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ELECTROMAGNETIC INTERACTIONS IN ELASTIC SOLIDS

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CHAPTER I

FOUNDATIONS OF LINEAR PIEZOELECTRICITY

by

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1.1. Electromagnetism.

Certain crystals, such as quartz, tourmaline, Seignette salt, when subject to a stress, become electrically polarized (J. and P. Currie 1880). This is the simple piezoelectric effect. Conversely, an external electromagnetic field produces in a piezoelectric crystal a deformation. This inverse piezoelectric effect was predicted on the basis of thermodynamic consideration by H.G. Lippmann (1) and confirmed experimentally by brothers J. and P. Currie (2) in 1881. The linear theory of piezoelectricity was created by W. Voigt (3).

The practical applications of piezoelectric effects are widely known; first of all in generation of ultrasonic waves, in conversion of electromagnetic energy into mechanical energy and conversely, in prospecting solids with piezoelectric properties, etc. (4).

We begin our considerations from the electromagnetic foundations of the problem.

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(1) H.G. Lippmann, Ann. Chim. 29/1881/, 145.
(2) J. and P. Currie. Compt. Rendus. 93 /1884/, 1137.
(3) W. Voigt. Lehrbuch der Kristallphysik. Teubner, Leipzig, 1910.
(4) M.P. Wolarowicz, G.A. Sobolev, Piezoelectric method of geophysical prospecting of quartz, (in Russian). Izd. Nauka, Moscow, 1969.

The Maxwell equations in the MKS system have the form (5)

$$\operatorname{rot} \underline{H} = \frac{\partial \underline{D}}{\partial t} + \underline{J}, \tag{1}$$

$$\operatorname{rot} \underline{E} = - \frac{\partial \underline{B}}{\partial t}, \tag{2}$$

where $\underline{H}$ is the vector of the magnetic field, $\underline{E}$ the vector of the electric field, $\underline{B}$ the vector of magnetic induction, $\underline{D}$ the vector of the electric displacement and $\underline{J}$ the vector of the conduction current. In a solid we have the following constitutive relations for the field vectors :

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}, \tag{3}$$

$$\underline{B} = \mu_0 (\underline{H} + \underline{M}). \tag{4}$$

Here $\underline{P}$ is the vector of electric polarization and $\underline{M}$ the magnetization vector. ε_0, μ_0 denote the constant electric and magnetic perimeabilities. Eqs. (1) and (2) should be completed by the Gauss equation

$$\operatorname{div} \underline{D} = \varrho_e, \tag{5}$$

and an equation following from Eq. (2), namely

$$\operatorname{div} \underline{B} = 0. \tag{6}$$

Eq. (5) defines electric charges ϱ_e. Eqs. (1) and (5) imply the equation of conservation of electric charges.

$$\frac{\partial \varrho_e}{\partial t} + \operatorname{div} \underline{J} = 0. \tag{7}$$

Consider a region B of the body. bounded by surface ∂B . In the interior of B there is an electromagnetic field, electric currents and Joule's heat is created.

(5) J.A. Stratton, Electromagnetic Theory, Mc Graw-Hill, New York, 1941.

Multiply Eq. (1) by $\underline{E}$, Eq. (2) by $\underline{H}$, subtract the result and integrate over the region B ; than

$$\int_B (\underline{E}\,\mathrm{rot}\,\underline{H} - \underline{H}\,\mathrm{rot}\,\underline{E})\,dv = \int_B (\underline{E}\dot{\underline{D}} + \underline{H}\dot{\underline{B}})\,\mathrm{div} + \int_B \underline{E}\,\underline{J}\,dv. \tag{8}$$

Taking into account that

$$\underline{E}\,\mathrm{rot}\,\underline{H} - \underline{H}\,\mathrm{rot}\underline{E} = -\,\mathrm{div}(\underline{E}\times\underline{H}),$$

and making use of the Gauss transformation we transform (8) to the form

$$-\int_{\partial B} \underline{n}\cdot\underline{h}\,da = \int_B (\underline{E}\dot{\underline{D}} + \underline{H}\dot{\underline{B}})\,dv + \int_B \underline{E}\,\underline{J}\,dv \tag{9}$$

where we have introduced the so-called Poynting vector

$$\underline{h} = \underline{E}\times\underline{H}\,.$$

Eq. (9) is a mathematical consequence of the Maxwell equations and can be physically interpreted as the balance of electromagnetic energy. Thus, the scalar $\underline{n}\cdot\underline{h}$ represents the flux of electromagnetic energy through the surface ∂B of the body, into the surrounding medium. The expression $\underline{E}\dot{\underline{D}} + \underline{H}\dot{\underline{B}}$ is identified with the time increment of the electromagnetic energy U_e . Finally $\underline{E}\,\underline{J}$ represents Joule's heat. Eq. (9) can be written in the form

$$\frac{\partial}{\partial t}\int_B U_e\,dv = -\int_{\partial B} \underline{n}\cdot\underline{h}\,da - \int_B \underline{E}\cdot\underline{J}\,dv \tag{10}$$

The energy balance (10) states that the time increment of the electromagnetic energy is equal to the energy increment flowing through the surface ∂B and the increment of the electromagnetic energy dissipated by means of conversion into heat. Eq. (10) expresses the law of energy conservation for the electromagnetic field.

In the next Section we shall present a generalized energy balance taking into account the deformation of the body.

Let us now return to the Maxwell equations. In accordance with Eq. (6) the vector $\underline{B}$ is solenoidal. it can therefore be expressed in terms of the rotation of a vector $\underline{A}_o$:

$$\underline{B} = \mathrm{rot}\,\underline{A}_o \tag{11}$$

However, (11) does not define the vector $\underline{A}_0$ uniquely. Thus, we have

$$\underline{B} = \mathrm{rot}\,\underline{A}\,, \qquad \underline{A} = \underline{A}_0 - \mathrm{grad}\,\psi \tag{12}$$

Introducing (11) and (12) into Eq. (2) we obtain

$$\mathrm{rot}(\underline{E} + \dot{\underline{A}}_0) = 0\,, \qquad \mathrm{rot}(\underline{E} + \dot{\underline{A}}) = 0\,, \tag{13}$$

whence

$$\underline{E} = -\dot{\underline{A}}_0 - \mathrm{grad}\,\varphi_0 \qquad \underline{E} = -\dot{\underline{A}} - \mathrm{grad}\,\varphi\,, \tag{14}$$

and the functions φ, φ_0, ψ are connected by the relation

$$\varphi - \varphi_0 = \frac{\partial \psi}{\partial t}\,. \tag{15}$$

In view of (12) and $(14)_2$ we can represent the Maxwell equations in terms of the vector potential $\underline{A}$ and the scalar potential φ :

$$\mathrm{rot}\,\underline{H} = \dot{\underline{D}} + \underline{J}\,, \tag{16}$$

$$\underline{B} = \mathrm{rot}\,\underline{A}\,, \tag{17}$$

$$\underline{E} = -\mathrm{grad}\,\varphi - \dot{\underline{A}} \tag{18}$$

The constitutive relations (3) and (4) and the Gauss equation (6) remain the same.

The Poynting vector $\underline{h}$ can be written in terms of the potentials $\underline{A}$, φ , namely we have

$$\underline{h} = \underline{E} \times \underline{H} = \varphi(\underline{J} + \dot{\underline{D}}) - \dot{\underline{A}} \times \underline{H}\,. \tag{19}$$

Consider now piezoelectric bodies (which are dielectrics). In general they are electrically neutral, contain the same amount of positive and negative charges and to not conduct current. An introduction of a dielectric into an electromagnetic field changes the latter. Consequently, the vectors $\underline{E}$ and $\underline{D}$ are not parallel and differ by the polarization vector $\underline{P}$. For piezoelectrics we introduce the same simplifications as for non-magnetizable dielectrics

$$\underline{J} = 0\,, \qquad \varrho_e = 0\,, \qquad \underline{M} = 0\,. \tag{20}$$

Under the above assumptions, the Maxwell equations (16)-(18) take the form

$$\mathrm{rot}\,\underline{H} = \dot{\underline{D}}\,, \tag{21}$$

$$\underline{B} = \mathrm{rot}\,\underline{A}\,, \tag{22}$$

$$\underline{E} = -\,\mathrm{grad}\,\varphi - \dot{\underline{A}}\,. \tag{23}$$

The constitutive rlation (3) remains the same, while the relation (9) is now

$$\underline{B} = \mu_0 \underline{H} \tag{24}$$

In view of the fact $\varrho_e = 0$, Eq. (6) is homogeneous

$$\mathrm{div}\,\underline{D} = 0\,. \tag{25}$$

and the Poynting vector takes the simpler form

$$\underline{h} = \varphi \dot{\underline{D}} - \dot{\underline{A}} \times \underline{H}\,. \tag{26}$$

A further simplification consists in neglecting the magnetic term ($\dot{\underline{A}} = 0$) in the expres– sion (23). Thus, we arrive at the relation

$$\underline{E} = -\,\mathrm{grad}\,\varphi \tag{27}$$

We also have at our disposal the equation

$$\mathrm{div}\,\underline{D} = 0\,, \tag{28}$$

and the constitutive relation

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}\,. \tag{29}$$

Neglecting also in the expression for the Poynting vector the magnetic term we obtain

$$h = \varphi \dot{\underline{D}}\,. \tag{30}$$

A justification of the above (experimentally confirmed) simplification was presented in an interesting paper by H.F. Tiersten(1).

This simplification is valid for electromagnetic waves which are not coupled with elastic waves and when we consider wave lengths close to the lengths of elastic waves (the latter are much shorter than the electromagnetic waves with the same frequency).

Let us now return to the energy balance (10). For piezoelectrics, when $\underline{B} = 0$, $\underline{J} = 0$, in view of (30) we have

$$\frac{\partial}{\partial t} \int_{B} U_e \, dv = - \int_{\partial B} n_i \dot{D}_i \varphi \, da$$

(1) H.F. Tiersten. The radiation and confinement of electromagnetic energy accompanying the oscillations of piezoelectric crystal plates. Rec. Advances in Engineering Science. Part I. Ed. A.C. Eringen. Gordon and Breach Science Publ. New York 1970.

or

$$\frac{\partial}{\partial t}\int_B U_e dv = -\int_B \varphi_{,i}\dot{D}_i dv \tag{31}$$

In performing the Gauss transformation we have made use here of Eq. (28). Thus, finally

$$\frac{\partial}{\partial t}\int_B U_e dv = \int_B E_i \dot{D}_i dv \tag{32}$$

In the next Section we shall take into account the deformation of the body in the energy balance.

1.2. Energy balance

Assume that the considered body undergoes deformation due to external loading and electromagnetic field, whoch may vary in time. Assume also that there are no heat sources in the body and no heat exchange by means of heat conduction (an adiabatic process). Apply to an arbitrary region B of the body bounded by surface ∂B the principle of energy conservation

$$\frac{\partial}{\partial t}\int_B\left(\frac{1}{2}\varrho v_i v_i + U\right) dv = \int_B X_i v_i dv + \int_{\partial B} p_i v_i da + \int_B E_i \dot{D}_i dv\,, \tag{1}$$

where

$$\mathscr{K} = \frac{1}{2}\int_B \varrho_i v_i v_i d\,v. \tag{2}$$

is the kinetic energy, U the internal (mechanical and electromagnetic) energy and denote by

$$\mathscr{L} = \int_B X_i v_i\, dv + \int_{\partial B} p_i v_i\, da \tag{3}$$

the mechanical power. The last integral in (1) is the flux of electromagnetic energy through the surface ∂B ,

$$\mathscr{D} = -\int_{\partial B}\varphi\,\dot{D}_i n_i da = \int_B E_i \dot{D}_i dv\,. \tag{4}$$

The principle of energy conservation states that the time increment of the kinetic and internal energies is equal to the power of the external forces and the electromagnetic energy flowing through the surface ∂B . Eq. (1) can be written in the form

$$\frac{d}{dt}(\mathcal{K}+\mathcal{U}) = \mathcal{L}+\mathcal{D}\ , \quad \mathcal{U} = \int_B U\,dv\ , \tag{5}$$

Let us transform in (1) the surface integral into a volume integral, making use of the relation

$$p = \sigma_{ji} n_j \tag{6}$$

where n_i is the normal vector directed outwards. Making use of the Gauss transformation we arrive at the equation

$$\int_B \dot{U}\,dv = \int_B \Big[(\sigma_{ji,j} + X_i - \varrho \dot{v}_i) v_i + \sigma_{ji} v_{i,j} + E_i \dot{D}_i \Big]\,dv\ , \tag{7}$$

which must be satisfied for every part of the body. Thus, we obtain the local principle of energy conservation

$$\dot{U} = \sigma_{ji} v_{i,j} + (\sigma_{ji,j} + X_i - \varrho \dot{v}_i) v_i + E_i \dot{D}_i \tag{8}$$

We shall demand that Eq. (8) be invariant under the rigid motion of the body[1] and first consider the translation

$$v_i \longrightarrow v_i + b_i \tag{9}$$

where b_i is an arbitrary constant vector. We assume that then the quantities ϱ . U , X_i , σ_{ji} . E_i, remain constant. Introducing (9) into (8) we have

$$\dot{U} = (v_i + b_i)(\sigma_{ji,j} + X_i - \varrho \dot{v}_i) + \sigma_{ji} v_{i,j} + E_i \dot{D}_i\ , \tag{10}$$

and subtracting (9) from (10) we are led to the equation

$$b_i(\sigma_{ji,j} + X_i - \varrho \dot{v}_i) = 0$$

which should be satisfied for arbitrary b_i . Thus, we arrive at the equation of motion

$$\sigma_{ji,i} + X_i = \varrho \dot{v}_i\ . \tag{11}$$

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(1) A.E. Green, R.S. Rivlin. Arch. Rat. Mech. Anal. 17/1964/. 113.

This equation considerably simplifies the energy balance, namely

$$\dot{U} = \sigma_{ji} v_{i,j} + E_i \dot{D}_i . \tag{12}$$

In accordance with our assumption the expression (12) must be invariant with respect to a rigid rotation. Therefore, assume that

$$v_i \to v_i - \varepsilon_{ikl} x_k \Omega_l , \quad v_{i,j} \to v_{i,j} - \varepsilon_{ijl} \Omega_l , \quad \underline{\Omega} = \mathrm{const}. \tag{13}$$

Introducing (13) into (12) we have

$$\dot{U} = \sigma_{ji}(v_{i,j} - \varepsilon_{ijl} \Omega_l) + E_i \dot{D}_i \tag{14}$$

Subtracting from Eq. (14) the expression (12) and bearing in mind the invariance of the quantities U , σ_{ji} , E_i we obtain

$$\Omega_i \varepsilon_{ijk} \sigma_{jk} = 0 . \tag{15}$$

This result implies the symmetry of the stress tensor σ_{ji} ,

$$\varepsilon_{ijk} \sigma_{jk} = 0 , \quad \sigma_{ji} = \sigma_{ij} . \tag{16}$$

Consequently we obtain the equation

$$\dot{U} = \sigma_{ji} v_{i,j} + E_i \dot{D}_i , \tag{17}$$

moreover, now we already know that the stress tensor is symmetric. Since, by definition

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{18}$$

we have

$$v_{i,j} = \dot{\varepsilon}_{ij} + \dot{\omega}_{ij}$$

where ε_{ij} is the symmetric strain tensor and ω_{ij} is the antisymmetric rotation tensor. Thus, $\sigma_{ij} \omega_{ij} = 0$. The energy balance (17) can be written in the form

$$U = \sigma_{ij} \dot{\varepsilon}_{ij} + E_i \dot{D}_i \tag{19}$$

It is evident that $U = U(\varepsilon_{ij}, D_i)$ and that

$$\dot{U} = \frac{\partial U}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} + \frac{\partial U}{\partial D_i} \dot{D}_i . \tag{20}$$

Hence, in view of Eqs (19) and (20) we obtain

$$\left(\sigma_{ij} - \frac{\partial U}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} + \left(E_i - \frac{\partial U}{\partial D_i}\right)\dot{D}_i = 0 . \tag{21}$$

This equation should hold for arbitrary values of $\dot{\varepsilon}_{ij}$, $\dot{D}_i$, whence

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}, \quad E_i = \frac{\partial U}{\partial D_i} \tag{22}$$

In further considerations it will be convenient to employ constitutive relations in which the quantities σ_{ij} and D_i depend on ε_{ij}, E_i . Therefore we introduce the electric entalpy H defined as follows :

$$H = U - E_i D_i . \tag{23}$$

Eliminating $\dot{U}$ from (19) and (23) we obtain the equation

$$\dot{H} = \sigma_{ij}\dot{\varepsilon}_{ij} - D_i \dot{E}_i . \tag{24}$$

Its is evident that $H \equiv H(\varepsilon_{ij}, E_i)$. Since

$$\dot{H} = \frac{\partial H}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial H}{\partial E_i}\dot{E}_i \tag{25}$$

Eqs (24) and (25) imply that

$$\left(\sigma_{ij} - \frac{\partial H}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} - \left(D_i + \frac{\partial H}{\partial E_i}\right)\dot{E}_i = 0 , \tag{26}$$

which again should be satisfied for arbitrary $\dot{\varepsilon}_{ij}$, $\dot{E}_i$. Hence

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}}, \qquad D_i = -\frac{\partial H}{\partial E_i} . \tag{27}$$

This relation will be employed in deriving the constitutive relations.

1.3. The constitutive relations

Let us expand the electric entalpy $H(\varepsilon_{ij}, E_i)$ into the Mac Laurin series in the vicinity of the natural state ($\varepsilon_{ij} = 0$, $E_i = 0$), neglecting terms higher than of the second order. For a homogeneous anisotropic body we obtain the following series :

$$H(\varepsilon_{ij}, E_i) = \frac{1}{2} c_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} - e_{kij}\, \varepsilon_{ij}\, E_k - \frac{1}{2} \epsilon_{ij}\, E_i\, E_j . \tag{1}$$

We assumed that in the body there are no initial stresses and no initial electric field. The relations

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}}, \quad D_i = -\frac{\partial H}{\partial E_i}, \tag{2}$$

lead to the constitutive equations

$$\sigma_{ij} = c_{ijkl}\, \varepsilon_{kl} - e_{kij}\, E_k , \tag{3}$$

$$D_i = e_{ikl}\, \varepsilon_{kl} + \epsilon_{ik}\, E_k , \tag{4}$$

where $c_{ijkl} = \overset{E}{c}_{ijkl}$ is the elastic stiffness and $E_i = \text{const.}$, e_{kij} is the piezoelectric constant and ϵ_{ij} the constant permittivity for $\varepsilon_{kl} = \text{const.}$

Observe that

$$\frac{\partial^2 H}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}} = \frac{\partial^2 H}{\partial \varepsilon_{kl}\, \partial \varepsilon_{ij}} \quad \text{or} \quad \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}},$$
$$\frac{\partial^2 H}{\partial E_i\, \partial E_j} = \frac{\partial^2 H}{\partial E_j\, \partial E_i} \quad \text{or} \quad \frac{\partial D_i}{\partial E_j} = \frac{\partial D_j}{\partial E_i} . \tag{5}$$

The above relations imply that

$$c_{ijkl} = c_{klij}, \qquad \epsilon_{ij} = \epsilon_{ji} . \tag{6}$$

whereas the symmetry of the tensors σ_{ij} and ε_{ij} leads to the relations

$$c_{ijkl} = c_{jikl}, \qquad c_{ijkl} = c_{ijlk}, \qquad e_{kij} = e_{kji}, \tag{7}$$

The tensor ϵ_{ij} is symmetric and the polar tensor e_{kji} is symmetric with respect to the indices j and i.

In the general case of triclinic crystal we have 21 elastic constants c_{ijkl} . 18 piezoelectric constants e_{kij} and 6 permittivity constants ϵ_{ij}.

It is known that the piezoelectric effect can occur in materials which do not have a center of symmetry. In bodies which do have a center of symmetry the polar tensor ϵ_{kij}

vanishes.

