilde{\Phi}(\xi, p) \equiv \int_0^\infty \Phi(\xi, \tau)e^{-p\tau}\, d\tau.$$

The quantities λ_1 and λ_2 are roots of the equation

$$\lambda^4 - \lambda^2 p(p + 1 + \varepsilon) + p^3 = 0, \qquad \varepsilon \equiv \eta' m,$$

i.e.,

$$\left.\begin{matrix}\lambda_1^2\\ \lambda_2^2\end{matrix}\right\} = \frac{p}{2}[p + 1 + \varepsilon \pm \sqrt{p^2 - 2p(1 - \varepsilon) + (1 + \varepsilon)^2}].$$

Consider the solution $G(\xi, \eta, \tau)$ of the equation

$$[(\nabla^2 - \partial_\tau^2)(\nabla^2 - \partial_\tau) - \varepsilon\, \partial_\tau \nabla^2]G(\xi, \eta, \tau) = -\delta(\xi - \eta)\delta(\tau) \tag{2.4}$$

in an infinite region. The right-hand side of the equation corresponds to an instantaneous point impulse applied at the point η. It is easy to see that this impulse is equivalent to an instantaneous heat source concentrated at that point.

Suppose that the initial conditions are homogeneous. Taking Laplace transforms, we obtain

$$\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{G}(\xi, \eta, p) = -\delta(\xi - \eta). \tag{2.5}$$

The solution to this equation is known [5], [6]. It is given by

$$\tilde{G}(\xi, \eta, p) = \frac{e^{-\lambda_1\rho} - e^{-\lambda_2\rho}}{4\pi\rho(\lambda_1^2 - \lambda_2^2)}, \qquad \rho^2 \equiv (\xi_i - \eta_i)(\xi_i - \eta_i). \tag{2.6}$$

The function G describes an outgoing spherical wave centered at η. Let $K(\xi, \eta, \tau)$ be the temperature corresponding to the potential G. Taking Laplace transforms of both sides of

$$(\nabla^2 - \partial_\tau^2)G(\xi, \eta, \tau) = mK(\xi, \eta, \tau) \tag{2.7}$$

we obtain

$$\tilde{K}(\xi, \eta, p) = \frac{1}{m}D_1\tilde{G}(\xi, \eta, p) = \frac{(\lambda_1^2 - p^2)e^{-\lambda_1\rho} - (\lambda_2^2 - p^2)e^{-\lambda_2\rho}}{4\pi\rho(\lambda_1^2 - \lambda_2^2)m} \tag{2.8}$$

(wherein the notation $D_1 \equiv \nabla^2 - p^2$ has been introduced).

In order to obtain an expression for the function Φ in B^+ (for $\eta \in B^+$) in terms of volume and surface integrals, we apply the identity

$$\begin{aligned}(2.9)\qquad &\int_V (\tilde{G}\Box^2_{\lambda_1}\Box^2_{\lambda_2}\tilde{\Phi} - \tilde{\Phi}\Box^2_{\lambda_1}\Box^2_{\lambda_2}\tilde{G})\,dV \\ &= \int_V [\tilde{G}\nabla^4\tilde{\Phi} - \tilde{\Phi}\nabla^4\tilde{G} - (\lambda_1^2 + \lambda_2^2)(\tilde{G}\nabla^2\tilde{\Phi} - \tilde{\Phi}\nabla^2\tilde{G})]\,dV,\end{aligned}$$

where $dV = d\xi_1\,d\xi_2\,d\xi_3$. Using the following formula for the iterated Laplacian:

$$\begin{aligned}(2.10)\qquad &\int_V (\tilde{G}\nabla^4\tilde{\Phi} - \tilde{\Phi}\nabla^4\tilde{G})\,dV \\ &= \int_A \left(\nabla^2\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial}{\partial n}\nabla^2\tilde{G} + \tilde{G}\frac{\partial}{\partial n}\nabla^2\tilde{\Phi} - \nabla^2\tilde{\Phi}\frac{\partial\tilde{G}}{\partial n}\right) dA\end{aligned}$$

and Green's theorem

$$(2.11)\qquad \int_V (\tilde{G}\nabla^2\tilde{\Phi} - \tilde{\Phi}\nabla^2\tilde{G})\,dV = \int_A \left(\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial\hat{G}}{\partial n}\right) dA,$$

we may write (2.9) in the form

$$\begin{aligned}(2.12)\qquad &\int_A (\tilde{G}\Box^2_{\lambda_1}\Box^2_{\lambda_2}\tilde{\Phi} - \tilde{\Phi}\Box^2_{\lambda_1}\Box^2_{\lambda_2}\tilde{G})\,dV \\ &= \int_A \left(\tilde{G}\frac{\partial}{\partial n}\Box^2\tilde{\Phi} - \tilde{\Phi}\frac{\partial}{\partial n}\Box^2\tilde{G}\right) dA - \int_A \left(\nabla^2\Phi\frac{\partial\tilde{G}}{\partial n} - \nabla^2\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n}\right) dA,\end{aligned}$$

where $\Box^2 \equiv \nabla^2 - (\lambda_1^2 + \lambda_2^2)$.

Now taking (2.3) and (2.5) into consideration, we obtain the following formula:

$$\begin{aligned}(2.13)\qquad \tilde{\Phi}(\eta, p) &= \frac{1}{m}\int_A \left[\tilde{G}(\xi, \eta, p)\frac{\partial}{\partial n}\Box^2\tilde{\Phi}(\xi, p) - \tilde{\Phi}(\xi, p)\frac{\partial}{\partial n}\Box^2\tilde{G}(\xi, \eta, p)\right] dA(\xi) \\ &\quad - \frac{1}{m}\int_A \left[\nabla^2\tilde{\Phi}(\xi, p)\frac{\partial}{\partial n}\tilde{G}(\xi, \eta, p) - \nabla^2\tilde{G}(\xi, \eta, p)\frac{\partial}{\partial n}\tilde{\Phi}(\xi, p)\right] dA(\xi).\end{aligned}$$

Formula (2.13) is valid for $\eta \in B^+$. If $\eta \in B^-$, $\tilde{\Phi}(\eta, p) \equiv 0$.

The temperature, as we know, is related to the potential Φ by the equation $(\nabla^2 - \partial_\tau^2)\Phi = m\Theta$. Taking Laplace transforms in this, we have

$$(2.14)\qquad \tilde{\Theta} = \frac{1}{m}D_1\tilde{\Phi}, \qquad D_1 \equiv \nabla^2 - p^2.$$

Using the last relation and introducing the operator $\Box_\lambda^2 \equiv \nabla^2 - (\lambda_1^2 + \lambda_2^2 - p^2)$, we arrive at the final formula for $\tilde{\Phi}(\eta, p)$:

$$\tag{2.15} \begin{aligned} \tilde{\Phi}(\eta, p) &= \int_A \left(\tilde{G}\frac{\partial \tilde{\Theta}}{\partial n} - \tilde{\Theta}\frac{\partial \tilde{G}}{\partial n} \right) dA \\ &\quad + \frac{1}{m}\int_A \left(\Box_\lambda^2 \tilde{G}\frac{\partial \tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial}{\partial n}\Box_\lambda^2 \tilde{G} \right) dA(\xi) \end{aligned}$$

for $\eta \in B^+$. If $\eta \in B^-$, $\tilde{\Phi}(\eta, p) \equiv 0$.

Formula (2.15) determines $\tilde{\Phi}(\eta, p)$ inside B^+ in terms of the values $\tilde{\Phi}(\xi, p)$, $\partial\tilde{\Phi}(\xi, p)/\partial n$, $\tilde{\Theta}(\xi, p)$ and $\partial\tilde{\Theta}(\xi, p)/\partial n$ on the surface A. Taking inverse Laplace transforms in (2.15), we obtain

$$\tag{2.16} \begin{aligned} \Phi(\eta, \tau) = \int_0^\tau d\tau' \Bigg\{ &\int_A \left[G(\xi, \eta, \tau - \tau')\frac{\partial \Theta(\xi, \tau')}{\partial n} \right. \\ &\left. - \Theta(\xi, \tau')\frac{\partial G(\xi, \eta, \tau - \tau')}{\partial n} \right] dA(\xi) \\ &+ \frac{1}{m}\int_A \left[\hat{G}(\xi, \eta, \tau - \tau')\frac{\partial \Phi(\xi, \tau')}{\partial n} \right. \\ &\left. - \Phi(\xi, \tau')\frac{\partial}{\partial n}\hat{G}(\xi, \eta, \tau - \tau') \right] dA(\xi) \Bigg\}, \end{aligned}$$

where

$$\hat{G} \equiv L^{-1}[\Box_\lambda^2 \tilde{G}] \equiv L^{-1}[\Box_\lambda^2 L(G)].$$

Formula (2.16), obtained within the framework of coupled thermoelasticity, is similar to Kirchhoff's theorem in classical elastokinetics. When the processes giving rise to the deformation are time-harmonic, formula (2.16) is the analogue of Helmholtz's theorem in classical elastokinetics [8].