Consider the particular case of isotropic body. We have the following isotropic tensors :

$$\begin{aligned} c_{ijkl} &= \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\,\delta_{ij}\delta_{kl}\,, \\ e_{kij} &= \epsilon_{kij}e\,, \quad \epsilon_{ij} = \delta_{ij}\,\epsilon\,. \end{aligned} \tag{8}$$

Here ϵ_{kij} is Ricci's antisymmetric tensor. Introducing (8) into (3) and (4) we obtain

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk} - e\epsilon_{kij}E_k\ , \tag{9}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \delta_{ij}\varepsilon E_k\,. \tag{10}$$

Since the tensors $\sigma_{ij}, \varepsilon_{ij}$ are symmetric and e_{kij} is antisymmetric, the piezoelectric term in Eqs (9) and (10) vanishes. We have

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\,\delta_{ij}\varepsilon_{kk}\,, \qquad D_i = \epsilon E_i\,. \tag{11}$$

Let us now return to the electric entalpy $H = U - E_i D_i$; hence

$$U = H + D_i E_i \tag{12}$$

Substituting for H from Eq. (1) and for D_i from the relation (4) we have

$$U = \frac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}\epsilon_{ij}E_iE_j \tag{13}$$

Since U is a non-negative scalar, the right-hand side of Eq. (13) should be a positive definite quadratic form, which ensures the stability of the solution.

1.4. The differential equations of piezoelectricity

Let us collect the equations and relations of piezoelectricity. Thus, we have the equations of motion and the equation for the electrical field

$$\sigma_{ji,j} + X_i = \varrho\ddot{u}_i\ , \tag{1}$$

$$D_{i,i} = 0\,. \tag{2}$$

completed by the constitutive equations

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{kij} E_k , \tag{3}$$

$$D_i = e_{ikl}\,\varepsilon_{kl} + \epsilon_{ik} E_k , \qquad E_k = -\varphi_{,k} , \tag{4}$$

and the definition of the strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) . \tag{5}$$

We introduce (3) and (4) into the differential equations (1)-(2) and make use of (5) ; then we arrive at four equations with the displacement vector u_i and the potentials φ as unknowns

$$c_{ijkl}\, u_{k,lj} + e_{kij}\,\varphi_{,kj} + X_i = \varrho \ddot{u}_i \tag{6}$$

$$e_{ikl}\, u_{k,li} - \epsilon_{ik}\,\varphi_{,ki} = 0 . \tag{7}$$

These differential equations should be completed by the boundary and initial conditions. If on a part of the body ∂B_1 displacements and on the complementary, part ∂B_2 tractions p_i are prescribed,

$$u_i = U_i(\underline{x},t) \text{ on } \partial B_1 , \qquad p_i = \sigma_{ji} n_j = P_i(\underline{x},t) \text{ on } \partial B_2 , \qquad \partial B_1 \cup \partial B_2 = \partial B \tag{9}$$

Suppose that on ∂B_3 the electric potential and on ∂B_4 surface charges $D_k n_k = -\sigma$ are given :

$$\varphi = \Phi(\underline{x},t) \text{ on } \partial B_3 , \quad D_k n_k = -\sigma \text{ on } \partial B_4 , \quad \partial B_3 \cup \partial B_4 = \partial B . \tag{10}$$

If we know the solution (u_i, φ) of the system of equations (6)-(7) then we can determine successively E_k from the formula $E_k = -\varphi_{,k}$, the strain ε_{ij} from the definition (5), the stress and the electric displacement from the constitutive relations (3) and (4). The knowledge of functions E_i and D_i makes it possible to calculate the electric polarization

$$P_i = D_i - \epsilon_0 E_i , \tag{11}$$

where ϵ_0 is the permittivity in cacuum. If the piezoelectric effect is absent, Eq. (6) takes the form

$$c_{ijkl}\, u_{k,lj} = \varrho \ddot{u}_i . \tag{12}$$

If, on the other hand, it does occur but the deformation is neglected then (7) reduces to

$$\epsilon_{ij}\varphi_{,ij} = 0 . \tag{13}$$

Eq. (12) is completed by the boundary conditions (9), while Eq. (13) by the boundary conditions (10).

1.5. Principle of virtual work. Uniqueness of solutions

Consider the celebrated Lagrange's principle for the deformable elastic body

$$\int_B (X_i - \varrho \ddot{u}_i)\,\delta u_i\,dv + \int_{\partial B} p_i\,\delta u_i\,da = \int_B \sigma_{ij}\,\delta\varepsilon_{ij}\,dv . \tag{1}$$

Here δu_i is a virtual displacement increment of u_i. We assume that δu_i is continuous and differentiable of class $C^{(1)}$. Moreover, we assume that the increment δu_i is compatible with the conditions restricting the motion of the body. On the part of the boundary on which the displacement u_1 is prescribed (i.e. on ∂B_1), $\delta u_i = 0$. If the tractions p_i are given on ∂B_2, then on this part of the surface the increment δu_i is arbitrary.

We shall apply the principle of virtual work (1) to piezoelectric bodies. Making use of the constitutive relation

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{kij}\,E_k \tag{2}$$

in (1) we obtain

$$\int_B (X_i - \varrho \ddot{u}_i)\,\delta u_i\,dv + \int_{\partial B} p_i\,\delta u_i\,da = c_{ijkl}\int_B \varepsilon_{kl}\,\delta\varepsilon_{ij}\,dv - e_{kij}\int_B E_k\,\delta\varepsilon_{ij}\,dv . \tag{3}$$

The second constitutive relation

$$D_k = e_{kij}\,\varepsilon_{ij} + \epsilon_{ij}E_j . \tag{4}$$

introduced into (3) yields the equation

$$\int_B (c_{ijkl}\,\varepsilon_{kl}\,\delta\varepsilon_{ij} + \epsilon_{kj}E_k\,\delta E_j)\,dv = \int_B (X_i - \varrho\ddot{u}_i)\,\delta u_i\,dv + \\ + \int_{\partial B} p_i\,\delta u_i\,da + \int_B E_k\,\delta D_k\,dv . \tag{5}$$

Making use of the relations for the electromagnetic field

$$E_k = -\varphi_{,k}, \qquad D_{i,i} = 0, \tag{6}$$

and introducing the notation

$$\mathcal{W} = \frac{1}{2}\int_B c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dv, \quad \mathcal{E} = \frac{1}{2}\int_B \epsilon_{ij}E_iE_j dv. \qquad \text{where} \qquad \mathcal{W} = \mathcal{E} + \mathcal{U},$$

we reduce Eq. (5) to the form

$$\delta(\mathcal{W}+\mathcal{E}) = \int_B (X_i - \varrho\ddot{u}_i)\delta u_i dv + \int_{\partial B}(p_i\delta u_i - \varphi\delta D_k n_k)da. \tag{7}$$

Thus, we have arrived at the principle of virtual work for a piezoelectric medium.

Consider a particular case of virtual increments δu_i, δD_i namely the real increments of the displacement u_i and the field D_i. We have

$$\delta u_i = \frac{\partial u_i}{\partial t}dt = v_i dt, \qquad \delta D_k = \frac{\partial D_k}{\partial t}dt = \dot{D}_k dt,$$
$$\delta\mathcal{W} = \dot{\mathcal{W}}dt, \qquad \delta\mathcal{E} = \dot{\mathcal{E}}dt. \tag{8}$$

Introducing the definition of the kinetic energy

$$\mathcal{K} = \frac{1}{2}\varrho\int_B v_iv_i dv, \qquad \delta\mathcal{K} = \varrho\int_B v_i\dot{v}_i dv = \dot{K}dt, \tag{9}$$

we reduce Eq. (7) to the form

$$\frac{d}{dt}(\mathcal{K}+\mathcal{W}+\mathcal{E}) = \int_B X_iv_i dv + \int_{\partial B}(p_iv_i - \varphi\dot{D}_k n_k)da. \tag{10}$$

It is readily observed that the above equation, called the fundamental energy balance of piezoelectricity, constitutes a version of the energy balance (see (1) in Sec. 1.2. and (13) in Sec. 1.3.).

Eq. (10) can serve for the proof of the theorem of uniqueness of the solution of the piezoelectricity equations.

Consider the equations

$$\sigma_{ji,i} + X_i = \varrho\ddot{u}_i, \tag{11}$$

with the boundary conditions

$$u_i = U_i(\underline{x}, t) \text{ on } \partial B_1 , \quad \sigma_{ji} n_i = p_i(\underline{x}, t) \text{ on } \partial B_2 , \quad \partial B_1 \cup \partial B_2 = \partial B \tag{12}$$

and the initial conditions

$$u_i(\underline{x}, 0) = f_i(\underline{x}) , \quad \dot{u}_i(\underline{x}, 0) = g_i(\underline{x}) \text{ on } B \tag{13}$$

Furthermore, consider the field equation

$$D_{i,i} = 0 \tag{14}$$

with the boundary conditions

$$\varphi = \Phi(\underline{x}, t) \text{ on } \partial B_3 , \quad D_k n_k = -\sigma(\underline{x}, t) \text{ on } \partial B_4 , \quad \partial B_3 \cup \partial B_4 = \partial B \tag{15}$$

We assume that two pairs of functions (u_i' , φ') and (u_i'' , φ'') satisfy Eqs. (11) and (14) and the appropriate boundary and initial conditions. Their difference $\hat{u}_i = u_i' - u_i''$, $\hat{\varphi} = \varphi' - \varphi''$ satisfies therefore the homogeneous equations (11) and (14) and the homogeneous boundary and initial conditions. Aq. (10) holds for the soulution $\hat{u}_i$, $\hat{\varphi}$ namely we have

$$\frac{d}{dt}(\hat{\mathscr{K}} + \hat{\mathscr{W}} + \hat{\mathscr{E}}) = \int_B \hat{X}_i \hat{v}_i \, dv + \int_{\partial B_1 + \partial B_2} \hat{p}_i \hat{v}_i \, da - \int_{\partial B_3 + \partial B_4} \hat{\varphi} \dot{\hat{D}}_i n_i \, da \tag{16}$$

Now, in view of the homogeneity of the equations and the boundary conditions, the right-hand side of (16) vanishes, for

$$\begin{gathered} \hat{X}_i = 0 , \quad \hat{u}_i = 0 \text{ on } \partial B_1 , \quad \hat{p}_i = 0 \text{ on } \partial B_2 , \\ \hat{\varphi} = 0 \text{ on } \partial B_3 , \quad \hat{D}_k n_k = 0 \text{ on } \partial B_4 . \end{gathered} \tag{17}$$

Thus, we have

$$\frac{d}{dt}(\hat{\mathscr{K}} + \hat{\mathscr{W}} + \hat{\mathscr{E}}) = 0 , \tag{18}$$

but in view of the homogeneity of the initial conditions

$$\hat{\mathscr{K}} + \hat{\mathscr{W}} + \hat{\mathscr{E}} = 0 , \tag{19}$$

or

$$\int_B (\varrho \hat{v}_i \hat{v}_i + c_{ijkl} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{kl} + \epsilon_{ij} \hat{E}_i \hat{E}_j)\, dv = 0 . \tag{20}$$

The integrand is a positive definite quadratic form. Hence

$$\hat{v}_i = 0, \qquad \hat{\varepsilon}_{ij} = 0, \qquad \hat{E}_i = 0 . \tag{21}$$

In view of the initial conditions for $\hat{u}_i$, $\hat{\varphi}$, to above relations imply that

$$u' = u'' + \text{linear term}, \qquad \varphi' = \varphi'' + \text{const.} \tag{22}$$

The linear term appearing in $(22)_1$ is a rigid displacement of the body. If the latter is clamped on ∂B_1, the linear term vanishes. The uniqueness of the strain $\varepsilon'_{ij} = \varepsilon''_{ij}$ and the field $E'_i = E''_i$ implies the uniqueness of the stress $\sigma'_{ij} = \sigma''_{ij}$ and the electric displacement $D'_i = D''_i$.

1.6. Hamilton's principle

Consider the functional

$$\pi = \int_B (H - X_i u_i)\, dv - \int_{\partial B} (p_i u_i - \sigma \varphi)\, da \tag{1}$$

where H is the electric entalpy, φ the electric potential and σ the charge on ∂B. Hamilton's principle generalized to piezoelectricity has the form

$$\delta \int_{t_1}^{t_2} (\mathcal{K} - \pi)\, dt = 0, \tag{2}$$

where $t_1 - t_2$ is the considered time interval, π the functional (1) and K the kinetic energy. The admissible motions of the body must be compatible with the conditions restricting the motion of the body. Moreover, the following conditions must be satisfied :

$$\delta u_i(\underline{x}, t_1) = \delta u_i(\underline{x}, t_2) = 0 . \tag{3}$$

The quantities subject to variation are the displacement u_i and the electric potential φ Performing the variations

$$\delta\int_{t_1}^{t_2}\mathscr{K}\,dt = \delta\int_{t_1}^{t_2}dt\int_B \frac{1}{2}\varrho v_i v_i\,dv = \int_{t_1}^{t_2}dt\int_B \varrho v_i\delta v_i\,dv =$$

$$= \int_{t_1}^{t_2}dt\int_B\Big[\frac{\partial}{\partial t}(\varrho\dot{u}_i\delta u_i) - \varrho\ddot{u}_i\delta u_i\Big]dv = \tag{4}$$

$$= \int_B\Big[\varrho\dot{u}_i\delta u_i\Big]_{t_1}^{t_2}dv - \int_{t_1}^{t_2}dt\int_B\varrho\ddot{u}_i\delta u\,dv\,.$$

In view of (3) we have

$$\delta\int_{t_1}^{t_2}\mathscr{K}\,dt = -\int_{t_1}^{t_2}dt\int_B\varrho\ddot{u}_i\delta u_i\,dv \tag{5}$$

Returning to Eq. (2)

$$\delta\int_{t_1}^{t_2}(\mathscr{K}-\pi)\,dt = -\int_{t_1}^{t_2}dt\Big\{\int_B\varrho\ddot{u}\;\delta u_i dv +$$

$$+\int_B\Big(\frac{\partial H}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{\partial H}{\partial E_i}\delta E_i - X_i\delta u_i\Big)dv - \int_{\partial B}(p_i\delta u_i - \sigma\delta\varphi)\,da\Big\} = 0\,, \tag{6}$$

Making use of the constitutive relations

$$\frac{\partial H}{\partial\varepsilon_{ij}} = \sigma_{ij} \qquad \frac{\partial H}{\partial E_i} = -D_i\,, \tag{7}$$

and the known relation

$$p_i = \sigma_{ij}n_j \quad \text{on} \quad \partial B \tag{8}$$

after simple transformations we arrive at the equation

$$\begin{aligned}&\int_{t_1}^{t_2}dt\,\Big\{\int_B\Big[\big[\sigma_{ji,j} + X_i - \varrho\ddot{u}_i\big]\delta u_i + D_{i,i}\delta\varphi\Big]dv - \\ &-\int_{\partial B}\Big[(\sigma_{ji}n_j - p_i)\delta u_i + (D_i n_i + \sigma)\delta\varphi\Big]da\Big\} = 0\,.\end{aligned} \tag{9}$$

Since the variations $\delta u_i, \delta\varphi$ are arbitrary, (9) leads to the following equations governing the motion and the electromagnetic field :

$$\sigma_{ji,j} + X_i = \varrho\ddot{u}_i \tag{10}$$

$$D_{i,i} = 0, \qquad \underline{x} \in B, \ t \geqslant 0. \tag{11}$$

Moreover, Eq. (9) yields the boundary conditions for Eqs. (10), (11)

$$p_i = \sigma_{ji} n_j \quad \text{on} \quad \partial B_2 \tag{12}$$

$$D_i n_i = -\sigma \quad \text{on} \quad \partial B_4 \tag{13}$$

If on ∂B_1 displacement is prescribed, then δu_i whereas if on δB_3 the potential is given, $\delta\varphi = 0$. Thus, the Hamilton principle yields the so-called natural boundary conditions. The generalized Hamilton principle was deduced by H.F. Tiersten[1].

The Hamilton principle presented here contains as a particular case the principle for an anisotropic body without piezoelectric properties. Then we replace the electric entalpy H by the internal energy U. Thus, (1) takes the form

$$\pi^* = \int_B (U - X_i u_i)\, dv - \int_{\partial B} p_i u_i \, da . \tag{14}$$

and the classical Hamilton principle is

$$\delta \int_{t_1}^{t_2} (\mathscr{K} - \pi^*)\, dt = 0 . \tag{15}$$

1.7. The recipocity theorem

Consider two sets of causes and effects. The causes are the action of body forces, prescribed displacements and tractions on the boundary, an electric potential or electric charges on ∂B and finally the action of initial conditions.

The effects are the displacement u_i and the electric potential φ. The second set of causes and effects will be denoted by primes.

We base on the equations of motion for both sets of causes and effects

$$\sigma_{ji,j} + X_i - \varrho \ddot{u}_i = 0, \tag{1}$$

$$\sigma'_{ji,j} + X'_i - \varrho \ddot{u}' = 0 . \tag{2}$$

(1) H.F. Tiersten. Linear piezoelectric plate vibrations. Plenum Press. New York. 1969.

Performing over both equations the Laplace transform defined as follows

$$\mathscr{L}\left[u_i(\underline{x},t)\right] = \bar{u}_i(\underline{x},p) = \int_0^{\infty} u_i(\underline{x},t)\, e^{-pt} dt, \tag{3}$$

we obtain

$$\bar{\sigma}_{ji,j} + \bar{X}_i = \varrho p^2 \bar{u} \tag{4}$$

$$\bar{\sigma}'_{ji,j} + \bar{X}'_i = \varrho p^2 \bar{u}' \tag{5}$$

where we assumed that the initial conditions for the displacements are homogeneous, i.e.

$$u_i(\underline{x},0) = 0, \qquad \dot{u}_i(\underline{x},0) = 0 \qquad \text{on } B \tag{6}$$

Obviously, the non-homogeneous initial conditions can be treated in the same manner. Multiply now Eq. (4) by $\bar{u}_i'$ and Eq. (5) by $\bar{u}_i$ substract the result and integrate over the region of the body ; thus

$$\int_B \left[(\bar{\sigma}_{ji,j} + \bar{X}_i)\bar{u}'_i - (\bar{\sigma}'_{ji,j} + \bar{X}'_i)\bar{u}_i\right] dv = 0. \tag{7}$$

Introducing the contract forces we transform Eq. (7) to the form

$$\int_B (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\, dv + \int_{\partial B} (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\, da = \\ = \int_B (\bar{\sigma}_{ji} \bar{\varepsilon}'_{ji} \quad \bar{\sigma}'_{ji} \bar{\varepsilon}_{ji})\, dv. \tag{8}$$

In view of the constitutive relation

$$\bar{\sigma}_{ij} = c_{ijkl}\bar{\varepsilon}_{kl} - e_{kij}\bar{E}_k, \qquad \bar{\sigma}'_{ij} = c_{ijkl}\bar{\varepsilon}'_{kl} - e_{kij}\bar{E}'_k, \tag{9}$$

we reduce Eq. (8) to the form

$$\int_B (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\, dv + \int_{\partial B} (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\, da \\ = - e_{kij} \int_B (\bar{\varepsilon}'_{ij} \bar{E}_k - \bar{\varepsilon}_{ij} \bar{E}'_k)\, dv. \tag{10}$$

We have at our disposal the equations for the electromagnetic field

$$D_{k,k} = 0, \qquad D'_{k,k} = 0, \tag{11}$$

$$E_k = -\varphi_{,k}\,, \qquad E'_k = -\varphi'_{,k}\,.$$

Let us perform over the above equations and relations the integral transform and consider the expression

$$\int_B (\bar{D}_{k,k}\bar{\varphi}' - \bar{D}'_{k,k}\bar{\varphi})\,dv = 0 \tag{12}$$

or

$$\int_{\partial B} (\bar{D}_k\bar{\varphi}' - \bar{D}'_k\bar{\varphi})n_k\,da + \int_B (\bar{D}_k\bar{E}'_k - \bar{D}'_k\bar{E}_k)\,dv = 0. \tag{12'}$$

Taking into account the constitutive equations

$$\bar{D}_k = e_{kij}\bar{\varepsilon}_{ij} + \epsilon_{kj}\bar{E}_j\,, \qquad \bar{D}'_k = e_{kij}\bar{\varepsilon}'_{ij} + \epsilon_{kj}\bar{E}'_j\,, \tag{13}$$

in Eq. (12′) we are led to the equation

$$\int_{\partial B} (\bar{D}'_k\bar{\varphi} - \bar{D}_k\bar{\varphi}')\,n_k\,da = -e_{kij}\int_B (\bar{\varepsilon}_{ij}\bar{E}'_k - \bar{\varepsilon}'_{ij}\bar{E}_k)\,dv \tag{14}$$

Finally, eliminating form Eqs (10) and (14) the common terms we arrive at the equation for reciprocity of work in the form

$$\begin{aligned} &\int_B \bar{X}_i\bar{u}'_i\,dv + \int_{\partial B} (\bar{p}_i\bar{u}'_i + \bar{D}_k n_k\bar{\varphi}')\,da = \\ &= \int_B \bar{X}'_i\bar{u}_i\,dv + \int_{\partial B} (\bar{p}'_i\bar{u}_i + \bar{D}'_k n_k\bar{\varphi})\,da\,. \end{aligned} \tag{15}$$