3. INTEGRAL REPRESENTATION OF THE SOLUTION TO THE HEAT EQUATION

To determine the temperature field $\Theta(\xi, \tau)$ associated with the potential $\Phi(\xi, \tau)$, consider first the solution, having a singularity, of the equation

$$\tag{3.1} [(\nabla^2 - \partial_\tau)(\nabla^2 - \partial_\tau) - \varepsilon\,\partial_\tau\nabla^2]H(\xi, \eta, \tau) = -(\nabla^2 - \partial_\tau^2)\delta(\xi - \eta)\delta(\tau)$$

in an infinite elastic space. We are assuming here that an instantaneous concentrated source has been applied at the point η and that H satisfies homogeneous initial conditions.

Taking Laplace transforms in (3.1), we obtain

$$\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{H}(\xi,\eta,p) = -D_1\delta(\xi-\eta). \tag{3.2}$$

Its solution is given by

$$\tilde{H}(\xi,\eta,p) = D_1\tilde{G}(\xi,\eta,p) = \frac{(\lambda_1^2-p^2)e^{-\rho\lambda_1}-(\lambda_2^2-p^2)e^{-\rho\lambda_2}}{4\pi\rho(\lambda_1^2-\lambda_2^2)}. \tag{3.3}$$

The identity analogous to (2.12) here has the form

$$\int_V (\tilde{H}\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{\Phi} - \tilde{\Phi}\square^2_{\lambda_1}\square^2_{\lambda_2}\tilde{H})\,dV = \int_A\left(\tilde{H}\frac{\partial}{\partial n}\square^2\tilde{\Phi} - \tilde{\Phi}\frac{\partial}{\partial n}\square^2\tilde{H}\right)dA - \int_A\left(\nabla^2\tilde{\Phi}\frac{\partial\tilde{H}}{\partial n} - \nabla^2\tilde{H}\frac{\partial\tilde{\Phi}}{\partial n}\right)dA.$$

By means of (2.3) and (3.2), we find

$$D_1\tilde{\Phi}(\eta,p) = \int_A\left(\tilde{H}\frac{\partial}{\partial n}\square^2\tilde{\Phi} - \tilde{\Phi}\frac{\partial}{\partial n}\square^2\tilde{H}\right)dA - \int_A\left(\nabla^2\tilde{\Phi}\frac{\partial\tilde{H}}{\partial n} - \nabla^2\tilde{H}\frac{\partial\tilde{\Phi}}{\partial n}\right)dA. \tag{3.5}$$

Applying (2.14) we obtain after some simple operations,

$$m\tilde{\Theta}(\eta,p) = m\int_A\left(\tilde{H}\frac{\partial\tilde{\Theta}}{\partial n} - \tilde{\Theta}\frac{\partial\tilde{H}}{\partial n}\right)dA + \int_A\left(\square^2_\lambda\tilde{H}\frac{\partial\tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial}{\partial n}\square^2_\lambda\tilde{H}\right)dA. \tag{3.6}$$

This result is totally similar to formula (2.15). Here too the value of $\tilde{\Theta}(\eta,p)$ at point $\eta\in B^+$ is expressed in terms of surface integrals of the functions $\tilde{\Theta}$, $\partial\tilde{\Theta}/\partial n$, $\tilde{\Phi}$, $\partial\tilde{\Phi}/\partial n$. Formula (3.6) is valid for $\eta\in B^+$. When $\eta\in B^-$, $\tilde{\Theta}(\eta,p)\equiv 0$.

Taking inverse Laplace transforms in (3.6), we arrive at

$$\Theta(\eta,\tau) = \int_0^\tau d\tau'\left\{\int_A\left[H(\xi,\eta,\tau-\tau')\frac{\partial\Theta(\xi,\tau')}{\partial n} - \Theta(\xi,\tau')\frac{\partial H(\xi,\eta,\tau-\tau')}{\partial n}\right]dA(\xi) + \frac{1}{m}\int_A\left[\hat{H}(\xi,\eta,\tau-\tau')\frac{\partial\Phi(\xi,\tau')}{\partial n} - \Phi(\xi,\tau')\frac{\partial}{\partial n}\hat{H}(\xi,\eta,\tau-\tau')\right]dA(\xi)\right\}, \tag{3.7}$$

where $\hat{H}\equiv L^{-1}(\square^2_\lambda\tilde{H})$.

Equation (3.7) may be regarded as a generalization of Green's theorem in the theory of heat conduction to the problems in coupled thermoelasticity.