How we invert the Laplace transform

$$\begin{aligned} &\int_B X_i * u'_i\,dv + \int_{\partial B} (p_i * u'_i + D_k * \varphi' n_k)\,da = \\ &= \int_B X'_i * u_i\,dv + \int_{\partial B} (p'_i * u_i + D'_k * \varphi n_k)\,da\,, \end{aligned} \tag{16}$$

using the notation

$$\mathscr{L}^{-1}(\bar{X}_i\bar{u}'_i) = X_i * u'_i = \int_0^t X_i(\underline{x}, t-\tau)\,u'_i(\underline{x}, \tau)\,d\tau, \quad \text{etc.}$$

Eq. (16) consitutes the theorem of reciprocity of work generalized to piezoelectricity. If the piezoelectric effect is absent $(\varphi = 0,\ D_i = 0)$ (16) implies the known D. Graffi's theorem[1].

$$\int_B X_i * u_i' dv + \int_{\partial B} p_i * u_i' da = $$
$$= \int_B X_i' * u_i dv + \int_{\partial B} p_i' * u_i da . \tag{17}$$

In the case of harmonic vibrations we have

$$u_i(\underline{x}, t) = u_i^*(\underline{x})\, \varepsilon^{-i\omega t} , \qquad X_i(\underline{x}, t) = X_i^*(\underline{x})\, \varepsilon^{-i\omega t} , \tag{18}$$

where $\omega > 0$ is the frequency. Eqs. (1) and (2) can now be written in terms of amplitudes

$$\sigma_{ji,j}^* + X_i^* + \varrho\,\omega^2 u^* = 0 \tag{19}$$

$$\sigma_{ji,j}'^* + X_i'^* + \varrho\,\omega^2 u'^* = 0 . \tag{20}$$

Repeating all operations performed before for transforms, we arrive at the following form of the reciprocity theorem :

$$\int_B X_i^* u_i'^* dv + \int_{\partial B} (p_i^* u_i'^* + D_k^* n_k \varphi'^*) da = $$
$$= \int_B X'^* u_i\, dv + \int_{\partial B} (p_i'^* u_i^* + D_k'^* n_k \varphi^*) da . \tag{21}$$

If the piezoelectric effect is absent, Eq. (21) is simplified to the known equation of dynamic elasticity

$$\int_B X_i^* u_i'^* dv + \int_{\partial B} p_i^* u_i'^* da = \int_B X_i'^* u_i^* dv + \int_{\partial B} p_i'^* u_i^* da . \tag{22}$$

If we regard the piezoelectric effect as uncoupled with the deformation of the body, Eq. (21) implies that

$$\int_{\partial B} D_i^* \varphi'^* n_i da = \int_{\partial B} D_i'^* \varphi^* n_i da . \tag{23}$$

(1) D. Graffi. Sui teoremi di reciprocita nei fenomeni non stazionari. Att. Acad. Sci. Bologna. 10, ser. 11 (1963). 33.

1.8. Orthogonality of eigenvibrations of piezoelectrics

Consider eigenvibrations of a bounded piezoelectric body. We assume that the vibrations are harmonic, the equations of motion and of the electromagnetic field are homogeneous and the boundary conditions have the same property. We shall investigate two forms of vibrations, denoting their frequencies by ω_m and ω_n, $\omega_m \neq \omega_n$. The equations of motion can now be written in the form

$$\overset{(m)}{\sigma}_{ji,j} + \varrho\omega_m^2 \overset{(m)}{u}_i = 0 \quad , \tag{1}$$

$$\overset{(n)}{\sigma}_{ji,j} + \varrho\omega_n^2 \overset{(n)}{u}_i = 0 \quad , \tag{2}$$

where $\overset{(m)}{u}_i$ is the amplitude of the eigenvibrations and $\overset{(m)}{\sigma}_{ji}$ the corresponding stress. Multiply Eq. (1) by $\overset{(n)}{u}_i$ and Eq. (2) by $\overset{(m)}{u}_i$, substracting the result and integrating over the region of the body we have

$$\int_B \left[\overset{(m)}{\sigma}_{ji,j} \overset{(n)}{u}_i - \overset{(n)}{\sigma}_{ji,j} \overset{(m)}{u}_i \right] dv = \varrho(\omega_n^2 - \omega_m^2) \int_B \overset{(n)}{u}_i \overset{(m)}{u}_i dv . \tag{3}$$

Consider now the field equations

$$\overset{(m)}{D}_{i,i} = 0 , \tag{4}$$

$$\overset{(n)}{D}_{i,i} = 0 \tag{5}$$

We shall also use hereafter the identity

$$\int_B \left[\overset{(m)}{D}_{i,i} \overset{(n)}{\varphi} - \overset{(n)}{D}_{i,i} \overset{(m)}{\varphi} \right] dv = 0 . \tag{6}$$

Transforming Eq. (3) and making use of the constitutive relations for $\overset{(m)}{\sigma}_{ji}$, $\overset{(n)}{\sigma}_{ji}$ we obtain

$$\int_{\partial B} (\overset{(m)}{p}_i \overset{(n)}{u}_i - \overset{(n)}{p}_i \overset{(m)}{u}_i) da + e_{kij} \int_B (\overset{(n)}{\varepsilon}_{ij} \overset{(m)}{E}_k - \overset{(m)}{\varepsilon}_{ij} \overset{(n)}{E}_k) dv =$$

$$= \varrho(\omega_n^2 - \omega_m^2) \int_B \overset{(m)}{u}_i \overset{(n)}{u}_i dv . \tag{7}$$

On the other hand, transforming Eq. (6), in view of the constitutive relations for the functions $\overset{(m)}{D}_i$, $\overset{(n)}{D}_i$ we obtain

$$\int_{\partial B} (\overset{(m)}{D}_i \overset{(n)}{\varphi} - \overset{(n)}{D}_i \overset{(m)}{\varphi}) n_i da + e_{kij} \int_B (\overset{(n)}{E}_k \overset{(m)}{\varepsilon}_{ij} - \overset{(m)}{E}_k \overset{(n)}{\varepsilon}_{ij}) dv = 0 . \tag{8}$$

Adding (7) and (8)

$$\varrho(\omega_n^2 - \omega_m^2)\int_B \overset{(m)}{u_i}\,\overset{(n)}{u_i}\,dv = \int_{\partial B}(\overset{(m)}{p_i}\,\overset{(n)}{u_i} - \overset{(n)}{p_i}\,\overset{(m)}{u_i})\,da\ +$$

$$+\int_B(\overset{(m)}{D_i}\,\overset{(n)}{\varphi} - \overset{(n)}{D_i}\,\overset{(m)}{\varphi})\,n_i da \tag{9}$$

We have assumed above that the body performs free vibrations, i.e. the boundary conditions are homogeneous. If on ∂B_2 we have $p_i = 0$ and on $\partial B_1 : u_i = 0$ and, moreover, on $\partial B_3 : \varphi = 0$ and on $\partial B_4 : D_i n_i = 0$, we have

$$\varrho(\omega_n^2 - \omega_m^2)\int_B \overset{(n)}{u_i}\,\overset{(m)}{u_i}\,dv = 0 \tag{10}$$

Since the frequencies are different

$$\int_B \overset{(n)}{u_i}\,\overset{(m)}{u_i}\,dv = 0\,, \qquad n \neq m\,. \tag{11}$$

Thus, we have deduced the property of orthogonality of piezoelectric vibrations.

1.9. Equations and relations of piezoelectricity in new notation

In what follows, in concrete problems it will be convenient to introduce new notation. We replace ij and $k\ell$ by p and q ; since i, j and k take the values 1, 2, 3; p and q run from 1 to 6. Thus, we have

$$c_{ijk\ell} = c_{pq}\,, \qquad e_{ik\ell} = e_{iq}\,, \qquad \sigma_{ij} = T_{ij} = T_p\,. \tag{1}$$

The constitutive equations (3) and (4) of Sec. 1.3. take the form

$$T_p = c_{pq} S_q - e_{kp} E_k\,, \tag{2}$$

$$D_i = e_{iq} S_q + \epsilon_{ik} E_k \qquad i,k = 1,2,3\,, \quad p\,q = 1,2,\ldots,6, \tag{3}$$

where

$$\begin{aligned} \varepsilon_{ij} &= S_p \qquad \text{for } i=j,\ p = 1,2,3 \\ 2\,\varepsilon_{ij} &= S_p \qquad \text{for } i \neq j,\ p = 4,5,6 \end{aligned} \tag{4}$$

In the constitutive relation (2) we have

$$T_1 = T_{11} = \sigma_{11}\,,\quad T_2 = T_{22} = \sigma_{22}\,,\quad T_3 = T_{33} = \sigma_{33}\,,\quad T_4 = T_{23} = \sigma_{23} = \sigma_{32}$$

$$T_5 = T_{31} = \sigma_{31} = \sigma_{13}\,,\quad T_6 = T_{12} = \sigma_{12} = \sigma_{31}\,.$$

The constitutive relations (2) and (3) can be written in the matrix form

$$\begin{bmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} S_1\\ S_2\\ S_3\\ S_4\\ S_5\\ S_6 \end{bmatrix} - \begin{bmatrix} e_{11} & e_{21} & e_{31}\\ e_{12} & e_{22} & e_{32}\\ e_{13} & e_{23} & e_{33}\\ e_{14} & e_{24} & e_{34}\\ e_{15} & e_{25} & e_{35}\\ e_{16} & e_{26} & e_{36} \end{bmatrix} \begin{bmatrix} E_1\\ E_2\\ E_3 \end{bmatrix} \tag{5}$$

and

$$\begin{bmatrix} D_1\\ D_2\\ D_3 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16}\\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26}\\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{bmatrix} S_1\\ S_2\\ S_3\\ S_4\\ S_5\\ S_6 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13}\\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23}\\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_1\\ E_2\\ E_3 \end{bmatrix} \tag{6}$$

The first relations of the group (5) and (6) are the following :

$$\begin{aligned} T_1 &= \sigma_{11} = c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + c_{15}(u_{3,1} + u_{1,3}) +\\ &+ c_{16}(u_{1,2} + u_{2,1}) + e_{11}\varphi_{,1} + e_{21}\varphi_{,2} + e_{31}\varphi_{,3}\,,\qquad E_k = -\varphi_{,k}\\ T_2 &= \;\ldots\,. \end{aligned} \tag{7}$$

and

$$\begin{aligned} D_1 &= e_{11}u_{1,1} + e_{12}u_{2,2} + e_{13}u_{3,3} + e_{14}(u_{2,3} + u_{3,2}) + e_{15}(u_{3,1} + u_{1,3}) +\\ &+ e_{16}(u_{1,2} + u_{2,1}) - \epsilon_{11}\varphi_{,1} - \epsilon_{12}\varphi_{,2} - \epsilon_{13}\varphi_{,3}\,,\\ D_2 &= \;\ldots\,. \end{aligned} \tag{8}$$

In the case of the most general anisotropic material

$$c_{pq} = c_{qp} , \qquad \epsilon_{ik} = \epsilon_{ki} .$$

Thus, we have 21 elastic constants c_{pq}, 6 dielectric constants ϵ_{ik} and 18 piezoelectric constants e_{kp}. Altogether there are 45 independent material constants. Their number becomes smaller when the crystal has a symmetry axis of n-th order. In the case of monoclinic crystal with x_1 as the diagonal axis we have the following matrices in the constitutive equations:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0 \\ c_{14} & c_{24} & c_{34} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} - \begin{bmatrix} e_{11} & 0 & 0 \\ e_{12} & 0 & 0 \\ e_{13} & 0 & 0 \\ e_{14} & 0 & 0 \\ 0 & e_{25} & e_{35} \\ 0 & e_{26} & e_{36} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{25} & e_{26} \\ 0 & 0 & 0 & 0 & e_{35} & e_{36} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & \epsilon_{23} \\ 0 & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{10}$$

Without going into details we refer the reader to the known monograph by J.F. Nye[1] and present now only the constitutive relations for two widely used piezoelectric materials,

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(1) J.F. Nye. Physical properties of crystals. Clarendon Press, Oxford, 1960.

namely for hexagonal crystals (6mm) and (622).

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{11}$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad c_{66} = \frac{1}{2}(c_{11} - c_{12}) \tag{12}$$

This system of constants (five elastic constants c_{pq} , three piezoelectric constants and two dielectric constants e_{kp} , altogether 10 independent constants) is characteristic for polarized ceramic ferroelectrics, i.e. materials with a strong piezoelectric coupling. The constitutive relations for crystals of class (622) have the form

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{14} & 0 & 0 \\ 0 & -e_{14} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad c_{66} = \frac{1}{2}(c_{11} - c_{12}) \tag{13}$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \qquad (14)$$

Here, too, we have five elastic constants c_{pq} but only one piezoelectric constant e_{ip} and two dielectric constants ϵ_{ij}. Altogether there are 8 independent material constants.

Let us now introduce the constitutive relations (11) and (12) for material of class (6mm) into the piezoelectricity equations

$$T_{ji,j} = \varrho \ddot{u}_{ij} \qquad D_{i,i} = 0 .$$

Then we arrive at the system of four equations

$$c_{66} \nabla_1^2 u_1 + (c_{66} + c_{12})(u_{1,11} + u_{2,12}) + c_{44} u_{1,33} + \\ + (c_{13} + c_{44}) u_{3,13} + (e_{31} + e_{15}) \varphi_{,13} = \varrho \ddot{u}_1 , \qquad (15)$$

$$c_{66} \nabla_1^2 u_2 + (c_{66} + c_{12})(u_{1,21} + u_{2,22}) + c_{44} u_{2,33} + \\ + (c_{13} + c_{44}) u_{2,23} + (e_{15} + e_{31}) \varphi_{,23} = \varrho \ddot{u}_2 , \qquad (16)$$

$$c_{44} \nabla_1^2 u_3 + c_{33} u_{3,33} + (c_{13} + c_{44})(u_{1,31} + u_{2,32}) + \\ + e_{15} \nabla_1^2 \varphi + e_{33} \varphi_{,33} = \varrho \ddot{u}_3 , \qquad (17)$$

$$e_{15}(u_{3,11} + u_{3,22} + u_{1,31} + u_{2,32}) + e_{31}(u_{1,13} + u_{2,23}) + \\ + e_{33} u_{3,33} - (\epsilon_{11} \nabla_1^2 \varphi + \epsilon_{33} \varphi_{,33}) = 0 . \qquad (18)$$

where

$$\nabla_1^2 = \partial_1^2 + \partial_2^2 \,, \qquad c_{66} = \frac{1}{2}\,(c_{11} - c_{12}) \;.$$

Consider a plane wave in a monoclinic medium. Assuming that the displacement and the electric potential depend only on the variables x_2 and t we obtain the following four equations

$$c_{66} u_{1,22} + e_{26}\varphi_{,22} = \varrho \ddot{u}_1 \tag{19}$$

$$c_{22} u_{2,22} + c_{24} u_{3,22} = \varrho \ddot{u}_2 \tag{20}$$

$$c_{24} u_{2,22} + c_{44} u_{3,22} = \varrho \ddot{u}_3 \tag{21}$$

$$e_{26} u_{1,22} - \epsilon_{22}\varphi_{,22} = 0 \,. \tag{22}$$

It is evident that only the displacement u_1 and the potential φ are coupled. In what follows we shall concentrate on Eqs. (19) and (22) assuming that $u_2 = u_3 = 0$. Consider first a plane wave moving in the direction x_2 with a constant velocity c. Setting in (19) and (22)

$$u_1 = U_0 e^{ik(x_2 - ct)} \,, \quad \varphi = \Phi_0 e^{ik(x_2 - ct)} \tag{23}$$

we obtain the equation

$$U_0(k^2 \bar{c}_{66} - \varrho\omega^2) = 0 \qquad \bar{c}_{66} = c_{66} + \frac{e_{26}^2}{\epsilon_{22}} \tag{24}$$

whence

$$c = \left(\frac{\bar{c}_{66}}{\varrho}\right)^{1/2}, \quad c = \frac{\omega}{k} \,. \tag{25}$$

Consider now a layer of thickness 2h, performing foced vibrations due to a potential applied at the boundaries

$$\varphi = \pm \varphi_0 e^{-i\omega t} \qquad \text{for} \qquad x_2 = \pm h \tag{26}$$

We assume that the boundary $x_2 \pm h$ is free of tractions. Therefore

$$\sigma_{21} = T_{21} = (c_{66}u_1 + e_{26}\varphi)_{,2} = 0 \,. \quad \text{for} \quad x_2 = \pm h \tag{27}$$

Setting in Eqs. (19) and (22)

$$u_1 = U_1(x_2)\, e^{-i\omega t}, \qquad \varphi = \Phi(x_2)\, e^{-i\omega t}, \tag{28}$$

we have

$$(\partial_2^2 + \eta^2)\, U_1 = 0 \,. \qquad \eta^2 = \frac{\varrho\omega^2}{\bar{c}_{66}} \tag{29}$$

and

$$\partial_2^2 \left(U_1 - \frac{\epsilon_{22}}{e_{26}}\Phi\right) = 0 \,. \tag{30}$$

The solution of Eqs. (29) and (30) has the form

$$\begin{aligned} U_1 &= A\cos\eta x_2 + B\sin\eta x_2 \,, \\ \Phi &= \frac{e_{26}}{\epsilon_{22}} U_1 + C + x_2 D \,. \end{aligned} \tag{31}$$

The boundary conditions (26) imply that the function $\Phi(x_2)$ is antisymmetric; hence $A = C = 0$ The boundary condition (26) for $x_2 = h$ yields the relation

$$\varphi_0 = \frac{e_{26}}{\epsilon_{22}} B \sin\eta h + Dh \,. \tag{32}$$

In view of the boundary condition (27) we obtain the equation

$$\frac{\bar{c}_{66}}{e_{26}}\eta B\cos\eta h + D = 0 \,. \tag{33}$$

Eliminating the constant D from Eqs. (32) and (33) we arrive at the relation

$$B\left(\bar{c}_{66}\lambda\cos\lambda - \frac{e_{26}^2}{\epsilon_{22}}\sin\lambda\right) = -\varphi_0 e_{26}\,, \qquad \lambda = \eta h \,. \tag{34}$$

(32) serves for the determination of the constant D and this completes the solution. In the case of resonance

$$\operatorname{tg}\lambda = \frac{\bar{c}_{66}\epsilon_{22}}{e_{26}^2} \tag{35}$$

Some problems concerning free and forced vibrations were solved by H.F. Tiersten[(1), (2)]. The propagation of Rayleigh's surface waves was investigated in the papers[(3) - (6)].

1.10 R.A. Toupin's piezoelectricity equations

In Toupin' theory of dielectrics[(7)] the independent variables are the strain ε_{ij}, the components of the vector of dielectric polarization P_i and the electric field $E_i = -\varphi_{,i}$. The displacement or induction vector D_i is related to the polarization vector P_i and the electric field E_i as follows :

$$P_i = D_i - \epsilon_0 E_i \ . \qquad (1)$$

Here ϵ_0 is the permeability in vacuum.

R.D. Mindlin [(8)] proved that Toupin's system of equations is equivalent to the systems of equations presented in Sec. 1.4. We shall now present Mindlin's considerations.

Let us decompose the internal energy U of the dielectric into the energy related to the deformation of the body and the polarization U^L and the energy related to the electric field

$$U = U^L(\varepsilon_{ij}, P_i) + \frac{1}{2}\epsilon_0 \varphi_{,i}\varphi_{,i} \ . \qquad (2)$$

Introducing the electric entalpy

$$H = U - E_i D_i \qquad (3)$$

we obtain from (2)

$$H = U^L\left(\varepsilon_{ij}, P_i\right) - \frac{1}{2}\epsilon_0 \varphi_{,i}\varphi_{,i} + \varphi_{,i} P_i \ . \qquad (4)$$

(1) H.F. Tiersten. Linear piezoelectric plate vibrations. Plenum Press, New York, 1969.
(2) H.F. Tiersten. J. Acoust. Soc. of America. 35, (1963). 234.
(3) J.L. Bleustein. J Acoust. Soc. of America. 45, (1969), 614.
(4) D.S. Drumheller, A. Kalnis. J. Acoust. Soc. of America. 47. (1970). 1343.
(5) J.L. Bleustein. Applied physics Letters, 13. (1968), 412.
(6) P.M. Drenkow, C.F. Long. Acta Mechanica. 3. (1966), 13.
(7) R.A. Toupin. The elastic dielectric. J. Rat. Mech. Anal. 5, (1956). 849.
(8) R.D. Mindlin. Polarization gradient in elastic dielectrics. Int. J. Solids. Structures. 4. (1968). 637.