A formula for $\tilde{\Theta}(\eta, p)$ may also be obtained in another way by using the function $\tilde{G}(\xi, \eta, p)$. Apply the operator $(1/m)D_1$ to (2.15) and make use of the relation $D_1\Box_\lambda^2 = \Box_1^2\Box_2^2 + \varepsilon p^3$. Now taking formulae (2.8) and (2.14) into account, we obtain

$$\tag{3.8} \begin{aligned} \tilde{\Theta}(\eta, p) = \int_A \left(\tilde{K}\frac{\partial\tilde{\Theta}}{\partial n} - \tilde{\Theta}\frac{\partial\tilde{K}}{\partial n} \right) dA \\ + \frac{\varepsilon p^3}{m^2}\int_A \left(\tilde{G}\frac{\partial\tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial\tilde{G}}{\partial n} \right) dA. \end{aligned}$$

4. UNCOUPLED PROBLEM IN THERMOELASTICITY

Discarding the term $\eta_0\partial_t\nabla^2\Phi$ in the heat equation (1.8), we obtain instead of (1.6)–(1.8) a simpler system of equations characterizing the uncoupled problem in thermoelasticity. It corresponds to the so-called engineering theory of thermal stress. Disregarding the indicated term has an insignificant effect on the temperature and stress distribution but this term is essential in the study of the wave motion. For the problems in coupled thermoelasticity, the waves Φ and Θ attenuate and scatter, whereas in the theory of thermal stress Φ defines a pure elastic and a diffusion wave and Θ has the character of a diffusion wave.

Within the framework of the theory of thermal stress, the formulae for Φ and Θ are given by the formulae in §§ 2 and 3 provided we set $\varepsilon = 0$ in them. Thus the Green's functions $\tilde{G}$, $\tilde{K}$, and $\tilde{H}$ become

$$\tag{4.1} \begin{aligned} \tilde{G}|_{\varepsilon=0} = \frac{m}{4\pi\rho}\frac{e^{-\rho p} - e^{-\rho\sqrt{p}}}{p(p-1)}, \\ \tilde{K}|_{\varepsilon=0} = \tilde{H}|_{\varepsilon=0} = \frac{e^{-\rho\sqrt{p}}}{4\pi\rho}. \end{aligned}$$

It is also easy to check that

$$\tag{4.2} \Box_\lambda^2\tilde{G}|_{\varepsilon=0} = \frac{me^{-\rho p}}{4\pi\rho}, \qquad \Box_\lambda^2\tilde{H}|_{\varepsilon=0} = 0.$$

Thus formula (2.15) has the form

$$\tag{4.3} \begin{aligned} \tilde{\Phi}(\eta, p) = \int_A \left(\tilde{G}\Big|_{\varepsilon=0}\frac{\partial\tilde{\Theta}}{\partial n} - \tilde{\Theta}\frac{\partial}{\partial n}\tilde{G}\Big|_{\varepsilon=0} \right) dA \\ + \frac{1}{4\pi}\int_A \left[\left(\frac{e^{-\rho p}}{\rho}\right)\frac{\partial\tilde{\Phi}}{\partial n} - \tilde{\Phi}\frac{\partial}{\partial n}\left(\frac{e^{-\rho p}}{\rho}\right) \right] dA. \end{aligned}$$

Here Θ is a known function obtained by solving the heat equation with $\eta = 0$.

We first examine the second surface integral and introduce the notation

$$[\tilde{\Phi}] = \tilde{\Phi}e^{-\rho p}, \qquad \left[\frac{\partial\tilde{\Phi}}{\partial n}\right] = \frac{\partial\tilde{\Phi}}{\partial n}e^{-\rho p}.$$

We transform the integral (for $\eta \in B^+$) into the form

$$\tilde{\Phi}_2(\eta, p) = \frac{1}{4\pi}\int_A \left\{[\tilde{\Phi}]\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right) - \frac{1}{\rho}\frac{\partial\rho}{\partial n}[p\tilde{\Phi}] - \frac{1}{\rho}\left[\frac{\partial\tilde{\Phi}}{\partial n}\right]\right\} dA. \tag{4.4}$$

Applying the formulae

$$L^{-1}\left(\frac{e^{-\rho p}}{\rho}\right) = \frac{1}{\rho}\delta(\rho - \tau),$$

$$L^{-1}[\tilde{\Phi}] = \int_0^\tau \Phi(\xi, \tau - \tau')\delta(\rho - \tau')\,d\tau' = \Phi(\xi, \tau - \rho) = [\Phi(\xi, \tau)],$$

$$L^{-1}[p\Phi] = \frac{\partial\Phi(\xi, \tau - \rho)}{\partial\tau} = \left[\frac{\partial\Phi(\xi, \tau)}{\partial\tau}\right],$$