Consider a body B of volume v bounded by the surface ∂B separating the body from the vacuum B'. Toupin's form of the Hamilton principle is the following :

$$\delta \int_{t_1}^{t_2} dt \int_{B^*} (K - H)\, dv + \int_{t_1}^{t_2} dt \left[\int_B (X_i \delta u_i + E_i^0 \delta P_i)\, dv + \int_{\partial B} p_i \delta u_i\, da \right] = 0 . \quad (5)$$

Here $B^* = B + B'$ and E_i^0 is the external electric field ;
$K = \frac{1}{2} \varrho v_i v_i$.

Observe that

$$\frac{1}{2} \delta \int_{t_1}^{t_2} \varrho \dot{u}_i \dot{u}_i\, dt = - \varrho \int_{t_1}^{t_2} \ddot{u}_i \delta u_i\, dt \text{ (see (4) and (5) of sec. 1.6)} \quad (6)$$

and

$$\delta H = \frac{\partial U^L}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} + \frac{\partial U^L}{\partial P_i} \delta P_i - \epsilon_0 \varphi_{,i} \delta \varphi_{,i} + \varphi_{,i} \delta P_i + \delta \varphi_{,i} P_i . \quad (7)$$

We define the stress σ_{ij} and the effective local electric force E_i^L by the formulae

$$\sigma_{ij} = \frac{\partial U^L}{\partial \varepsilon_{ij}} , \qquad E_i^L = - \frac{\partial U^L}{\partial P_i} . \quad (8)$$

Then

$$\delta H = \sigma_{ij} \delta \varepsilon_{ij} + (\varphi_{,i} - E_i^L) \delta P_i - \varepsilon_0 \varphi_{,i} \delta \varphi_{,i} + P_i \delta \varphi_{,i} . \quad (7')$$

Consequently

$$\sigma_{ij} \delta \varepsilon_{ij} = (\sigma_{ij} \delta u)_{,j} - \sigma_{ij,j} \delta u_i ,$$
$$\varphi_{,i} \delta \varphi_{,i} = (\varphi_{,i} \delta \varphi)_{,i} - \varphi_{,i,i} \delta \varphi ,$$
$$P_i \delta \varphi_{,i} = (P_i \delta \varphi)_{,i} - P_{i,i} \delta \varphi ,$$

Introducing (7′) into (5), after simple transformations we arrive at the equation

$$\int_{t_1}^{t_2} dt \int_B \left[(\sigma_{ji,j} + X_i - \varrho \ddot{u}_i) \delta u_i + (E_i^L - \varphi_{,i} + E_i^0) \delta P_i + (-\epsilon_0 \varphi_{,ii} + P_{i,i}) \delta \varphi \right] dv +$$

$$- \int_{t_1}^{t_2} dt \int_B \epsilon_0 \varphi_{,ii} \delta \varphi\, dv + \int_{t_1}^{t_2} dt \int_{\partial B} \left[(p_i - \sigma_{ji} n_j) \delta u_i + (\epsilon_0 | \varphi_{,i} | - P_i) n \delta \varphi \right] da = 0 . \quad (9)$$

where $|\varphi_{,i}|$ is the jump of the function $\varphi_{,i} = - E_i$ on the surface ∂B. In view of the arbitrariness of the vibrations $\delta \varphi$, δP_i we obtain Euler's equations

$$\left.\begin{array}{l} \sigma_{ji,j} + X_i = \varrho \ddot{u}_i \, , \\ E_i^L - \varphi_{,i} + E_i^0 = 0 \, , \\ -\epsilon_0 \varphi_{,ii} + P_{i,i} = 0 \, . \end{array}\right\} \text{on } B \, , \quad \begin{array}{r} (10) \\ (11) \\ (12) \end{array}$$

and

$$-\epsilon_0 \varphi_{,ii} = 0 \qquad \text{on } B' . \tag{13}$$

These equations are completed by the natural boundary conditions

$$\left.\begin{array}{l} \sigma_{ji} n_j = p_i \, , \\ (-\epsilon_0 |\varphi_{,i}| + P_i) n_i = 0 \, , \end{array}\right\} \text{on } \partial B \, . \quad \begin{array}{r} (14) \\ (15) \end{array}$$

Thus, we have derived the equation of motion (10) and Eq. (12), identical with the equations $D_{i,i} = 0$.

Eq. (11) did not appear before. It constitutes the balance of intermolecular forces, deduced by R.A. Toupin on the basis of considerations concerning the equilibrium o of electric forces. This eauqtion is not connected with any boundary conditions.

Eqs. (10) – (12) and the natural boundary conditions (14)-(15) constitute the linear form of the equations for elastic dielectrics, given by R.A. Toupin.

Assume not the energy $\bar{U}^L = U^L(\varepsilon_{ij}, P_i)$ in the form

$$U^L = \frac{1}{2} \overset{P}{c}_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} \overset{\varepsilon}{a}_{ij} P_i P_j + f_{kij} \varepsilon_{ij} P_k . \tag{16}$$

The relations (8) imply the constitutive equations

$$\sigma_{ij} = \overset{P}{c}_{ijkl} \varepsilon_{kl} + f_{kij} P_k \, , \tag{17}$$

$$-E_i^L = f_{ikl} \varepsilon_{kl} + \overset{\varepsilon}{a}_{ik} P_k \, . \tag{18}$$

R.D. Mindlin[1] derived relations between the constants c_{ijkl}, e_{kij}, ϵ_{ij} and the constants

(1) R.D. Mindlin. Elasticity, piezoelectricity and crystal lattice dynamics. J. of Elasticity, 2, 4, (1972), 217.

c^P_{ijkl} , a^ε_{ij} , f_{kij} and proved that in view of the constitutive relations (17) – (18), Eqs. (10) – (12) lead to the equations of classical piezoelectricity (6) – (7) of Sec. 1.4.

Let us now present a generalization of Toupin's piezelectricity equations, carried out by R.D. Mindlin[(2)]. This generalization consists in taking into account in the electric entalpy, the gradient of the polarization vector

$$H = U^L(\varepsilon_{ij}, P_i, P_{j,i}) - \frac{1}{2}\epsilon_0 \varphi_{,i}\varphi_{,i} + \varphi_{,i}P_i . \tag{19}$$

Introducing the new definition

$$E_{ij} = \frac{\partial U^L}{\partial P_{j,i}} , \tag{20}$$

we represent Hamilton's principle in the form

$$\int_{t_1}^{t_2} dt \int_B \left[(\sigma_{ji,j} + X_i - \varrho \ddot{u}_i)\delta u_i + (E_{ji,j} + E^L_i - \varphi_{,i} + E^0_i)\delta P_i + (-\epsilon_0 \varphi_{,ii} + P_{i,i})\delta\varphi \right] dv +$$
$$- \int_{t_1}^{t_2} dt \int_{B'} \epsilon_0 \varphi_{,ii}\delta\varphi \, dv + \int_{t_1}^{t_2} dt \int_{\partial B} \left[(p_i - \sigma_{ji}n_j)\delta u_i - E_{ji}n_j \delta P_i + n_i(\epsilon_0|\varphi_{,i}| - P_i)\delta\varphi \right] da = 0 \tag{21}$$

In view of the arbitrariness of the virtual increments we obtain Euler's equations

$$\left.\begin{aligned} &\sigma_{ji,j} + X_i - \varrho \ddot{u}_i = 0 , &(22)\\ &E_{ji,j} + E^L_i - \varphi_{,i} + E^0_i = 0, &(23)\\ &-\epsilon_0 \varphi_{,ii} + P_{i,i} = 0 &(24) \end{aligned}\right\} \text{ on } \partial B.$$

and

$$\varphi_{,ii} = 0 \qquad \text{on } B' . \tag{25}$$

(2) R.D. Mindlin, see p. 29.

These equations should be completed by the natural boundary conditions following from Eq. (21)

$$\sigma_{ji} n_j = p_i \ , \tag{26}$$

$$E_{ji} n_j = 0 \ , \tag{27}$$

$$(-\epsilon_0 |\varphi_{,i}| + P_i) n_i = 0. \tag{28}$$

The surface integral (21) implies the admissible boundary conditions. Besides the condition (26) for tractions we may assume the displacement condition (where $\delta u_i = 0$). Similarly, besides $E_{ji} n_j = 0$ we may take a condition for the polarization P_i . Finally, besides the condition (28) prescribing the charge on the surface we may prescribe the potential φ .

Let us take the energy $U^L(\varepsilon_{ij}, P_i, P_{i,j})$ in the form

$$U^L = \overset{0}{b}_{ij} P_{j,l} + \frac{1}{2} \overset{\varepsilon G}{a}_{ij} P_i P_j + \frac{1}{2} \overset{eP}{b}_{ijkl} P_{j,i} P_{l,k} + \frac{1}{2} \overset{P,G}{c}_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \\ + \overset{P}{d}_{ijkl} P_{j,i} \varepsilon_{kl} + \overset{G}{f}_{ijk} P_i \varepsilon_{jk} + \overset{\varepsilon}{\dot{j}}_{ijk} P_i P_{k,j} . \tag{29}$$

Here the indices P, G. ε denote a fixed polarization, the polarization gradient and the strain. In view of the relations

$$\sigma_{ij} = \frac{\partial U^L}{\partial \varepsilon_{ij}} \ , \quad E_i^L = -\frac{\partial U^L}{\partial P_i} \ , \quad E_{ij} = \frac{\partial U^L}{\partial P_{j,i}} \ , \tag{30}$$

we obtain the constitutive equations

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl} + f_{kij} P_k + d_{klij} P_{l,k} \ , \tag{31}$$

$$-E_j^L = f_{jkl} \varepsilon_{kl} + a_{jk} P_k + j_{jkl} P_{l,k} \ , \tag{32}$$

$$E_{ij} = d_{ijkl} \varepsilon_{kl} + j_{kij} P_k + b_{ijkl} P_{l,k} + \overset{0}{b}_{ij} \ . \tag{33}$$

Introducing the relations (31) –(33) into Eqs. (22) – (24) we arrive at a system of seven differential equations with the following unknowns : the polarization P_i the displacement u_i and the electric potential φ . Observe that introduction of the polarization gradient does not rise the order of the differential equations. It is also noteworthy that the electromechanical coupling appears also in a body with central symmetry. Although in this particular case $f_{ijk} = j_{ijk} = 0$ (since odd tensors do not appear in bodies with central symmetry), the constants d_{ijkl} do not vanish. It follows from (31) and (32) that these constants play the role of couplings between the mechanical and electric fields.

1.11. Thermopiezoelectricity. Fundamental relations and differential equations[1]

In the preceding considerations we assumed that the process in adiabatic. Now we discard this restriction. Thus, there flows through surface elements heat represented by its flux $\underline{q}$ referred to a unit area and unit time. In the interior of the body there act heat sources W referred to a unit volume of the body and unit time. Consequently, there arises in the body a temperature increment θ equal to the temperature difference $\theta = T - T_0$ where T is the absolute temperature and T_0 the temperature of the natural state in which there are no strains or stresses.

We shall deal with the energy balance taking into account the thermal terms

$$\frac{d}{dt}\int_B \left(\frac{1}{2}\varrho v_i v_i + U\right) dv = \int_B (X_i v_i + E_i \dot{D}_i + W) dv + \int_{\partial B} (p_i v_i - q_i n_i) da , \qquad (1)$$

and the Clausius-Duhen inequality

$$\dot{S} + \left(\frac{q_i}{T}\right)_{,i} - \frac{W}{T} \geqslant 0 . \qquad (2)$$

The energy balance contains the non-mechanical power, the flux of heat through the surface of the body and the energy generated by heat in the interior of the body.

In the inequality (2) S is the entropy referred to unit volume.

The contact forces in (1) can be expressed in terms of stresses $(p_i = \sigma_{ji} n_j)$; transforming then surface integrals into volume integrals we arrive at the local form of the

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(1) R.D. Mindlin. On the equations of motion of piezoelectric crystals. Problems of Continuum Mechanics. SIAM, Philadelphia, Pennsylvania, 1961.

energy balance

$$\dot{U} = \sigma_{ij}\dot{\varepsilon}_{ij} + E_i\dot{D}_i - q_{i,i} + W \ . \tag{3}$$

We have made use here of the equation of motion

$$\sigma_{ji,j} + X_i - \varrho\ddot{u}_i = 0 \ . \tag{4}$$

In what follows we introduce the free energy F and the electric entalpy H,

$$F = U - ST, \qquad H = F - E_i D_i \ . \tag{5}$$

Making use of these definitions we arrive at the new form of the energy balance

$$\dot{H} = \sigma_{ij}\dot{\varepsilon}_{ij} - D_i\dot{E}_i - S\dot{T} - \dot{S}T - q_{i,i} + W \ . \tag{6}$$

Let us eliminate from the inequality (2) and Eq. (6) the heat sources. Then we obtain the inequality

$$-(\dot{H} + S\dot{T}) + \sigma_{ij}\dot{\varepsilon}_{ij} - D_i\dot{E}_i - \frac{q_i T_{,i}}{T} \geqslant 0 . \tag{7}$$

Assume now that $H = H(\varepsilon_{ij}, E_i, T, T_{,i})$. Then

$$\dot{H} = \frac{\partial H}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial H}{\partial E_i}\dot{E}_i + \frac{\partial H}{\partial T}\dot{T} + \frac{\partial H}{\partial T_{,i}}\dot{T}_{,i} \quad , \tag{8}$$

From (7) and (8) we have

$$\left(\sigma_{ij} - \frac{\partial H}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} - \left(D_i + \frac{\partial H}{\partial E_i}\right)\dot{E}_i - \left(\frac{\partial H}{\partial T} + S\right)\dot{T} - \\ - \frac{\partial H}{\partial T_{,i}}\dot{T}_{,i} - \frac{q_i T_{,i}}{T} \geqslant 0 \ . \tag{9}$$

This inequality should be satisfied for all variations of the variables $\dot{E}_i, \dot{T}, \dot{T}_{,i}$. Consequently the coefficients of these variables must vanish :

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}} , \qquad D_i = -\frac{\partial H}{\partial E_i} , \qquad S = -\frac{\partial H}{\partial T} , \qquad \frac{\partial H}{\partial T_{,i}} = 0 \tag{10}$$

Thus. the entalpy H is independent of the temperature gradinet. The remaining inequality has the form

$$-\frac{q_i T_{,i}}{T} \geqslant 0 \quad , \tag{11}$$

which is satisfied by assuming that

$$q_i = -k_{ij} T_{,j} \quad , \tag{12}$$

i.e. the Fourier law for anisotropic bodies. The quantity $\Omega = -q_i T_{,i} \geq 0$ should be a positive definite quadratic form,

$$\Omega = k_{ij} T_{,i} T_{,j} > 0 \quad . \tag{13}$$

This inequality (in view of Silvester's theorem!) leads to restrictions on the symmetric coefficients of heat conductivity k_{ij} .

Expanding the entalpy H into the Taylor series in the vicinity of the natural state

$$H = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - e_{kij} \varepsilon_{ij} E_k + \frac{1}{2} \epsilon_{ij} E_i E_j - \\ - \gamma_{ij} \varepsilon_{ij} \theta - g_i E_i \theta - \frac{c_\varepsilon}{2T_0} \theta^2 \quad , \tag{14}$$

and making use of the relations (10) we arrive at the constitutive equations

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl} - \gamma_{ij} \theta - e_{kij} E_k \quad , \tag{15}$$

$$S = \gamma_{ij} \varepsilon_{ij} + \frac{c_\varepsilon}{T_0} \theta + g_i E_i \quad , \tag{16}$$

$$D_i = e_{ikl} \varepsilon_{kl} + g_i \theta + \epsilon_{ik} E_k \quad . \tag{17}$$

Eq. (15) is the Duhamel–Neumann equations generalized to piezoelectricity, the second is an expression for the entropy in terms of the variables ε_{ij} , θ , E_k and the last is an expression for the electric displacement.

Observe that (14) – (17) lead to the constitutive relations

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} \, , \qquad \frac{\partial D_i}{\partial E_j} = \frac{\partial D_j}{\partial E_i} \quad , \tag{18}$$

and

$$\frac{\partial \sigma_{ij}}{\partial E_k} = -\frac{\partial D_k}{\partial E_j} \, , \quad \frac{\partial \sigma_{ij}}{\partial T} = -\frac{\partial S}{\partial \varepsilon_{ij}} \, , \quad \frac{\partial S}{\partial E_i} = \frac{\partial D_i}{\partial T} \quad . \tag{19}$$

The relations (18) imply that

$$\kappa_{ijkl} = \kappa_{klij} , \qquad \epsilon_{ij} = \epsilon_{ji} \tag{20}$$

The symmetry of the tensors σ_{ij} , ε_{ij} leads (15)) to the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{ijlk} , \qquad \gamma_{ij} = \gamma_{ji} , \qquad e_{kij} = e_{kji} , \tag{21}$$

In the case of general anisotropy we have 21 constants c_{ijkl} , 18 piezoelectric constants e_{kij} , six constants ϵ_{ij} and γ_{ij} and three constants g_i . There also appear the constant c_ε , which has the meaning of the specific heat at constant strain ε_{ij} and constant E_i . Altogether there are 55 material constants. Observe that meaning of the coefficients c_{ijkl}, e_{kij} , . . . is now different than that in Sec. 1.3. The latter referred to the adiabatic state while the new coefficients are measured in the isothermal state.

In the adiabatic state $q = 0, W=0$. The entropy balance

$$T\dot{S} = -q_{i,i} + W , \tag{22}$$

implies that in the adiabatic state $\dot{S} = 0$. Eq. (16) for the adiabatic case leads to the relation

$$\theta = -\frac{T_0}{c_\varepsilon}\left(\gamma_{ij}\varepsilon_{ij} + g_i E_i \right) . \tag{23}$$

This relation may serve to eliminate the temperature increment from the constitutive equations (15) – (17). Thus we obtain the constitutive relations for the adiabatic state

$$\sigma_{ij} = \check{c}_{ijkl}\varepsilon_{kl} - \check{e}_{kij}E_k , \tag{24}$$

$$D_i = \check{e}_{ikl}\varepsilon_{kl} + \check{\epsilon}_{ik}E_k \tag{25}$$

where

$$\check{c}_{ijkl} = c_{ijkl} + \frac{T_0}{c_\varepsilon}\gamma_{ij}\gamma_{kl} , \qquad \check{e}_{kij} = e_{kij} - \frac{T_0}{c_\varepsilon}\gamma_{ij}g_k , \qquad \check{\epsilon}_{ik} = \epsilon_{ik} - \frac{T_0}{c_\varepsilon}g_i g_k$$

In the isothermal state $\theta = 0$ or $T = T_0$. In this case the constitutive equation has the form

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k , \tag{26}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k , \tag{27}$$

valid for stationary state.

Introduce now the constitutive relations (15) – (17) into the equations of motion and the equation for the electric field

$$\sigma_{ji,j} + X_i = \varrho \ddot{u}_i , \qquad D_{i,i} = 0 . \tag{28}$$

Taking into account that $E_k = -\varphi_{,k}$ we obtain

$$c_{ijkl}u_{k,lj} + e_{kij}\varphi_{,kj} - \gamma_{ij}\theta_{,j} = \varrho \ddot{u}_i , \tag{29}$$

$$e_{kil}u_{k,li} - \epsilon_{ik}\varphi_{,ki} + g_i\theta_{,i} = 0 . \tag{30}$$

These equations should be completed by the equation of heat conduction. It is derived on the basis of the entropy balance (22) taking into account the constitutive relation (16) and the Fourier law (12). Thus we have

$$T(\gamma_{ij}\dot{\varepsilon}_{ij} + \frac{c_\varepsilon}{T_0}\dot{\theta} + g_i\dot{E}_i) = k_{ij}T_{,ij} + W . \tag{31}$$

Bearing in mind that

$$T = T_0(1 + \frac{\theta}{T_0}) ,$$

and assuming that $|\theta/T_0| \ll 1$ we arrive at the linear heat conduction equation in the form

$$k_{ij}\theta_{,ij} - c_\varepsilon\theta - T_0(\gamma_{ij}\varepsilon_{ij} - g_i\varphi_{,i}) = -W . \tag{32}$$

Eqs. (29), (30) and (32) constitute the complete set of equations of thermopiezoelectricity. The considered set of eauqtions is coupled. In the case of a stationary problem Eq. (29) becomes the Poisson equation

$$k_{ij}\theta_{,ij} = -W , \tag{33}$$

while Eqs. (29) and (30) are still coupled. The function θ appearing here is already known from Eq. (33).

1.12. General theorems of thermopiezoelectricity

To prove the uniqueness of the solution of the differential equations of thermopiezoelectricity we need a modified energy balance. It follows from the principle of virtual work

$$\int_B (X_i - \rho \ddot{u}_i)\,\delta u_i\,dv + \int_{\partial B} p_i\,\delta u_i\,da = \int_B \sigma_{ij}\,\delta\varepsilon_{ij}\,dv \;, \tag{1}$$

in which the virtual increments have been replaced by the real increments

$$\delta u_i = v_i\,dt\,, \qquad \delta\varepsilon_{ij} = \dot{\varepsilon}_{ij}\,dt\;, \;\ldots \qquad \text{etc.}$$

Thus, we obtain the fundamental energy equation

$$\int_B (X_i - \rho\dot{v}_i)\,v_i\,dv + \int_{\partial B} p_i v_i\,da = \int_B \sigma_{ij}\,\dot{\varepsilon}_{ij}\,dv \;, \tag{2}$$

into which we introduce the constitutive relations

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{kij}E_k - \gamma_{ij}\theta \;, \tag{3}$$

Hence

$$\frac{d}{dt}(\mathcal{K}+\mathcal{W}) = \int_B X_i v_i\,dv + \int_{\partial B} p_i v_i\,da + \int_B (\gamma_{ij}\dot{\varepsilon}_{ij}\theta + e_{kij}\dot{\varepsilon}_{ij}E_k)\,dv \;, \tag{4}$$

Where $\mathcal{K}$ is the kinetic energy and $\mathcal{W}$ the work of deformation (see the definition in (Sec. 1.5. $\mathcal{W} = 1/2\int_B c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\,dv$) .

To eliminate the term $\int_B \gamma_{ij}\dot{\varepsilon}_{ij}\theta\,dv$ we consider the heat conduction equation (setting $W = 0$)

$$\frac{1}{T_0}\,(k_{ij}\theta_{,ij} - c_\varepsilon\dot{\theta}) = \gamma_{ij}\,\dot{\varepsilon}_{ij} + g_k\,\dot{E}_k \;. \tag{5}$$

Multiplying it by θ and integrating over the region of the body, after simple transformations we obtain

$$\int_B \gamma_{ij}\dot{\varepsilon}_{ij}\theta\,dv = \frac{k_{ij}}{T_0}\int_{\partial B}\theta\,\theta_{,j}n_i\,da - \frac{d\mathcal{P}}{dt}\chi_\theta - g_k\int_B \theta\,\dot{E}_k\,dv \;, \tag{6}$$

where

$$\mathcal{P} = \frac{c_\varepsilon}{2T_0}\int_B \theta^2 dv\,, \qquad \chi_\theta = \frac{k_{ij}}{T_0}\int_B \theta_{,i}\theta_{,j}dv\,.$$

Substituting form (6) into (4) we are led to the equation

$$\frac{d}{dt}(\mathcal{K}+\mathcal{W}+\mathcal{P}) + \chi_\theta = \int_B X_i v_i dv + \int_{\partial B} p_i v_i da + \\ + \frac{k_{ij}}{T_0}\int_{\partial B}\theta\,\theta_{,j}n_i da + \int_B (e_{kij}\dot{\varepsilon}_{ij}E_k - g_k\dot{E}_k\theta)\,dv\,. \tag{7}$$

To eliminate the term $e_{kij}\dot{\varepsilon}_{ij}E_k$ from the last integral of Eq. (7) we make use of the constitutive relation

$$D_k = e_{kij}\varepsilon_{ij} + g_k\theta + \epsilon_{kj}E_j\,. \tag{8}$$

After simple transformations we obtain

$$\int_B (e_{kij}E_k\dot{\varepsilon}_{ij} - g_k\theta\dot{E}_k)\,dv = -\int_B \dot{D}_k\varphi_{,k}dv - \\ - \frac{dE}{dt} - \frac{d}{dt}(g_k\int_B \theta E_k dv)\,. \tag{9}$$

where we have introduced the notation $E = 1/2\,\epsilon_{ij}\int_B E_j E_k dv$.
In view of (9), bearing in mind that $D_{i,i} = 0$ we arrive at the modified energy balance

$$\frac{d}{dt}(\mathcal{K}+\mathcal{W}+E+\mathcal{P}+g_k\int_B \theta E_k dv) + \chi_\theta = \int_B X_i v_i dv + \\ + \int_{\partial B} p_i v_i da + \frac{k_{ij}}{T_0}\int_{\partial B}\theta_{,j}\theta n_i da - \int_{\partial B}\dot{D}_i\varphi n_i da\,. \tag{10}$$

The energy blance (10) makes it possible to carry out the proof of uniqueness of the solution. As in Sec. 1.5. we assume that there are two distinct solution $(u_i', \varphi', \theta')$ and $(u'', \varphi'', \theta'')$, we construct their difference $\hat{u}_i = u_i' - u_i'', \ldots$ and proceed as before. The difference (denoted by "roof") satisfies the homogeneous thermopiezoelectricity equations with homogeneous boundary and initial conditions. In view of the homogeneity of the equations and the boundary conditions, the right-hand side of Eq. (10) vanishes.

Hence

$$\frac{d}{dt}(\hat{\mathscr{K}}+\hat{\mathscr{W}}+\hat{\mathscr{P}}+\hat{\mathscr{E}}+g_k\int_B \hat{\theta}\hat{E}_k dv) = -\hat{\chi}_\theta \leq 0 . \tag{11}$$

where we have made use of the fact that the integrand of the energy dissipation function $\hat{\chi}_\theta$ is a positive definite quadratic form. The integral in the left-hand side of Eq. (11) vanishes at the initial instant, for the functions $\hat{u}_i$, $\hat{\theta}$, $\hat{\varphi}$, $\hat{E}_k$ satisfy the homogeneous initial conditions. On the other hand, the inequality (11) proves that its left-hand side is either negative or zero. The second possibility occurs if the integrand is a sum of the squares. Consequently we assume that

$$\hat{\mathscr{K}} = 0, \quad \hat{\mathscr{W}} = 0, \quad \hat{\mathscr{P}}+\hat{\mathscr{E}}+g_k\int_B \hat{\theta}\hat{E}_k dv \geq 0 . \tag{12}$$

These results imply that

$$\hat{v}_i = 0, \quad \hat{\varepsilon}_{ij} = 0, \quad \hat{\theta} = 0, \quad \hat{E}_k = 0 . \tag{13}$$

We still have to deduce relations between the constants c_ε, p_i and $\in_{ij}$ ensuring that the inequality $(12)_3$[(1)] is true.

Eqs. (13") imply the uniqueness of the solutions of the thermopiezoelectricity equations, e.e.

$$u_i' = u_i'', \quad \varphi' = \varphi'', \quad \theta' = \theta'', \quad E_k' = E_k'' . \tag{14}$$

Moreover, it follows from the constitutive relations that

$$\sigma_{ij}' = \sigma_{ij}'', \quad D_i' = D_i'', \quad S' = S'' . \tag{15}$$

(1) J. Ignaczak deduced the following sufficient condition (private communication). Assume that $\in_{ij}$ is a known positive definite symmetric tensor, g_i a vector and $c = c_\varepsilon/T_0 > 0$, and consider the function

$$A(\theta, E_i) = c\theta^2 + 2\theta\, g_i E_i + \in_{ij} E_i E_j .$$

A is non negative $(A \geq 0)$ for every real pair (θ, E_i), provided

$$|g_i|^2 \leq c\lambda_m$$

where λ_m is the smallest positive eigenvalue of the tensor $\in_{ij}$.

Consider now the generalized Hamilton's principle. We define two functionals

$$\pi = \int_B (H + ST - X_i u_i)\,dv - \int_{\partial B} (p_i u_i - \sigma\varphi)\,da \ , \tag{16}$$

$$\psi = \int_B (\Gamma - S\dot{T} - WT)\,dv + \int_{\partial B} T Q_i n_i\,da \ , \tag{17}$$

Where H is the electric entalpy, φ the electric potential, σ the electric charge on ∂B and Γ the potential of the heat flow

$$\Gamma = \frac{1}{2} k_{ij} T_{,i} T_{,j} \ , \qquad q_i = \frac{\partial \Gamma}{\partial T_{,i}} = - k_{ij} T_{,j} \ , \tag{18}$$

The generalized Hamilton principle has the form

$$\delta \int_{t_1}^{t_2} (K - \pi)\,dt = 0 \ ; \qquad \delta \int_{t_1}^{t_2} \psi\,dt = 0 \ . \tag{19}$$

This form of Hamilton's principle was first stated for the problems of coupled thermoelasticity by H. Parkus[1] and for the adiabatic problem of piezoelectricity by H.T. Tiersten[2].

Returning to Eq. (11) we find that the following conditions must be satisfied

$$\begin{aligned} &\delta \underline{u}(\underline{x}, t_1) = \delta \underline{u}(\underline{x}, t_2) = 0 \ , \\ &\delta \theta(\underline{x}, t_1) = \delta \theta(\underline{x}, t_2) = 0 \ . \end{aligned} \tag{20}$$

The displacement u_i, the potential φ and the temperature θ are subject to variation. Performing the variations in accordance with Eq. $(19)_1$, making use of the constitutive relations (10) in Sec. 1.10. and bearing in mind that

$$\delta H = \frac{\partial H}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} + \frac{\partial H}{\partial T} \delta T + \frac{\partial H}{\partial E_i} \delta E_i \ ,$$

(1) H. Parkus. Uber die Erweiterung des Hamilton'schen Prinzipes auf thermoelastische Vorgänge. Federhofer-Girkman Festschrift, Wien, 1950, Verlag. F. Deuticke.

(2) H.F. Tiersten. loc. cit. p. 18.

we obtain the equation

$$\int_{t_1}^{t_2} dt \Big\{ \int_B \big[(\sigma_{ji,j} + X_i - \varrho \ddot{u}_i)\,\delta u_i + D_{i,i}\,\delta\varphi \big] dv + \int_{\partial B} \big[(\sigma_{ji} n_j - p_i)\,\delta u_i + (D_i n_i + \sigma)\,\delta\varphi \big] da \Big\} = 0 . \tag{21}$$

Since the variations δu_i, $\delta\varphi$ are arbitrary, we obtain from (21) the equations governing the motion and the electric field, completed by the appropriate boundary conditions. These equations and boundary conditions are identical with those presented in Sec. 1.6. (Eqs.(10)–(13)).

Performing the required variation in Eq. $(19)_2$

$$\delta \int_{t_1}^{t_2} \psi\, dt = \int_{t_1}^{t_2} dt \Big\{ \int_B \Big(\frac{\partial \Gamma}{\partial T_{,i}} \delta T_{,i} - \dot{S}\,\delta T - S\,\delta\dot{T} - W\delta T \Big) dv + \int_{\partial B} \theta_i n_i \,\delta T\, da \tag{22}$$

and taking into account

$$q_i = -\frac{\partial \Gamma}{\partial T_{,i}} , \qquad q_i \,\delta T_{,i} = -q_{i,i}\,\delta T + (q_i \,\delta T)_{,i}$$

we transform Eq. (22) to the form

$$\int_{t_1}^{t_2} dt \Big\{ \int_B \Big((q_{i,i} - W + \dot{S} T)\,\delta T - \dot{\overline{(S T \delta T)}} \Big) dv - \int_{\partial B} (q_i - \theta_i) n_i \,\delta T \Big\} = 0 . \tag{23}$$

In view of the assumption $(20)_{5,6}$ we have

$$\int_{t_1}^{t_2} \dot{\overline{(S T \delta T)}}\, dt = \Big| S T \delta T \Big|_{t_1}^{t_2} = 0 . \tag{24}$$

We still have the equation

$$\int_{t_1}^{t_2} dt \Big\{ \int_B (q_{i,i} - W + T\dot{S})\,\delta T\, dv + \int_{\partial B} (Q_i - q_i) n_i \,\delta T\, da \Big\} = 0 \tag{25}$$

valid for arbitrary variation δT satisfying the conditions (24).

Eq. (25) yields the entropy balance

$$T\dot{S} = -q_{i,i} + W , \qquad \underline{x} \in B . \tag{26}$$

and the boundary condition for the heat flow

$$q_i = Q_i\,, \qquad \underline{x} \in \partial B\,. \tag{27}$$

Consider now the theorem of reciprocity of work. It constitutes a generalization of the reciprocity theorem deduced in Sec. 1.7. As before, the point of departure is the set of equations of motion, transformed by Laplace. We have the identity (cf. Eq. (18) of Sec. 1.7.)).

$$\int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\,dv + \int_{\partial B} (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\,da = \int_B (\bar{\sigma}_{ij} \bar{\varepsilon}_{ij}' - \bar{\sigma}_{ij}' \bar{\varepsilon}_{ij})\,dv\,. \tag{28}$$

In view of the constitutive equations

$$\bar{\sigma}_{ij} = c_{ijkl} \bar{\varepsilon}_{kl} - \gamma_{ij} \bar{\theta} - e_{kij} \bar{E}_k\,, \qquad \bar{\sigma}_{ij}' = c_{ijkl} \bar{\varepsilon}_{kl}' - \gamma_{ij} \bar{\theta}' - e_{kij} \bar{E}_k' \tag{29}$$

we obtain

$$\int_B (\bar{X}_i \bar{u}_i' - \bar{X}_i' \bar{u}_i)\,dv + \int_{\partial B} (\bar{p}_i \bar{u}_i' - \bar{p}_i' \bar{u}_i)\,da + \\ + \int_B \left[\gamma_{ij} (\bar{\theta} \bar{\varepsilon}_{ij}' - \bar{\theta}' \varepsilon_{ij}) + e_{kij} (\bar{E}_k \bar{\varepsilon}_{ij}' - \bar{E}_k' \bar{\varepsilon}_{ij}) \right] dv = 0 \tag{30}$$

In what follows we shall make use of the heat conduction equation for both systems of loadings

$$\frac{1}{T_0} (k_{ij} \bar{\theta}_{,ij} - c_\varepsilon p \bar{\theta}) - p (\gamma_{ij} \bar{\varepsilon}_{ij} + g_i \bar{E}_i) = - \frac{\bar{W}}{T_0}\,, \tag{31}$$

$$\frac{1}{T_0} (k_{ij} \bar{\theta}_{,ij}' - c_\varepsilon p \bar{\theta}') - p (\gamma_{ij} \bar{\varepsilon}_{ij}' + g_i \bar{E}_i') = - \frac{\bar{W}'}{T_0}\,. \tag{32}$$

Multiply now Eq. (31) by $\bar{\theta}'$, Eq. (32) by $\bar{\theta}$, subtract the result and integrate over the region of the body. After transformations we obtain the equation

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W. Nowacki. A reciprocity theorem for coupled mechanical and thermoelectric fields in piezoelectric crystal. Proc. Vibr. Problems, 6, 1 (1965).

$$\frac{1}{T_0} k_{ij} \int_{\partial B} (\bar{\theta}' \bar{\theta}_{,j} - \bar{\theta} \bar{\theta}'_{,j}) n_i \, da - p \int_B \Big[(\gamma_{ij} \bar{\varepsilon}_{ij} + g_k \bar{E}_k) \bar{\theta}' -$$
$$- (\gamma_{ij} \bar{\varepsilon}'_{ij} + g_k \bar{E}'_k) \bar{\theta} \Big] da + \frac{1}{T_0} \int_B (\bar{W} \bar{\theta}' - \bar{W}' \bar{\theta}) dv = 0 \,. \tag{33}$$

Finally, let us make use of the equations for the electric field

$$\bar{D}_{k,k} = 0 \,, \qquad \bar{D}'_{k,k} = 0 \,. \tag{34}$$

Multiplying the first by $\bar{\varphi}'$, the second by $\bar{\varphi}$ substracting the result and integrating over the region of the body we have

$$\int_{\partial B} (\bar{D}_k \bar{\varphi}' - \bar{D}'_k \bar{\varphi}) n_i \, da + \int_B (\bar{D}_k \bar{E}'_k - \bar{D}'_k \bar{E}_k) dv = 0. \tag{35}$$

Introducing the constitutive relation

$$\bar{D}_k = e_{kij} \bar{\varepsilon}_{ij} + g_k \bar{\theta} + \epsilon_{ij} \bar{E}_j \tag{36}$$

and a similar relation for $\bar{D}'_k$, into the volume integral, we transform Eq. (35) to the form

$$\int_{\partial B} (\bar{D}_k \bar{\varphi}' - \bar{D}'_k \bar{\varphi}) n_k da + \int_B \Big[e_{kij} (\bar{\varepsilon}_{ij} \bar{E}'_k - \varepsilon'_{ij} \bar{E}_k) + g_k (\bar{\theta} \bar{E}'_k - \bar{\theta}' \bar{E}_k) \Big] dv = 0 \,. \tag{37}$$

Eliminating the common terms from Eqs. (30), (33) and (37) we arrive at one common equation of reciprocity of work containing all causes and effects

$$T_0 p \left\{ \int_B (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i) dv + \int_{\partial B} \Big[(\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i) + (\bar{D}_k \bar{\varphi}' - \bar{D}' \bar{\varphi}) n_k \Big] da \right\}$$
$$+ \int_B (\bar{W}' \bar{\theta} - \bar{W} \bar{\theta}') dv + k_{ij} \int_{\partial B} (\bar{\theta} \bar{\theta}'_{,j} - \bar{\theta}' \bar{\theta}_{,j}) n_i \, da = 0 \,. \tag{38}$$

Inverting in this equation the Laplace transform we obtain

$$T_0\Big\{\int_B (X_i\odot u_i' - X_i'\odot u_i)dv + \int_{\partial B}\Big[(p_i\odot u_i') - (p_i'\odot u_i) + (D_k\odot\varphi' - D_k'\odot\varphi)n_k da\Big]\Big\} +$$

$$+\int_B (W'*\theta - W*\theta')dv + k_{ij}\int_{\partial B}(\theta*\theta_{,j}' - \theta'*\theta_{,j})n_i da = 0 . \qquad (39)$$

Where we have introduced the notation

$$X_i\odot u_i' = \int_0^t X_i(\underline{x},\underline{t}-\tau)\,\frac{\partial u_i(\underline{x},\tau)}{\partial\tau}\,d\tau , \;\ldots\ldots\; \text{etc}$$

$$W*\theta' = \int_0^t W(\underline{x},t-\tau)\theta'(\underline{x},\tau)\,d\tau , \;\ldots\ldots\; \text{etc}$$

Similarly to thermoelasticity(1) we can investigate the action of concentrated forces, instantaneous and moving concentrated sources, we can derive the Somigliana and Green formulae generalized to thermopiezoelectricity, etc.

As in Sec. 1.7. it is easy to deduce the theorem of reciprocity of work for harmonic vibrations and stationary problems.

1.13. Coupling of elastic and electromagnetic waves

In the preceding considerations we dealt with the coupling between the quasi-static electric field and the motion of the elastic body. In this theory the equations of motion of the elasticity theory are coupled with the Gauss equation $\operatorname{div}\underline{D}$ by means of the piezoelectric constants.

We now proceed to a more general problem, namely the dynamic elastic and electromagnetic problem. We confine ourselves to the adiabatic process.

We shall now discard the previous assumption

$$\operatorname{rot}\underline{E} = 0 , \qquad \underline{E} = -\operatorname{grad}\varphi , \qquad (1)$$

(1) W. Nowacki. Dynamic Problems of thermoelasticity. PWN-Warszawa, Nordhoff Int. Publ. Leyden.

implying the quasistatic nature of the electric field, and consider the complete set of Maxwell's equations (assuming that $\varrho_e = 0, \underline{J} = 0, \underline{M} = 0$)

$$\operatorname{rot} \underline{H} = \dot{\underline{D}}, \qquad \operatorname{rot} \underline{E} = -\dot{\underline{B}}, \tag{2}$$

$$\operatorname{div} \dot{\underline{D}} = 0, \qquad \operatorname{div} \dot{\underline{B}} = 0, \tag{3}$$

completed by the constitutive relations

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}, \qquad \underline{B} = \mu_0 \underline{H}. \tag{4}$$

Performing over Eq. $(2)_2$ the operation of rotation, making use of Eq. $(2)_1$ and the relation $(4)_2$ we arrive at the wave equation[1]

$$\operatorname{rot}\operatorname{rot} \underline{E} = -\mu_0 \frac{\partial^2 D}{\partial t^2}. \tag{5}$$

Next, we represent the vector $\underline{D}$ by means of the constitutive relation for the quasistatic problem (the formula $(4)_7$ of Sec. 1.3.)

$$D_i = e_{ik\ell}\,\varepsilon_{k\ell} + \epsilon_{ik} E_k, \tag{6}$$

Then, substituting from (6) into (5) we obtain a system of three wave equations with the unknown functions u_i and E_i. The remaining three equations are deduced from the quations of motion

$$\sigma_{ji,i} + X_i = \varrho \ddot{u}_i \tag{7}$$

The stresses σ_{ij} are given by the constitutive relation for the quasistatic problem

$$\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell} - e_{kij} E_k. \tag{8}$$

(1)J.J. Kayme. Conductivity, and viscosity effects on wave propagation in piezoelectric crystals. J. Acoust. Soc. Amer. 26, /19 990.