$$L^{-1}\left(\frac{\partial\Phi}{\partial n}\right) = \int_0^\tau \frac{\partial\Phi(\xi, \tau - \tau')}{\partial n}\delta(\rho - \tau')\,d\tau' = \frac{\partial\Phi(\xi, \tau - \rho)}{\partial n} = \left[\frac{\partial\Phi(\xi, \tau)}{\partial n}\right],$$

we arrive at the following expression for $\Phi_2(\eta, \tau)$:

$$\begin{aligned}\Phi_2(\eta, \tau) = \frac{1}{4\pi}\int_A &\left\{[\Phi(\xi, \tau)]\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\right.\\ &\left. - \frac{1}{\rho}\frac{\partial\rho}{\partial n}\left[\frac{\partial\Phi(\xi, \tau)}{\partial\tau}\right] - \frac{1}{\rho}\left[\frac{\partial\Phi(\xi, \tau)}{\partial n}\right]\right\} dA(\xi).\end{aligned} \tag{4.5}$$

The function $\Phi_2(\eta, \tau)$ is in terms of the retarded potential $[\Phi(\xi, \tau)]$ and its derivatives. Formula (4.5) is the familiar analytic form of Kirchhoff's theorem.

Taking the inverse Laplace transform of the function $\tilde{G}(\xi, \eta, p)|_{\varepsilon=0}$, we obtain

$$G(\xi, \eta, \tau)|_{\varepsilon=0} = \frac{1}{4\pi\rho}(e^{\tau-\rho} - 1)H(\tau - \rho) - \frac{1}{4\pi\rho}\left[U(\rho, \tau) - \operatorname{erfc}\left(\frac{\rho}{2\sqrt{\tau}}\right)\right], \tag{4.6}$$

where

$$U = \frac{e^\tau}{2}\left[e^{-\rho}\operatorname{erfc}\left(\frac{\rho}{2\sqrt{\tau}} - \sqrt{\tau}\right) + e^\rho\operatorname{erfc}\left(\frac{\rho}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right],$$

and $H(x)$ is the Heaviside function. The first term in (4.6) represents an outgoing spherical elastic wave centered at η propagating with velocity $c_1 = \sqrt{(\lambda + 2\mu)/\rho_0}$. The second term $U(\rho, \tau) - \operatorname{erfc}(\frac{1}{2}\rho/\sqrt{\tau})$ has the nature of a thermal diffusion wave.

Taking inverse Laplace transforms in the first surface integral of (4.3), we obtain

$$\Phi_1(\eta, \tau) = \int_0^\tau d\tau' \int_A \left[G(\xi, \eta, \tau - \tau')\Big|_{\varepsilon=0} \frac{\partial \Theta(\xi, \tau')}{\partial n} \right.$$
$$\left. - \Theta(\xi, \tau') \frac{\partial}{\partial n}(\xi, \eta, \tau - \tau')\Big|_{\varepsilon=0} \right] dA. \tag{4.7}$$

The function $G|_{\varepsilon=0}$ in this is given by (4.6).

Within the framework of the theory of thermal stress, the thermoelastic displacement potential Φ consists of two terms, a diffusion wave Φ_1 and a thermoelastic wave Φ_2.

We shall in addition consider the temperature field for the case $\varepsilon = 0$. On the basis of (3.6) or (3.8), we obtain (for $\eta \in B^+$)

$$\tilde{\Theta}(\eta, p) = \frac{1}{4\pi} \int_A \left[\frac{e^{-\rho\sqrt{p}}}{\rho} \frac{\partial \tilde{\Theta}}{\partial n} - \tilde{\Theta} \frac{\partial}{\partial n} \left(\frac{e^{-\rho\sqrt{p}}}{\rho} \right) \right] dA(\xi). \tag{4.8}$$

Taking inverse Laplace transforms in this, we find

$$\Theta(\eta, \tau) = \frac{1}{4\pi} \int_0^\tau d\tau' \int_A \left[F(\xi, \eta, \tau - \tau') \frac{\partial \Theta(\xi, \tau')}{\partial n} \right.$$
$$\left. - \Theta(\xi, \tau') \frac{\partial}{\partial n} F(\xi, \eta, \tau - \tau') \right] dA(\xi), \tag{4.9}$$

where

$$F(\xi, \eta, \tau) = \frac{1}{8(\pi\tau)^{3/2}} \exp\left(-\frac{\rho^2}{4\tau} \right).$$

Equation (4.9) permits the determination of the temperature Θ at point η and time τ in terms of prescribed values of Θ and $\partial\Theta/\partial n$ on the surface A. Note that the quantities λ and μ occurring in the formulae of this section through the variables ξ and η are referred to an isothermal state.

5. PASSAGE TO CLASSICAL ELASTOKINETICS

The variables x and t will be more convenient in the discussion of this section. Consider the equation for the thermoelastic displacement potential

$$\left(\nabla^2 - \frac{1}{c_1^2} \partial_t^2 \right) \Phi(x, t) - m_0 \Theta(x, t) = 0, \tag{5.1}$$

having a regular solution in B^+. Here

$$c_1 = \sqrt{\frac{\lambda_T + 2\mu_T}{\rho_0}}, \qquad m_0 = \frac{(3\lambda_T + 2\mu_T)\alpha_t}{\rho_0},$$

and λ_T and μ_T are referred to isothermal conditions.

For the purpose of solving (5.1), consider first the solution (having a singularity) of the equation

$$\left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right) G^*(x, x', t) - m_0 K^*(x, x', t) = -\frac{1}{c_1^2}\delta(x - x')\delta(t) \tag{5.2}$$

in infinite space. Suppose the function G^* and K^* satisfy homogeneous initial conditions. Let $\Theta(x, t)$ be the temperature associated with the potential Φ and let K^* be the temperature associated with the function G^*. Taking Laplace transforms in (5.1) and (5.2) and carrying out appropriate operations, we obtain

$$\int_V (\tilde{G}^*\nabla^2\tilde{\Phi} - \tilde{\Phi}\nabla^2\tilde{G}^*)\, dV - m_0 \int_V (\tilde{\Theta}\tilde{G}^* - \tilde{K}^*\tilde{\Phi})\, dV = \frac{1}{c_1^2}\tilde{\Phi}(x', p). \tag{5.3}$$

Classical elastokinetics are based, as we know, on the assumption that the exchange of heat between individual body particles happens slowly; the thermodynamic process is considered to be adiabatic. In consequence of this, the following relation holds between the temperature and volume change:

$$\Theta(x, t) = -\eta_T \varkappa u_{i,i} = -\eta_T \varkappa \nabla^2 \Phi, \qquad \eta_T \equiv \frac{\gamma_T T_0}{c_\varepsilon \varkappa}. \tag{5.4}$$

Holding similarly is the relation

$$K^*(x, x', t) = -\eta_T \varkappa \nabla^2 G(x, x', t). \tag{5.5}$$

Relations (5.4) and (5.5) replace the heat equation in classical elastokinetics. Substituting (5.4) and (5.5) in (5.3), we obtain

$$\begin{aligned} \tilde{\Phi}(x', p) &= c^2 \int_V (\tilde{G}^*\nabla^2\tilde{\Phi} - \tilde{\Phi}\nabla^2\tilde{G}^*)\, dV, \\ c^2 &\equiv c_1^2(1 + \eta_T m_0 \varkappa). \end{aligned} \tag{5.6}$$

Using (5.5) in (5.2) and taking Laplace transforms in the result, we have

$$\left(\nabla^2 - \frac{p^2}{c^2}\right)\tilde{G}^* = -\frac{1}{c^2}\delta(x - x'). \tag{5.7}$$

The solution of this equation is given by

$$\tilde{G}^*(x, x', p) = \frac{1}{4\pi c^2 r} e^{-rp/c}. \tag{5.8}$$

Applying Green's theorem, we can reduce (5.6) to the form

$$\tilde{\Phi}(x', p) = c^2 \int_A \left(\tilde{G}^* \frac{\partial \tilde{\Phi}}{\partial n} - \tilde{\Phi} \frac{\partial \tilde{G}}{\partial n} \right) dA(x), \qquad x' \in B^+. \tag{5.9}$$

Considering (5.8), we obtain

$$\tilde{\Phi}(x', p) = \frac{1}{4\pi} \int_A \left[\left(\frac{e^{-rp/c}}{r} \right) \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n} \left(\frac{e^{-rp/c}}{r} \right) \right] dA(x). \tag{5.10}$$

Here c is the propagation velocity of the longitudinal wave and is related to the adiabatic constants through [7, p. 75]

$$c^2 = c_1^2 + \frac{\eta_{0T} \varkappa \gamma_T}{\rho_0}.$$

Taking inverse Laplace transforms in expression (5.10) we arrive at the classical analytic form of Kirchhoff's formula

$$\begin{aligned} \Phi(x', t) = \frac{1}{4\pi} \int_A \Bigg\{ & \Phi(x, t) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{cr} \frac{\partial r}{\partial n} \left[\frac{\partial \Phi(x, t)}{\partial t} \right] \\ & - \frac{1}{r} \left[\frac{\partial \Phi(x, t)}{\partial n} \right] \Bigg\} dA(x), \qquad x' \in B^+. \end{aligned} \tag{5.11}$$

In this formula $[\Phi(x, t)] = \Phi(x, t - r/c)$ is the retarded potential.