Consider a simple example referring to the propagation of a monochromatic elastic and electromagnetic wave in amonium dihydrogen phosphate (ADF). This crystal belongs to the tetragonal system (of class $\bar{4}$ 2 m) in which we have ten independent elastic, piezoelectric and dielectric constants. Assume that the wave is propagated in the direction x_1 ; then derivatives with respect to x_2 and x_3 vanish. The constitutive relations in the considered case have the form

$$\sigma_{11} = c_{11} u_{1,1} , \qquad \sigma_{13} = c_{44} u_{3,1} + e_{41} E_2 , \qquad \sigma_{12} = c_{66} u_{2,1} + e_{63} E_3 , \tag{9}$$

$$D_1 = e_{11} E_1 , \qquad D_2 = - e_{41} u_{3,1} + \epsilon_{11} E_2 , \qquad D_3 = - e_{63} u_{2,1} + \epsilon_{33} E_3 . \tag{10}$$

Substituting (9) 1nd (10) into the wave equations (5) and (7) we obtain a system of five equations

$$(c_{11} \partial_1^2 - \varrho \partial_t^2) u_1 = 0 , \tag{11}$$

$$\left. \begin{aligned} (c_{66} \partial_1^2 - \varrho \partial_t^2) u_2 + e_{63} \partial_1 E_3 = 0 , \\ (\partial_1^2 - \mu_0 \epsilon_{33} \partial_t^2) E_3 + \mu_0 e_{63} \partial_t^2 \partial_1 u_2 = 0 , \end{aligned} \right\} \tag{12}$$

$$\left. \begin{aligned} (c_{44} \partial_1^2 - \varrho \partial_t^2) u_3 + e_{41} \partial_1 E_2 = 0 , \\ (\partial_1^2 - \mu_0 \epsilon_{11} \partial_t^2) E_2 + \mu_0 e_{41} \partial_t^2 \partial_1 u_3 = 0 . \end{aligned} \right\} \tag{13}$$

The sixth equation does not appear, for $E_1 = 0 \;\; D_1 = 0$. Observe that the longitudinal wave u_1 is unperturbed by the electromagnetic field.

The waves u_2 , E_3 and u_3 , E_2 are coupled. Hence we have five different wave velocities. The first phase velocity for the longitudinal wave is

$$v = \left(\frac{c_{66}}{\varrho} \right)^{1/2} .$$

The system of equations (12)–(13) can be transformed to the form

$$\left[(\partial_1^2-\mu_0\epsilon_{33}\partial_t^2)(c_{66}\partial_1^2-\varrho\partial_t^2)-\mu_0 e_{63}^2\partial_1^2\partial_t^2\right](u_2,E_3)=0 \tag{14}$$

$$\left[(\partial_1^2-\mu_0\epsilon_{11}\partial_t^2)(c_{44}\,\partial_1^2-\varrho\,\partial_t^2)-\mu_0\,e_{41}^2\,\partial_1^2\partial_t^2\right](u_3,E_2)=0 \tag{15}$$

For a monochromatic wave propagated in the x_1 – direction we have

$$\begin{aligned} u_i &= \hat{u}_i e^{-i(\omega t-kx_1)}, \\ E_i &= \hat{E}_i e^{-i(\omega t-kx_1)}. \end{aligned} \tag{16}$$

Introducing these functions into Eqs. (14) we obtain the following characteristic equation for the phase velocity v of the plane wave :

$$\zeta^4-\zeta^2\left(\frac{1}{c^2}+\frac{1}{V^2}+\eta\right)+\frac{1}{c^2V^2}=0,\quad v=\frac{\omega}{k}=\frac{1}{\zeta}. \tag{17}$$

We have introduced here the notation

$$\zeta=\frac{k}{\omega},\quad c=\left(\frac{c_{66}}{\varrho}\right)^{1/2},\quad V=\left(\frac{1}{\mu_0\epsilon_{33}}\right)^{1/2},\quad \eta=\frac{e_{63}^2\mu_0}{c_{66}}\ .$$

The biquadratic equation (17) yields

$$k^2=\frac{\omega^2}{2}\left\{\frac{1}{c^2}+\frac{1}{V^2}+\eta\pm\left[\left(\frac{1}{c^2}+\frac{1}{V^2}+\eta\right)^2-\frac{4}{c^2V^2}\right]^{1/2}\right\}, \tag{18}$$

Thus, if the determinant Δ of the equation is greater than zero, $k_1^2>0,\ k_2^2>0$.

The expression (18) can be represented in the form

$$k_{,}^2=\frac{\omega^2}{2}\left\{\frac{1}{c^2}+\frac{1}{V^2}+\eta\pm\left[\left(-\frac{1}{c^2}+\frac{1}{V^2}+\eta\right)^2+\frac{4\eta}{c^2}\right]^{1/2}\right\}, \tag{19}$$

where it is ensured that $\Delta>0$.

The solution of Eq. (14) for the monochromatic wave has the form

$$u_2=e^{-i\omega t}\left\{Ae^{ik_1x_1}+Be^{-ik_1x_1}+Ce^{ik_2x_1}+De^{-ik_2x_1}\right\}, \tag{20}$$

$$E_3 = e^{-i\omega t}\left\{(Ae^{ik_1x_1} - Be^{-ik_1x_1})\chi_1 + (Ce^{ik_2x_1} - De^{-ik_2x_1})\chi_2\right\},$$

where

$$\chi_1 = i\,\frac{\nu^2 - k_1^2}{\tau k_1}, \qquad \chi_2 = i\,\frac{\nu^2 - k_2^2}{\tau k_2}, \qquad \nu = \frac{\omega}{c}, \qquad \tau = \frac{e_{63}}{c_{66}},$$

The transverse wave u_2 and the electromagnetic wave E_3 are propagated with the same phase velocity $v = \zeta^{-1} = \omega / k$. In view of the existence of two roots k_1, k_2 we are faced with two waves.

Since k_1, k_2 are constants, the waves do not undergo dispersion and since they are real, the waves are not damped. An analogous reasoning holds for Eq. (15). Since the latter is analogous to Eq. (14) the only difference consists in different values of the constants.

The knowledge of the displacement $\underline{u}$ and the field $\underline{E}$ makes it possible to determine the vector $\underline{D}$ from the constitutive relations (10) and the stress from the formulae (9). The components of the field $\underline{H}$ are calculated on the basis of Eq. $(2)_1$ and the vector $\underline{B}$ from the formula $(4)_2$.

Assume now that the conduction current $\underline{J} \neq 0$. Then we are faced with the system of equations

$$\text{rot}\,\underline{H} = \dot{\underline{D}} + \underline{J}, \qquad \text{rot}\,\underline{E} = -\dot{\underline{B}} \tag{21}$$

$$\text{div}\,\underline{D} = 0, \qquad \text{div}\,\underline{B} = 0 \tag{22}$$

We take the constitutive relations in the form

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}, \qquad \underline{B} = \mu_0 \underline{H} \qquad \underline{J} = \sigma \underline{E}, \tag{23}$$

Thus, we assumed the proportionality of the vectors $\underline{J}$ and $\underline{E}$ and, moreover, that the material is isotropic with respect to the electric conductivity. σ is a (constant) coefficient of electric conductivity.

Eqs. $(21)_{1,2}$ lead to the wave equation

$$\text{rot rot}\,\underline{\underline{E}} = -\mu_0 \frac{\partial^2 \underline{\underline{D}}}{\partial t^2} - \mu_0 \frac{\partial \underline{J}}{\partial t} \tag{24}$$

The further procedure is analogous to that in the preceding problem. The difference as compared with the plane wave in amonium dihydrogen phosphate, in Eqs. (14) and (15). We have

$$\left[(c_{66}\partial_1^2 - \varrho\,\partial_t^2)(\partial_1^2 - \mu_0\epsilon_{33}\partial_t^2 - \underline{\mu_0\,\sigma\,\partial_t}) - \mu_0 e_{63}^2 \partial_1^2 \partial_t^2\right](u_2, E_3) = 0, \tag{25}$$

$$\left[(c_{44}\partial_1^2 - \varrho\,\partial_t^2)(\partial_1^2 - \mu_0\epsilon_{11}\partial_t^2 - \underline{\mu_0\,\sigma\,\partial_t}) - \mu_0 e_{41}^2 \partial_1^2 \partial_t^2\right](u_3, E_2) = 0. \tag{26}$$

Consider the wave (25) assuming (16) to be true. The phase velocity of the propagation of the waves u_2, E_3 is calculated from the equations

$$\zeta^4 - \zeta^2\left(\frac{1}{c^2} + \frac{1}{V^2} + \eta - \underline{\frac{i\,\sigma\mu_0}{\omega}}\right) + \frac{1}{c^2V^2} - \frac{\sigma}{\omega^2c^2}, \qquad v = \frac{1}{\xi} = \frac{\omega}{k}, \tag{27}$$

which yields

$$k^2 = \frac{\omega}{2}\left\{\frac{1}{c^2} + \frac{1}{V^2} + \eta - \frac{i\sigma\mu_0}{\omega} \pm \left[\left(-\frac{1}{c^2} + \frac{1}{V^2} + \eta\right)^2 + \frac{4\eta}{c^2}\right]^{1/2}\right\} \tag{28}$$

It is evident that the roots k_i ($i = 1, 2$) are complex. Namely, we have

$$k_\beta = a_\beta + i\vartheta_\beta, \qquad \beta = 1, 2. \tag{29}$$

Thus the wave is damped. The phase velocity and the damping coefficient can be found from the formulae

$$v_\beta = \frac{\omega}{a_\beta} = \frac{\omega}{R_e(k_\beta)}, \qquad \vartheta_\beta = J_m(k_\beta),\ \beta=1,2 \tag{30}$$

The roots k_β depend on the frequency of vibrations. Therefore, the waves u_2, E_3 are dispersed.

The solution of Eqs. (25) has the form

$$u_2 = Ae^{-i\omega(t-\frac{x_1}{v_1})}e^{-\vartheta_1 x_1} + Be^{-i\omega(t+\frac{x_1}{v_1})}e^{\vartheta_1 x_1} + \\ + Ce^{-i\omega(t-\frac{x}{v_2})}e^{-\vartheta_2 x_1} + De^{-i\omega(t+\frac{x_1}{v_2})}e^{\vartheta_2 x_1}, \tag{31}$$

$$E = \hat{A}e^{-i\omega(t-\frac{x_1}{v_1})}e^{-\vartheta_1 x_1} + \hat{B}e^{-i\omega(t+\frac{x_1}{v_1})}e^{\vartheta_1 x_1} + \\ + \hat{C}e^{-i\omega(t-\frac{x_1}{v_2})}e^{-\vartheta_2 x_1} + De^{-i\omega(t+\frac{x_1}{v_2})}e^{\vartheta_2 x_1}. \tag{32}$$

The relations between the constants $A, B, \ldots$ and $\hat{A}, \hat{B}, \ldots$ can be found on the basis of one of the following two equations :

$$(c_{66}\partial_1^2 - \varrho\partial_t^2)u_2 + e_{63}\partial_1 E_3 = 0, \\ (\partial_1^2 - \mu_0 \epsilon_{33}\partial_t^2 + \underline{\mu_0 \sigma \partial_t})E_3 + \mu_0 e_{63}\partial_t^2 u_2 = 0, \tag{33}$$

which constitute a generalization of Eqs. (12).

The above considerqtions can be generalized to thermopiezoelectricity. In this case we are faced with a system of 3 equations

$$\text{rot rot } E = -\mu_0 \frac{\partial^2 \underline{D}}{\partial t^2} - \mu_0 \frac{\partial \underline{J}}{\partial t} \tag{34}$$

$$\sigma_{ji,j} - \varrho \ddot{u}_i = 0, \tag{35}$$

$$k_{ij}\theta_{,ij} - c_\varepsilon\dot{\theta} - T_0\gamma_{ij}\dot{\varepsilon}_{ij} - T_0 g_i \dot{E}_i = 0. \tag{36}$$

The last equation is the heat conduction equation. The constitutive relations of thermo-piezoelectricity (the formulae (15), (17) of Sec. 1.7.) have the form

$$\sigma_{ji} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \gamma_{ij}\theta, \tag{37}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k + g_i\theta. \tag{38}$$

Introducing them into Eqs. (34)–(36) we arrive at a system of seven equations containing as unknowns three components of the displacement vector $\underline{u}$, three components of the field $\underline{E}$ and the temperature increment θ . Oberve that in view of the heat coupling and the presence of electric conductivity. all waves are damped and dispersed.

CHAPTER II

MAGNETOELASTICITY

2.1. The field equations and the constitutive equations of magnetoelasticity

In the last 20 years a new field has been developing, called magnetoelasticity, in which we investigate the interaction between the strain and electromagnetic fields in a solid elastic body. The theory is essentially an extension of linear elasticity and linear electrodynamics of slowly moving media.

If a body placed in a strong initial magnetic field is moved by external loading, besides the strain field, there arises an electromagnetic field. These two fields are coupled and interact with each other.

A stimulus for the development of magnetoelasticity was its application in geophysics, in some branches of acoustics and in investigating the damping of acoustic waves in a magnetic field.

The first paper on the subject was written by L. Knopoff[(1)], where the author investigated the propagation of elastic field in presence of Earth's magnetic field. We should also mention papers by A. Baños[(2)] and P. Chadwick[(3)]. There are also papers by S. Kaliski and J. Petykiewicz[(4, 5)], important in the development of magnetoelasticity. A somewhat different approach to the subject of magnetoelasticity is presented in the paper by

(1) L. Knopoff. The interaction between elastic wave motions and a magnetic field in electrical conductors. J. of Geophysical Research, 60, 4(1955), 441.

(2) A. Baños Jr., Phys. Rev. 104. 2(1956), 300.

(3) P. Chadwick. Ninth Int. Congr. Appl. Mech. 7/1957), 143.

(4) S. Kaliski and J. Petykiewicz. Dynamical equations of motion coupled with the field of temperature and resolving functions for elastic and viscoelastic anisotropic bodies in the magnetic field. Proc. Vibr. Problems, 1, 4(1960), 3.

(5) S. Kaliski and J. Petykievicz. Equations of motion coupled with the field of temperature in a magnetic field involving mechanical and electromagnetic relaxations for anisotropic bodies. Proc. Vibr. Problems, 1. 4(1959).

J.W. Dunkin and A.C. Eringen(1).

Consider an elastic body in a strong initial magnetic field. The mechanical (impact) and thermal (e.g. thermal shock) causes generate in the body a strain and a coupled electromagnetic field. In all above mentioned papers it was assumed that the influence of the electromagnetic field on the strain occurs by means of Lorentz forces appearing in the equations of motion. The Ohm law contains a term describing the increment of the density of the electric field depending on the velocity of the material particles moving in the magnetic field.

For this simplified model L. Knopoff proved that the magnetoelastic interactions play an insignificant role in the propagation of elastic waves in the presence of Earth's magnetic field. However, there exist instruments working in a strong initial magnetic field. G.A. Alers and P.A. Fleury(2) proved experimentally that in these cases the influence of magnetoelastic interactions is considerable.

Consider a homogeneous isotropic material medium, possessing a good electric conductivity. Assume first that the body is at rest with respect to the free space which we identify with an initial frame. In this case Maxwell's equations have the form(3,4)

$$\operatorname{rot} \underline{H} = \dot{\underline{D}} + \underline{J} \ , \qquad \operatorname{rot} \underline{E} = -\dot{\underline{B}} \ , \tag{1}$$

$$\operatorname{div} \underline{D} = \varrho_e \ , \quad \operatorname{div} \underline{B} = 0 \ . \tag{2}$$

Here the vectors $\underline{E}$, $\underline{H}$, $\underline{D}$, $\underline{B}$, $\underline{J}$, denote the electric field, the magnetic field, the electric displacement, the magnetic induction and the density of the electric field, respectively. Finally ϱ_e is the density of the electric charge. The quantities $\underline{E}$, $\underline{H}$, $\underline{D}$, $\underline{B}$, $\underline{J}$ are observed in the laboratory reference frame.

(1) J.W. Dunkin and A.C. Eringen. On the propagation of waves in an electromagnetic elastic solid. J. Engn. Sci. 1, 4(1963), 461.

(2) G.A. Alers and P.A. Fleury. Modification of the velocity of sound in metals by magnetic fields. Phys. Rev. 129, 6(1963) 2435.

(3) A. Sommerfeld. Electrodynamics. Academic Press, New York, 1952.

(4) J.A. Stratton. Electromagnetic theory. Mc Graw Hill, New York, 1941.

Eqs. $(1)_1$ and $(2)_1$ yield the electric continuity equation

$$\operatorname{div} \underline{J} + \frac{\partial \varrho_e}{\partial t} = 0 . \tag{3}$$

The system of equations (1) – (2) is not complete, for we have to add the constitutive relations. For the quantities $\underline{D}$ and $\underline{B}$ they have the form of the non-linear relations

$$\underline{D} = \underline{d}(\underline{E}, \underline{H}) , \qquad \underline{B} = \underline{b}(\underline{E}, \underline{H}) . \tag{4}$$

We shall confine ourselves to linear relations which for an isotropic body have the form

$$\underline{D} = \varepsilon \underline{E} , \qquad \underline{B} = \mu_e \underline{H} \tag{5}$$

The quantities ε and μ_e denote the electric and magnetic permeabilities, respectively.

The last constitutive relations, the Ohm law, is the relation between the vectors $\underline{J}$ and $\underline{E}$

$$\underline{J} = \sigma \underline{E} \tag{6}$$

where σ is the coefficient of electric conductivity.

In vacuum we have the constitutive relations

$$\underline{D} = \varepsilon_0 \underline{E} , \qquad \underline{B} = \mu_0 \underline{H} , \qquad \underline{J} = 0 , \tag{7}$$

where ε_0 , μ_0 are universal constants.

The Maxwell equations (1) – (2) hold both for the interior and the exterior of the body. On the surface separating the body from the vacuum there are the boundary conditions[2]

$$\begin{aligned} &\underline{n} \times [\underline{H}] = \underline{i} - v [\underline{D}] , \qquad &&\underline{n} \times [\underline{E}] = v [\underline{B}] , \\ &\underline{n} \cdot [\underline{D}] = 0 , &&\underline{n} \cdot [\underline{B}] = 0 . \end{aligned} \tag{8}$$

(1) In general the magnetic permeability is denoted by μ , we prefer here however μ_e , reserving μ for the Lamé constant.

(2) H. Parkus. Magneto-thermoelasticity, CISM, Udine, Springer-Verlag, Wien, 1972.

where $[\![\underline{H}]\!]$ denotes the jump $\underline{H}^+ - \underline{H}^-$ of the vector $\underline{H}$ through the surface separating the body form the vacuum. **V** is the component of the velocity vector in the direction of the vector $\underline{n}$. The vector $\underline{j}$ is the density of surface currents.

Let us return to the Maxwell equations (1) – (2). By eliminations making use of (5) and (6) we can transform them to the simple wave equations

$$\begin{aligned} &\nabla^2\underline{H} - \mu_e\varepsilon\ddot{\underline{H}} - \sigma\mu_e\dot{\underline{H}} = 0 \\ &(\nabla^2 - \text{grad div})\underline{E} - \varepsilon\mu_e\ddot{\underline{E}} - \sigma\mu_e\dot{\underline{E}} = 0 \end{aligned} \tag{9}$$

The structure of the above hyperbolic equations indicates that the propagating waves are damped and dispersed. For the vacuum, from the Maxwell equations and the constitutive relations (7) we obtain the wave equations

$$\nabla^2\underline{H} - \mu_0\varepsilon_0\ddot{\underline{H}} = 0\,, \qquad \nabla^2\underline{E} - \mu_0\varepsilon_0\ddot{\underline{E}} = 0\,, \tag{10}$$

or

$$(\nabla^2 - \frac{1}{c^2}\partial_t^2)(H,E) = 0\,, \qquad c = (\mu_0\varepsilon_0)^{-1/2}, \tag{11}$$

where c is the light velocity. The electromagnetic waves in vacuum are neither damped nor dispersed.