6. APPROXIMATE FORMULAE FOR THE POTENTIALS Φ AND Θ IN COUPLED THERMOELASTICITY

The determination of the potentials Φ and Θ from formulae (2.16) and (3.7) involves considerable difficulty relating to the evaluation of the inverse Laplace transforms of $\tilde{G}$, $\tilde{H}$ and their derivatives. These difficulties may be circumvented by applying a perturbation scheme in which the small parameter is the quantity $\varepsilon = \eta_0 m_0 \varkappa$ characterizing the coupling of the temperature field and strain field.[1]

Thus we start from the representations

$$\begin{aligned} \Phi &= \Phi_0 + \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + \cdots, \\ G &= G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \cdots. \end{aligned} \tag{6.1}$$

[1] For copper $\varepsilon = 0.0168$, for aluminum $\varepsilon = 0.0356$ and for steel $\varepsilon = 0.00297$.

We substitute these expressions into (2.16) and use the fact that the function $\hat{G}$ in it may be expressed in terms of G in the following way:

$$\hat{G} = [\nabla^2 - (1 + \varepsilon)\partial_\tau]G.$$

Retaining two terms in each of the series in (6.1), we obtain

$$\begin{aligned}\Phi(\eta, \tau) = {} & \int_0^\tau d\tau' \left\{ \int_A \left(G_0 \frac{\partial \Theta}{\partial n} - \Theta \frac{\partial G_1}{\partial n} \right) dA \right. \\ & \left. + \frac{1}{m} \int_A \left[N G_0 \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n}(N G_0) \right] dA \right\} \\ & + \varepsilon \int_0^\tau d\tau' \left\{ \int_A \left(G_1 \frac{\partial \Theta}{\partial n} - \Theta \frac{\partial G_1}{\partial n} \right) dA \right. \\ & \left. + \frac{1}{m} \int_A \left[(N G_1 - \partial_{\tau'} G_0) \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n}(N G_1 - \partial_{\tau'} G_0) \right] dA \right\},\end{aligned} \tag{6.2}$$

wherein $N \equiv \nabla^2 - \partial_{\tau'}$.

When $\varepsilon = 0$, (6.2) becomes the sum of the formulae (4.4) and (5.7) of the theory of thermal stress. Substituting (6.1) and

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots \tag{6.3}$$

into (3.7) and taking into account two terms in the expansions (6.1) and (6.3), we find

$$\begin{aligned}\Theta(\eta, \tau) = {} & \int_0^\tau d\tau' \int_A \left(H_0 \frac{\partial \Theta}{\partial n} - \Theta \frac{\partial H_0}{\partial n} \right) dA \\ & + \varepsilon \int_0^\tau d\tau' \left\{ \int_A \left(H_1 \frac{\partial \Theta}{\partial n} - \Theta \frac{\partial H_1}{\partial n} \right) dA \right. \\ & \left. + \frac{1}{m} \int_A \left[(N H_1 - \partial_{\tau'} H_0) \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n}(N H_1 - \partial_{\tau'} H_0) \right] dA \right\}.\end{aligned} \tag{6.4}$$

It remains for us to determine the functions G_0, G_1 and H_0, H_1. Expanding the quantities $\lambda_1^2(\varepsilon, p)$ and $\lambda_2^2(\varepsilon, p)$ in Maclaurin series with respect to ε and retaining the first two terms, we obtain

$$\begin{gathered}\lambda_1^2 \approx p^2 + \frac{p^2}{p-1}\varepsilon, \qquad \lambda_2^2 \approx p - \frac{p}{p-1}\varepsilon, \\ \lambda_1 \approx p + \frac{p}{2(p-1)}\varepsilon, \qquad \lambda_2 \approx p^{1/2}\left[1 - \frac{\varepsilon}{2(p-1)}\right], \\ \frac{1}{\lambda_1^2 - \lambda_2^2} \approx \frac{1}{p(p-1)}\left[1 - \varepsilon\frac{p+1}{(p-1)^2}\right].\end{gathered} \tag{6.5}$$

Insertion of the values of λ_1 and λ_2 into the expressions $e^{-\rho\lambda_1}$ and $e^{-\rho\lambda_2}$ yields

$$
\begin{aligned}
e^{-\rho\lambda_1} &\approx \left(1 - \frac{\varepsilon\rho}{2}\,\frac{p}{p-1}\right)e^{-\rho p}, \\
e^{-\rho\lambda_2} &\approx \left(1 - \frac{\varepsilon\rho}{2}\,\frac{p^{1/2}}{p-1}\right)e^{-\rho\sqrt{p}}.
\end{aligned} \tag{6.6}
$$