In Sec. 1.1. we derived the balance of electromagnetic energy

$$\frac{\partial}{\partial t}\int_B U_e dv = -\int_{\partial B}\underline{n}\cdot(\underline{E}\times\underline{H})da - \int_B \underline{E}\,\underline{J}dv\,, \tag{12}$$

where

$$\frac{\partial U_e}{\partial t} = \underline{E}\dot{\underline{D}} + \underline{H}\dot{\underline{B}}\,.$$

Consider now a body the material points of which move in an external magnetic or electric field. Denote by $\underline{E}', \underline{H}', \underline{D}', \underline{B}', \underline{J}', -\varrho_e'$ the magnetic quantities observed in a coordinate system X' connected with a moving material point. In this moving coordinate system we have the Maxwell equations (note the invariance with respect to the Lorentz transformation)

$$\text{rot}\,\underline{H}' = \underline{J}' + \frac{d_c\underline{D}'}{dt}\,, \qquad \text{rot}\,\underline{E}' = -\frac{d_cB'}{dt}\,, \tag{13}$$

$$\operatorname{div} \underline{D}' = \varrho_e' , \qquad \operatorname{div} \underline{B}' = 0$$

Here $d_c\underline{f}/dt$ is the convective time derivative of the flux vector $\underline{f}$:

$$\frac{d_c \underline{f}}{dt} = \frac{\partial \underline{f}}{\partial t} + \underline{v} \operatorname{div} \underline{f} + \operatorname{rot} (\underline{f} \times \underline{v}) . \tag{14}$$

Since the system (1) – (2) and (13) must be identical, in view of (14) we have the equations

$$\left.\begin{aligned} &\operatorname{rot}(\underline{H}' + \underline{v} \times \underline{D}') - \underline{J}' - \varrho_e \underline{v} - \frac{\partial \underline{D}'}{\partial t} = \operatorname{rot} \underline{H} - \underline{J} - \frac{\partial \underline{D}}{\partial t} \\ &\operatorname{rot}(\underline{E}' - \underline{v} \times \underline{B}') + \frac{\partial \underline{B}'}{\partial t} = \operatorname{rot} \underline{E} + \frac{\partial \underline{B}}{\partial t} , \\ &\operatorname{div} \underline{D}' - \varrho_e' = \operatorname{div} \underline{D} - \varrho_e , \\ &\operatorname{div} \underline{B}' = \operatorname{div} \underline{B} ; \end{aligned}\right\} \tag{15}$$

implying the relations

$$\begin{gathered} \underline{H}' + \underline{v} \times \underline{D}' = \underline{H}, \qquad \underline{E}' - \underline{v} \times \underline{B}' = \underline{E} \\ \underline{D}' = \underline{D}, \quad \underline{B}' = \underline{B}, \quad \underline{J}' - \varrho_e \underline{v} = \underline{J} , \end{gathered} \tag{16}$$

or

$$\underline{H}' = \underline{H} - \underline{v} \times \underline{D}, \qquad \underline{E}' = \underline{E} + \underline{v} \times \underline{B}, \qquad \underline{J}' = \underline{J} + \varrho_e \underline{v} . \tag{17}$$

The constitutive relations of electrodynamic of slowly moving media are not invariant with respect to the Lorentz transformation. Their form is analogous to (5) and (6), namely

$$\underline{D}' = \varepsilon \underline{E}', \qquad \underline{B}' = \mu_e \underline{H}', \qquad \underline{J}' = \sigma \underline{E}'. \tag{18}$$

In view of (17) they take the form

$$\begin{gathered} \underline{D} = \varepsilon(\underline{E} + \underline{v} \times \underline{B}) , \qquad \underline{B} = \mu_e(\underline{H} - \underline{v} \times \underline{D}) , \\ J = \sigma(\underline{E} + \underline{v} \times \underline{B}) + \varrho_e \underline{v} . \end{gathered} \tag{19}$$

Thus. we have arrived at the complete set of equations and relations of electrodynamics of slowly moving media. It contains the Maxwell equations (1)–(2) and the constitutive relations (19). The relation $(19)_3$ is the modified Ohm law : its last term is the influence of the velocity of the particle (moving through the electromagnetic field) on the density of the electric current.

Consider now a deformable body. We assume that the only influence of the mechanical field on the electromagnetic field is an the Lorentz force

$$\underline{f} = \varrho_e \underline{E} + \underline{J} \times \underline{B} , \tag{20}$$

which we introduce into the principle of conservation of energy as a volume force,

$$\int_B (X_i + f_i) dv + \int_{\partial B} \sigma_{ji} n_j da = \frac{d}{dt} \int_B \varrho v_i dv , \tag{21}$$

Making use of the Gauss transformation for the surface integral in (21) we obtain the local equation

$$\sigma_{ji,j} + X_i + f_i = \varrho \dot{v}_i , \tag{22'}$$

or

$$\sigma_{ji,j} + X_i + \varrho_e \underline{E} + \underline{J} \times \underline{B} = \varrho \ddot{u}_i . \tag{22''}$$

The mechanical constitutive relation is taken in the form of the generalized Hooke law. For the considered isotropic bodies we have

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{kk} , \tag{23}$$

where μ , λ are the Lamé constants, referred to the adiabatic state. We assumed above that there are no initial stresses due to the initial magnetic field in the body. We have neglected in the above relations additional terms of higher order due to the influence of the electromagnetic field on the mechanical field ; thus, we assumed that the relations (23) are the same in both systems of reference.

Let us return to the equations of motion (22′) ; in the case of magnetoelasticity they contain the volume Lorentz force f_i. Introducing the strain

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{24}$$

substituting the Hooke law into the equations of motion and eliminating the strain by means of (24) we obtain the displacement equations

$$\mu \nabla^2 u_i + (\lambda + \mu) u_{j,ji} + X_i + f_i = \varrho \ddot{u}_i \tag{25}$$

which can be written in the vector form

$$\mu \nabla^2 \underline{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \underline{u} + \underline{X} + \underline{f} = \varrho \ddot{\underline{u}} \tag{26}$$

Eqs. (26), the equations of electrodynamics (1) – (2) and the constitutive equations (19) constitute a complete set of differential equations of magnetoelasticity. The equations should be completed by the boundary and initial conditions.

The boundary conditions for the elctromagnetic field can be derived following the procedure given by J.W. Dunkin and A.C. Eringen(1).

Let us transform Eqs. (1), (2) to an equivalent form

$$\left.\begin{aligned} &\operatorname{rot}(\underline{E} + \underline{v} \times \underline{B}) = -\left(\frac{\partial \underline{B}}{\partial t} + \underline{v} \operatorname{div} \underline{B} - \operatorname{rot}(\underline{v} \times \underline{B})\right), \\ &\operatorname{rot}(\underline{H} - \underline{v} \times \underline{D}) = \left(\frac{\partial \underline{D}}{\partial t} + \underline{v} \operatorname{div} \underline{D} - \operatorname{rot}(\underline{v} \times \underline{D})\right) + \underline{j} - \varrho_e \underline{v} \\ &\operatorname{div} \underline{D} = \varrho_e \qquad \operatorname{div} \underline{B} = 0 \end{aligned}\right\} \tag{27}$$

We apply to the first two equations the Stokes transformation and to the last two, the Gauss transformation. Thus

$$\left.\begin{aligned} &\int_c (\underline{E} + \underline{v} \times \underline{B})\, d\underline{c} = -\frac{d_c}{dt} \int_{\mathcal{B}} \underline{B}\, d\underline{a}\,, \\ &\int_c (\underline{H} - \underline{v} \times \underline{D})\, d\underline{c} = -\frac{d_c}{dt} \int_{\mathcal{B}} \underline{D}\, d\underline{a} + \int_{\mathcal{B}} (\underline{j} - \varrho_e \underline{v})\, d\underline{a}\,, \end{aligned}\right\} \tag{28}$$

$$\int_{\partial B} \underline{B}\, d\underline{a} = 0 \qquad \int_{\partial B} \underline{D}\, d\underline{a} = \int_B \varrho_e dv\,. \tag{29}$$

(1) see footnote on p. 55.

In integrating $(27)_{1,2}$ we have used the convective time derivative

$$\frac{d_c}{dt}\int_{\partial B} \underline{b}\, d\underline{a} = \int_{\partial B}\left(\frac{d\underline{b}}{dt} + \underline{v}\,\mathrm{div}\,\underline{b} - \mathrm{rot}\,\underline{v} \times \underline{b}\right) d\underline{a}\,. \tag{30}$$

Observe that the curve c, the surface ∂B and the region of the body B are moving. Let us choose ∂B to be a small rectangular area, perpendicular to the boundary surface of the body (Fig. 1.). The contour of the rectangle is denoted by c. If now the dimension of the rectangle perpendicular to the boundary surface is decreased and tends to zero, Eqs. $(28)_{1,2}$ yield the boundary conditions in the form

$$[\![\underline{E} + \underline{v} \times \underline{B}]\!]_t = 0\,, \qquad [\![\underline{H} - \underline{v} \times \underline{D}]\!]_t = \hat{j}_m - \hat{\varrho}_e v_m\,. \tag{31}$$

where $[\![\underline{A}]\!]_t$ denotes the difference of the tangential components (parallel to the direction t) of the vector $\underline{A}$ inside and outside of the body. $\hat{j}_m$ and $\hat{\varrho}_e$ denote the density of the surface current and the surface density of the electric charge, respectively

Suppose now that the region B is the region of a cylinder the axis of which coincides with the direction of the normal (Fig. 2). If the height of the cylinder tends to zero, Eqs. $(29)_{1,2}$ yield the boundary conditions

$$[\![\underline{B}]\!]_n = 0\,, \qquad [\![\underline{D}]\!]_n = \hat{\varrho}_e \tag{32}$$

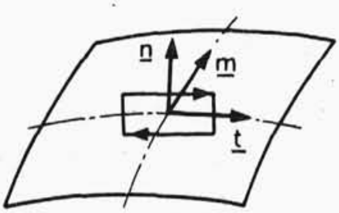

Fig. 1

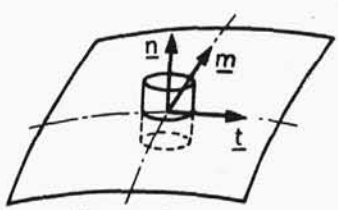

Fig. 2

where $[\![A]\!]_n$ denotes the jump of the normal components of the vector $\underline{A}$.

Integrating the continuity equation for the current density $\underline{J}$.

$$\mathrm{div}\,\underline{J} + \frac{\partial \varrho_e}{\partial t} = 0 \tag{33}$$

over the region B and proceeding as with Eq. $(29)_{1,2}$ we obtain the last boundary condition

$$\left[\!\left[\sigma(\underline{E} + \underline{v} \times \underline{B} \right]\!\right] = -\frac{\partial \hat{\varrho}_e}{\partial t} \quad . \tag{34}$$

Besides the above electrodynamic boundary conditions (31), (32) and (34) we have the boundary conditions for the equation of motion (26).

2.2. Linearization of the equations and relations of magnetoelasticity

Let us write down the equations of magnetoelasticity. Thus, we have the Maxwell equations

$$\text{rot}\,\underline{H} = \underline{J} + \underline{\dot{D}} \,, \qquad \text{rot}\,\underline{\dot{E}} = -\underline{\dot{B}} \,, \tag{1}$$

$$\text{div}\,\underline{D} = \varrho_e \,, \qquad \text{div}\,\underline{B} = 0 \,, \tag{2}$$

the constitutive relations

$$\begin{gathered} \underline{D} = \varepsilon(\underline{E} + \underline{v} \times \underline{B}) , \quad B = \mu_e(\underline{H} - \underline{v} \times \underline{D}) , \\ \underline{J} = \sigma(\underline{E} + \underline{v} \times \underline{B}) + \varrho_e \underline{v} \,, \end{gathered} \tag{3}$$

and the displacement equations

$$\mathscr{L}(\underline{u}) + \underline{X} + \varrho_e \underline{E} + \underline{J} \times \underline{B} = 0 \,, \tag{4}$$

where $\mathscr{L}(u)$ is the differential operator

$$\mathscr{L}(..) = \left(\mu \nabla^2 + (\lambda + \mu)\text{grad div} - \varrho \partial_t^2\right)(..) \tag{5}$$

The system of equations (1) – (4) is non-linear and most complicated. A considerable simplification is obtained if we assume the following : the body is subject to a strong initial magnetic field $\underline{H}_0$ = const and at the instant $t = 0$ we apply an external loading ; then in view of the coupling between the strain field and the electromagnetic field there appear small fluctuations $\underline{e}$, $\underline{h}$ described by the relations

$$\underline{H}(\underline{x},t) = \underline{H}^0 + \underline{h}(\underline{x},t) \qquad \underline{E}(\underline{x},t) = \underline{e}(\underline{x},t) \tag{6}$$

Substituting the above formulae into the constitutive relations (13) and neglecting products of the quantities h_i, e_i, v_i and products of their derivatives, we obtain

$$\begin{gathered} \underline{D} = \varepsilon \underline{e} + \alpha \underline{v} \times \underline{H}^0 , \qquad \underline{B} = (\underline{H}^0 + \underline{h})\mu_e , \\ \underline{J} = \sigma(\underline{e} + \mu_e \underline{v} \times \underline{H}^0), \quad \alpha = \varepsilon \mu_e , \end{gathered} \tag{7}$$

where we also neglected ϱ_e . Introducing the relations (7) into the Maxwell equations (1) – (2) we have

$$\text{rot}\, \underline{h} = \sigma(\underline{e} + \mu_e \underline{v} \times \underline{H}^0) + \varepsilon \dot{\underline{e}} + \alpha \dot{\underline{v}} \times \underline{H}^0, \qquad \text{rot}\, \underline{e} = -\mu_e \dot{\underline{h}} , \tag{8}$$

$$\varepsilon\, \text{div}\, \underline{e} + \alpha\, \text{div}(\underline{v} \times \underline{H}^0) = 0 , \qquad \text{div}\, \underline{h} = 0 . \tag{9}$$

Substituting from (7) into the displacement equations (4) we arrive at the vector equation

$$\mathscr{L}(\underline{u}) + \underline{X} + \mu_e \underline{J} \times \underline{H}^0 = 0 , \tag{10}$$

or, taking into account $(7)_3$ and $(8)_1$,

$$\mathscr{L}(\underline{u}) + \underline{X} + (\text{rot}\, \underline{h} \times \underline{H}^0 - \varepsilon \dot{\underline{e}} \times \underline{H}^0 - \alpha \dot{\underline{v}} \times \underline{H}^0 \times \underline{H}^0) = 0 . \tag{11}$$

Observe that eliminating the function $\underline{e}$ from the Maxwell equations (8) – (9) we are led to the wave equation for the function $\underline{h}$:

$$(\nabla^2 - \beta \partial_t - \beta_0 \partial_t^2)\underline{h} = -\beta\, \text{rot}(\underline{v} \times \underline{H}^0) - \alpha\, \text{rot}(\dot{\underline{v}} \times \underline{H}^0) . \tag{12}$$

Here $\beta = \sigma\mu$, $\beta_0 = \mu_e \varepsilon$.

Eq. (12) constitutes a generalization of Eq. (9) of Sec. 1.2. ; it is an equation of hyperbolic type.

The frequencies related to vibrations and mechanical waves are much smaller than the frequencies of electromagnetic waves with the same wave length. Thus, when we investigate machanical waves we may regard the electromagnetic fields as quasistatic. Mathematically it means that $\underline{D} = 0$. $\partial \underline{D}/\partial t = 0$ and then Eqs. $(7)_1$ imply that $\varepsilon = 0$, $\alpha = 0$, $\beta_o = 0$. Thus, we arrive at the simplified system of Maxwell equations

$$\text{rot}\,\underline{h} = \underline{J} \qquad \text{rot}\,\underline{e} = -\mu_e \dot{\underline{h}} \qquad \text{div}\,\underline{h} = 0\,, \tag{13}$$

and the constitutive relations

$$\underline{J} = \sigma(\underline{e} + \mu_e \underline{v} \times \underline{H}^o)\,, \quad \underline{B} = \mu_e(\underline{H}^o + \underline{h}) \tag{14}$$

The equations of motion are also considerably simplified, namely we have

$$\mathscr{L}(\underline{u}) + \underline{X} + \mu_e \text{rot}\,\underline{h} \times \underline{H}^o = 0\,. \tag{15}$$

The equations of motion (15) and the simplified field equation (12)

$$(\nabla^2 - \beta \partial_t)\underline{h} = -\beta\,\text{rot}(\underline{v} \times \underline{H}^o) \tag{16}$$

constitute a complete set of equations of magnetoelasticity. Eq. (16) is a diffusion equation rather than a wave equation.

The solution of the system of equations (15) and (16) yields the functions $\underline{u}$ and $\underline{h}$. The remaining functions are deduced from Eqs. (13) and (14).

A further simplification follows form the assumtion that the body is a perfect conductor. Then $\sigma = \infty$, $\beta = \infty$ and Eq. (16) takes the simpler form

$$\underline{h} = \text{rot}(\underline{u} \times \underline{H}^o) \tag{17}$$

Introducing the above formula into the equations of motion (15) we arrive at the uncoupled system of displacement equations

$$\mu \nabla^2 \underline{u} + (\lambda + \mu)\text{grad div}\,\underline{u} + \mu_e \text{rot rot}(\underline{u} \times \underline{H}^o) \times \underline{H}^o + \underline{X} = \varrho \ddot{\underline{u}}\,. \tag{18}$$

A solution of Eq. (18) yields the displacement $\underline{u}$ while the relation (17) makes it possible to calculate the function $\underline{h}$. The relation $(14)_1$ with $\sigma = \infty$ yields

$$\underline{e} = -\mu \underline{v} \times \underline{H}^o. \tag{19}$$

Finally, form Eq. (13) we find that

$$\underline{J} = \text{rot } \underline{h} . \tag{20}$$

The boundary conditions for the system of equations (15) – (16) are also considerably simplified. They can be derived from the boundary conditions (31) and (32) of Sec. 2.1. Assuming that $\underline{D} = 0, \varepsilon = 0, \alpha = 0$ we have

$$\begin{aligned} &[\![\underline{e} + \mu_e \underline{v} \times \underline{H}^o]\!]_t = 0 , \qquad [\![\underline{h}]\!]_t = 0 , \\ &\mu_e [\![\underline{h}]\!]_n = 0 , \qquad [\![\underline{e} + \mu_e \underline{v} \times \underline{H}^o]\!]_n = 0 . \end{aligned} \tag{21}$$

Finally, observe that Eqs. (15) and (16) can be written in the form

$$\begin{aligned} &\mu \nabla^2 \underline{u} + (\lambda + \mu) \text{grad div } \underline{u} + \frac{1}{\mu_e} \text{rot } \underline{b} \times \underline{B}^o + \underline{X} = \varrho \ddot{\underline{u}} , \\ &(\nabla^2 - \beta \partial_t) \underline{b} = -\beta \text{rot}(\underline{v} \times \underline{B}^o). \end{aligned} \tag{22}$$

They refer to the case in which there appears an initial field of magnetic induction $B^o =$ const. Then

$$\underline{B} = \underline{B}^o + \underline{b}(\underline{x}, t) , \qquad \underline{E} = e(\underline{x}, t) . \tag{23}$$

2.3. The fundamental equations and relations of magnetothermoelasticity

The point of departure of our considerations is the balance of the mechanical, electromagnetic and thermal energy, which in the spatial notation has the form[1]

$$\begin{aligned} \frac{d}{dt} \int_B \left(\frac{1}{2} \varrho v_i v_i + \varrho U + U_e \right) dv &= \int_B (X_i v_i + \varrho r) dv + \\ &+ \int_{\partial B} (p_i v_i - q_i n_i - (\underline{E} \times \underline{H})_i + U_e v_i n_i) da . \end{aligned} \tag{1}$$

The integration is over the region B and surface ∂B of the deformed body. The first term

1 H. Parkus. Magnetothermoelasticity. CISM. Udine. Springer Verlag. Wien. 1972.

in the left-hand side is the time increment of the kinetic energy $\mathcal{K} = 1/2\int_B \varrho v_i v_i \, dv$, the second is the time increment of the mechanical energy $\int_B \varrho U \, dv$, U being the specific energy referred to unit mass. Finally U_e is the electromagnetic energy referred to unit volume ϱ is the density and v the velocity of material point. The first term in the right-hand side is the power of body forces and the thermal power. r is the quantity of heat generated per unit mass and unit time. The first term of the surface integral is the power of surface (contact) forces, the second term the heat flux through the surface ∂B, the third the flux of electric energy and the last term is the flux of electromagnetic energy produced by the motion of the body in an external magnetic field. q denotes the heat flux vector referred to unit surface and $(E \times H)_i$ is the component of the Poynting vector.