Finally, applying (6.5) and (6.6), we find from (2.6) that

$$
\tilde{G}_0 = \tilde{G}|_{\varepsilon=0} = \frac{1}{4\pi\rho}\,\frac{e^{-\rho p} - e^{-\rho\sqrt{p}}}{p(p-1)},
$$

$$
\tilde{G}_1 = -\frac{1}{4\pi\rho}\,\frac{1}{p(p-1)^2}\left[\left(\frac{p+1}{p-1} + \frac{\rho p}{2}\right)e^{-\rho p} - \left(\frac{p+1}{p-1} + \frac{\rho\sqrt{p}}{2}\right)e^{-\rho\sqrt{p}}\right].
$$

The quantity $G_0 = G|_{\varepsilon=0}$ is given by (4.6). For the function $G_1(\rho, \tau)$ we obtain the following formula [3]:

$$
\begin{aligned}
G_1(\rho,\tau) = -\frac{1}{4\pi\rho}\Bigg\{&\left[\left((\tau-\rho)^2 + (\tau-\rho)\left(\frac{\rho}{2} - 1\right) + 1\right)e^{\tau-\rho} - 1\right]H(\tau-\rho) \\
&- L(\rho,\tau)\frac{\rho}{2} - \left[\frac{1}{2}\left(\tau^2 + \frac{\rho^2}{4}\right)U + \frac{1}{2}\left(\frac{\rho}{4} - \tau\rho\right)V\right. \\
&\left. - \frac{\rho}{4}\sqrt{\frac{\tau}{\pi}}\exp\left(-\frac{\rho^2}{4\tau}\right)\right] - \left(\frac{\tau^2}{2} - \tau + \frac{\rho^2}{8} + 1\right)U \\
&+ \left(\frac{5}{8}\rho - \frac{\tau\rho}{2}\right)V + \operatorname{erfc}\left(\frac{\rho}{2\sqrt{\tau}}\right) - \frac{\rho}{4}\sqrt{\frac{\tau}{\pi}}\exp\left(-\frac{\rho^2}{4\tau}\right)\Bigg\},
\end{aligned} \tag{6.7}
$$

where

$$
V = \frac{e^{\tau}}{2}\left[e^{-\rho}\operatorname{erfc}\left(\frac{\rho}{2\sqrt{\tau}} - \sqrt{\tau}\right) - e^{\rho}\operatorname{erfc}\left(\frac{\rho}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right],
$$

$$
L(\rho,\tau) = \int_0^{\tau}\left[(\tau_0 + \tfrac{1}{2})V(\rho,\tau_0) - \frac{\rho}{2}U(\rho,\tau_0) + \sqrt{\frac{\tau_0}{\pi}}\exp\left(-\frac{\rho^2}{4\tau_0}\right)\right]d\tau_0
$$

($U(\rho, \tau)$ was defined in § 4).

The function $G_1(\rho, \tau)$, as well as G_0, represents two types of waves, elastic and diffusion.

Substitution of (6.5) and (6.6) in the expression (3.3) for $\tilde{H}(\rho, p)$ yields

$$
\begin{aligned}
\tilde{H}_0(\rho,p) &= \frac{1}{4\pi\rho}e^{-\rho\sqrt{p}}, \\
\tilde{H}_1(\rho,p) &= \frac{p}{4\pi\rho(p-1)^2}\left[e^{-\rho p} + \left(\rho\frac{p-1}{2\sqrt{p}} - 1\right)e^{-\rho\sqrt{p}}\right].
\end{aligned} \tag{6.8}
$$

Determining the inverse Laplace transforms of these functions, we find

$$H_0(\rho,\tau) = \frac{1}{8(\pi\tau)^{3/2}}\exp\left(-\frac{\rho^2}{4\tau}\right),$$

$$\begin{aligned} H_1(\rho,\tau) = \frac{1}{4\pi\rho}\Bigg\{&(\tau-\rho+1)e^{\tau-\rho}H(\tau-\rho) \\ &-\left[(\tau+1)U(\rho,\tau)-\frac{\rho}{2}V\right] \\ &+\frac{\rho}{2}\left[V+\sqrt{\frac{1}{\pi\tau}}\exp\left(-\frac{\rho^2}{4\tau}\right)\right]\Bigg\}. \end{aligned} \tag{6.9}$$

The function H_0 is diffusive in nature. The expression for $H_1(\rho, \tau)$ contains terms characterizing both an elastic wave and a diffusion wave.

In considering stresses and strains in machine parts and structural members one may disregard the coupling of the temperature field and strain field and thus set $\varepsilon = 0$ in (6.2) and (6.4). In that event, we obtain the formulae discussed in § 4.

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