Let us now employ the known relations

$$\frac{d}{dt}\int_B U_e dv = \int_B \frac{\partial U_e}{\partial t} dv + \int_{\partial B} U_e v_i n_i da \quad , \tag{2}$$

and perform the differentiation

$$\frac{d}{dt}\int_B \left(\frac{1}{2}\varrho v_i v_i + \varrho U\right) dv = \int_B \left[\varrho(\dot{U} + v_i \dot{v}_i) + (U + \frac{1}{2}\varrho v_i v_i)(\dot{\varrho} + \varrho v_{i,i})\right] dv, \tag{3}$$

where

$$\dot{\varrho} = \frac{\partial \varrho}{\partial t} \text{ , e.t.c.}$$

Taking into account the equation of mass continuity

$$\frac{\partial \varrho}{\partial t} + \varrho v_{i,i} = 0 \tag{4}$$

and the relation between the contact forces and stress

$$p_i = \sigma_{ij} n_j \tag{5}$$

we arrive at the following form of the energy balance :

$$\int_B (\varrho\dot{U} + \dot{U}_e)\,dv = \int_B \left[(\sigma_{ji,j} + X_i - \varrho\dot{v}_i)v_i + \sigma_{ji}v_{i,j} - q_{i,i} + r - (\underline{E}\times\underline{H})_{i,i}\right] dv. \quad (6)$$

We have made use above of the Gauss transformation. Bearing in mind that Eq. (6) should hold for an arbitrary volume of the body we obtain the local equation

$$\varrho\dot{U} + \dot{U}_e = (\sigma_{ji,j} + X_i - \varrho\dot{v}_i)v_i + \sigma_{ji}v_{i,j} - q_{i,i} + \varrho r - (\underline{E}\times\underline{H})_{i,i}\,. \quad (7)$$

Consider first the expression U_e ; let us express the electromagnetic energy U_e in terms of the components of the electric and magnetic fields $\underline{E}$ and $\underline{H}$:

$$U_e = \frac{1}{2}(\varepsilon E^2 + \mu H^2)\,, \qquad E^2 = E_i E_i\,, \qquad H^2 = H_i H_i\,, \quad (8)$$

we assumed here that the constitutive equations for bodies which are isotropic and electrically and magnetic linear, have the form

$$\underline{D} = \varepsilon\underline{E}\,, \qquad \underline{B} = \mu\underline{H}\,, \quad (9)$$

Introducing the time derivative of the function U_e ,

$$\dot{U}_e = \varepsilon\underline{E}\dot{\underline{E}} + \mu\underline{H}\dot{\underline{H}} \quad (10)$$

into the Maxwell equations

$$\text{rot}\,\underline{H} = \underline{J} + \dot{\underline{D}}\,, \qquad \text{rot}\,\underline{E} = -\dot{\underline{B}}\,, \quad (11)$$

and the constitutive equations (9) we have

$$\varepsilon\dot{\underline{E}} = \text{rot}\,\underline{H} - \underline{J}\,, \qquad \mu\dot{\underline{H}} = -\,\text{rot}\,E\,. \quad (12)$$

Substituting (12) into the expression (10) we obtain

$$U_e = \underline{E}(\text{rot}\,\underline{H} - \underline{J}) - \underline{H}\,\text{rot}\,\underline{E} = -\,\text{div}(\underline{E}\times\underline{H}) - \underline{J}\underline{E}\,. \quad (13)$$

This quantity can be further transformed by means of the Ohm law : the latter, in electrodynamics of slowly moving bodies, has the form

$$\underline{J} = \sigma(\underline{E} + \underline{v} \times \underline{B} - \pi_0 \,\mathrm{grad}\, \theta), \qquad \theta = T - T_0 \tag{14}$$

The term containing $\underline{v}$ indicates that the current is modified by the deformation of the body. The classical Ohm law $\underline{J} = \sigma\underline{E}$ is modified by the term $\underline{v} \times \underline{B}$ and by the temperature flow. Multiplying Eq. (14) by $\underline{J}$ and solving for $\underline{J}\cdot\underline{E}$ we have

$$\underline{J}\underline{E} = \frac{1}{\sigma} J^2 + (\underline{J} \times \underline{B})\underline{v} - \pi_0 \underline{J}\,\mathrm{grad}\,\theta \tag{15}$$

The first term in the right-hand side is the power dissipated by means of the Joule heat. Introducing (15) into (13)

$$\dot{U}_e = -(\underline{E} \times \underline{H})_{i,i} - \frac{1}{\sigma^2} J^2 + (\underline{J} \times \underline{B})_i v_i - \pi_0 J_i \theta_{,i}\,, \tag{16}$$

and the result (16) into the energy balance (17) we obtain the equation

$$\varrho\dot{U} = (\sigma_{ji,j} + (\underline{J} \times \underline{B})_i + X_i - \varrho\dot{v}_i) v_i + \sigma_{ji} v_{i,j} + \varrho r -$$
$$- q_{i,i} + \frac{1}{\sigma} J^2 + \pi_0 J_i \theta_{,i}\,. \tag{17}$$

The energy balance (17) should be invariant with respect to a rigid translation of the body. Setting in (17)

$$\underline{v} \longrightarrow \underline{v} + \underline{b}\,, \qquad \underline{b} = \mathrm{const.}$$

and substracting the original equation we obtain for $\underline{b} \neq 0$ the first Cauchy equation of motion

$$\sigma_{ji,i} + (J \times B)_i + X_i - \varrho\dot{v}_i = 0\,. \tag{18}$$

We note that the expression $(\underline{J} \times \underline{B})_i$ is the component of the Lorentz force(1). The energy balance (17) also should be invariant with respect to the rigid rotation of the body. Setting therefore in (17)

$$\underline{v} \longrightarrow \underline{v} + \underline{\Omega} \times \underline{r}\,, \qquad v_{i,j} \longrightarrow v_{i,j} - \epsilon_{ijp}\Omega_p\,,$$

(1) If we take into account in deriving the expression (17), the flux of electric charge, then the Lorentz ponderomotive force has the form $f_i = \varrho_e \underline{E} + \underline{J} \times \underline{B}$

we arrive at the second Cauchy equation of motion

$$\sigma_{ij} = \sigma_{ji} . \tag{19}$$

Let us return to the energy balance (17). In view of the equations of motion (18), (19) it can be considerably simplified and there remains the expression

$$\varrho \dot{U} = \sigma_{ij}\dot{\varepsilon}_{ij} + \varrho r - q_{i,i} + \frac{1}{\sigma} J^2 + \pi_0 J_i \theta_{,i} \tag{20}$$

We add to the energy balance the Clausius-Duhem inequality

$$\varrho\dot{\eta} + \left(\frac{q_i}{T}\right)_{,i} - \frac{\varrho r}{T} \geq 0 , \tag{21}$$

where η is the entropy per unit mass. We introduce in (20) the free energy $\varrho\psi$ defined by the relation

$$\psi = U - \eta T . \tag{22}$$

Eliminating from Eq. (20) and the inequality (21) the term $q_{i,i}$ we obtain the inequality

$$\begin{aligned} &- \frac{\varrho}{T}(\dot{\psi} + \eta\dot{\theta}) + \frac{1}{T}(v_{i,j}\,\sigma_{ij}) + \\ &+ \frac{1}{T} J^2\sigma^{-1} + \pi_0 J_i \theta_i - \frac{q_i\theta_{,i}}{T} \geq 0 . \end{aligned} \tag{23}$$

We have considered so far the non-linear problem, assuming that the strain may be finite, now however we confine ourselves to the linear problem.

Assume that the free energy has the form

$$\psi = \psi(\dot{\varepsilon}_{ij}, \theta, \theta_{,k}) . \tag{24}$$

Since

$$\dot{\psi} = \frac{\partial\dot{\psi}}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\theta_{,k}}\dot{\theta}_{,k} , \tag{25}$$

we transform the inequality (23) to the form

$$\begin{aligned} &\frac{1}{T}\left[\left(\sigma_{ij} - \varrho\frac{\partial\psi}{\partial\varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} - \varrho\left(\eta + \frac{\partial\psi}{\partial\theta}\right)\dot{\theta} + \varrho\frac{\partial\psi}{\partial\theta_{,k}}\dot{\theta}_{,k}\right] + \\ &+ \frac{1}{T}\left(\sigma^{-1}J^2 + \pi_0 J_i\theta_{,i} - \frac{q_i\theta_{,i}}{T}\right) \geq 0 . \end{aligned} \tag{26}$$

It is postulated that the inequality (26) holds for all independent processes and the quantities $\dot{\varepsilon}_{ij}, \dot{\theta}$, $\dot{\theta}_k$ acquire independent variations. Since the inequality (26) is linear in all variables, we obtain

$$\sigma_{ij} = \varrho \frac{\partial \psi}{\partial \varepsilon_{ij}}, \qquad \eta = -\frac{\partial \psi}{\partial \theta}, \qquad \frac{\partial \psi}{\partial \theta_{,k}} = 0, \tag{27}$$

and

$$\sigma^{-1} J^2 + \pi_0 J_i \theta_{,i} - \frac{q_i \theta_{,i}}{T^2} \geq 0. \tag{28}$$

Consider first the free energy $\varrho\psi$ and the related constitutive equations. In accordance with (24) we assume that for a homogeneous isotropic body

$$\varrho\psi = \mu \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} \varepsilon_{kk}\varepsilon_{nn} - \gamma \varepsilon_{kk}\theta - \frac{n}{2}\theta^2. \tag{29}$$

Taking into account the relations (27) we have

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma\theta)\delta_{ij}, \qquad \gamma = (3\lambda + 2\mu)\alpha_t, \tag{30}$$

$$\varrho\eta = \gamma\varepsilon_{kk} + \frac{c_\varepsilon}{T_0}\theta, \tag{31}$$

where α_t is the coefficient of linear thermal expansion.

The inequality (28) is satisfied if

$$q_i = -k_0\theta_{,i} + \pi_0 J_i T \approx -k_0\theta_{,i} + \pi_0 J_i T_0. \tag{32}$$

The above relation constituting a generalization of the Fourier lax was given by Landau and Lifschitz[(1)]. The heat conduction equation follows from the entropy balance

$$\varrho T_0 \dot{\eta} = -q_{i,i} + \varrho r. \tag{33}$$

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(1) L.O. Landau and E.M. Lifschitz. Elektrodynamik der Kontinua. Akademie Verlag. Berlin. 1965.

In view of the constitutive relations (31) and (32) we obtain the equation

$$k_0 \nabla^2 \theta - c_\varepsilon \dot{\theta} - \gamma T_0 \dot{\varepsilon}_{kk} - \pi_0 T_0 J_{i,i} = - \varrho r \,. \tag{34}$$

Making use of the charge continuity equation

$$\operatorname{div} \underline{J} + \frac{\partial \varrho_\varepsilon}{\partial t} = 0 \,, \tag{35}$$

and eliminating from (34) and (35) the quantity $\operatorname{div} \underline{J}$ we arrive at the equation

$$(k_0 \nabla^2 - c_\varepsilon \partial_t)\theta - \hat{\eta} \operatorname{div} \dot{\underline{u}} + \pi_0 T_0 \dot{\varrho}_e = - \varrho r \,, \qquad \hat{\eta} = \gamma T_0 \,. \tag{36}$$

In what follows we assume that $\varrho_e = 0$. Then Eq. (36) takes the form analogous to that in coupled thermoelasticity

$$(\nabla^2 - \frac{1}{\varkappa} \partial_t)\theta - \eta \operatorname{div} \dot{u} = - \frac{\varrho r}{k} \,, \qquad \eta = \frac{\hat{\eta}}{k_0} \,, \qquad \varkappa = \frac{k_0}{c_\varepsilon} \,. \tag{37}$$

Consider the Maxwell equations with the constitutive relations (9)

$$\begin{aligned} \operatorname{div} \underline{H} &= \underline{J} + \varepsilon \dot{\underline{E}} \,, \qquad \operatorname{rot} \underline{E} = - \mu_e \dot{\underline{H}} \\ \operatorname{div} \underline{E} &= 0 \,, \qquad \operatorname{div} \underline{H} = 0 \,. \end{aligned} \tag{38}$$

Eliminating the vector $\underline{E}$ we have

$$(\nabla^2 - \beta \partial_t)\underline{H} - \beta_0 \partial_t^2 \underline{H} = - \beta \operatorname{rot}(v \times \underline{H}) \,, \qquad \beta = \mu_e \sigma \,, \quad \beta_0 = \mu_e \varepsilon \,. \tag{39}$$

In deriving the above equation we took into account the Ohm law (14). Introducing into the equations of motion (18) the constitutive relations (30) and the Lorentz force ($\underline{J} \times \underline{B}$), we obtain the vector form of the elasticity equations

$$\begin{aligned} \mu \nabla u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u + X + \mu_e \left[(\operatorname{rot} \underline{H} - \varepsilon \dot{\underline{E}} \times \underline{H} \right] &= \\ = \varrho \ddot{\underline{u}} + \gamma \operatorname{grad} \theta \,. \end{aligned} \tag{40}$$

Eqs. (37), (39) and (40) constitute the complete set of equations of magnetothermoelasticity.

These equations contain non-linear terms. they can however easily be linearized in the following particular case. Assume that the body is subject to a strong initial magnetic field $\underline{H}^{\circ}$ = const and the instant $t = 0$ the body is set into motion by mechanical or thermal causes ; then for $t > 0$,

$$\underline{H}(\underline{x},\underline{t}) = \underline{H}^{0} + \underline{h}(\underline{x},\underline{t}), \qquad \underline{E} = \underline{e}(\underline{x},\underline{t}), \qquad \underline{J} = \underline{j}(\underline{x},\underline{t}), \tag{41}$$

where $\underline{h}$, $\underline{e}$, $\underline{j}$ are small fluctuations. The Maxwell equations take the form

$$\begin{aligned} \mathrm{rot}\ \underline{h} &= \underline{j} + \varepsilon\dot{\underline{e}}, \qquad \mathrm{rot}\ \underline{e} = -\mu_e\dot{\underline{h}}, \\ \mathrm{div}\ \underline{h} &= 0, \qquad \mathrm{div}\ \underline{e} = 0. \end{aligned} \tag{42}$$

Linearizing (39) we have

$$(\nabla^2 - \beta\partial_t)\underline{h} + \beta_0\partial_t^2\underline{h} = -\beta\,\mathrm{rot}(\underline{v}\times\underline{H}^{0}). \tag{43}$$

The linearized equation of motion (40) has the form

$$\begin{aligned} \square_2\underline{u} + (\lambda+\mu)\,\mathrm{grad\ div}\ \underline{u} + \underline{X} + \mu_e\left[(\mathrm{rot}\ \underline{h} - \varepsilon\dot{\underline{e}})\times\underline{H}\right] &= \gamma\,\mathrm{grad}\,\theta, \\ \square_2 = \mu\nabla^2 - \varrho\partial_t^2. \end{aligned} \tag{44}$$

The only non-linear constitutive equation is the Ohm law; linearizing it we have

$$\underline{j} = \sigma\left[\underline{e} + \mu_e(\underline{v}\times\underline{H}^{\circ}) - \pi_0\,\mathrm{grad}\,\theta\right]. \tag{45}$$

If we regard the electrodynamic problem as quasistatic, we should neglect in Eq. (43) the term $\beta_0\ddot{\underline{h}}$ and in (44) the term $\varepsilon\dot{e}$. In this case the fundamental magnetothermoelasticity equations are the following :

$$(\nabla^2 - \frac{1}{\varkappa}\partial_t)\theta - \eta\,\mathrm{div}\,\dot{\underline{u}} = -\frac{\varrho r}{k_0}, \tag{46}$$

$$(\nabla^2 - \beta\partial_t)\underline{h} = -\beta\,\mathrm{rot}(\underline{v}\times\underline{H}^{\circ}), \tag{47}$$

$$\square_2\underline{u} + (\lambda+\mu)\,\mathrm{grad\ div}\ \underline{u} + \underline{X} + \mu_e\,\mathrm{rot}\,\underline{h}\times\underline{H}^{0} = \gamma\,\mathrm{grad}\theta \tag{48}$$

If a constant field of magnetic induction $\underline{B}^{\circ}$ is given, then

$$\underline{B} = \underline{B}^{0} + \underline{b}, \qquad \underline{E} = \underline{e},$$

Eqs. (46)–(48) take the form

$$(\nabla^2 - \frac{1}{\varkappa}\partial_t)\theta - \eta \,\mathrm{div}\, \dot{\underline{u}} = -\frac{\varrho r}{k_0}\,, \tag{46'}$$

$$(\nabla^2 - \beta\partial_t)\underline{b} = -\beta\,\mathrm{rot}(\underline{v} \times \underline{B}^\circ)\,, \tag{47'}$$

$$\Box_2 \underline{u} + (\lambda+\mu)\,\mathrm{grad\,div}\,\underline{u} + \underline{X} + \frac{1}{\mu_e}\,\mathrm{rot}\,\underline{b} \times \underline{B}^\circ = \gamma\,\mathrm{grad}\,\theta. \tag{48'}$$

The fundamental magnetothermoelasticity equations are considerably simplified in the case of a perfect electric conductor. Then $\sigma = \infty$, $\beta = \infty$ and Eq. (47′) takes the simpler form

$$\underline{b} = \mathrm{rot}\,(\underline{v} \times \underline{B}^\circ) \tag{49}$$

whereas Eq. (48′) is now

$$\Box_2 \underline{u} + (\lambda + \mu)\,\mathrm{grad\ div}\,\underline{u} + \underline{X} + \frac{1}{\mu_e}\left[\mathrm{rot\ rot}(\dot{\underline{u}} \times \underline{B}^\circ) \times \underline{B}^\circ\right] = \gamma\,\mathrm{grad}\,\theta \tag{50}$$

In this case only Eqs. (46′) and (50) are coupled.

2.4. Propagation of plane magnetoelastic wave

Consider the propagation of a plane magnetoelastic wave in an infinite space. In this particular case the system of the magnetoelasticity equations takes the form

$$\mu \nabla \underline{u} + (\lambda + \mu)\,\mathrm{grad\ div}\,\underline{u} + \frac{1}{\mu_e}(\mathrm{rot}\ \underline{b}) \times \underline{B}^\circ = \varrho \ddot{\underline{u}} \tag{1}$$

$$(\nabla^2 - \beta\partial_t)\underline{b} = -\beta\,\mathrm{rot}(\dot{\underline{u}} \times \underline{B}^\circ), \qquad \beta = \sigma\mu_e\,. \tag{2}$$

We consider a plane wave propagated in the direction of the x_1-axis. (1) and (2) yield the system of six equations

$$\left[(\lambda + 2\mu)\,\partial_1^2 - \varrho\,\partial_t^2\right]u_1 - \frac{1}{\mu_e}\,\partial_1(B_2^\circ b_2 - B_3^\circ b_3) = 0\,, \tag{3}$$

$$\left[\mu\partial_1^2 - \varrho\partial_t^2\right]u_2 + \frac{1}{\mu_e}\,B_1^\circ\partial_1 b_3 = 0\,, \tag{4}$$

$$\left[\mu\partial_1^2 - \varrho\partial_t^2\right] u_3 + \frac{1}{\mu_e} B_1^0 \partial_1 b_3 = 0 \tag{5}$$

and

$$b_1 = 0 \tag{6}$$

$$(\partial_1^2 - \beta\partial_t) b_2 = \beta\partial_1 \partial_t (B_2^0 u_1 - B_1^0 u_2), \tag{7}$$

$$(\partial_1^2 - \beta\partial_t) b_3 = -\beta\partial_1 \partial_t (B_1^0 u_3 - B_3^0 u_1) \tag{8}$$

Assuming that $b_3 = 0$ we find that $B_3^0 = 0$, $u_3 = 0$. There remains a system of three equations containing the function u_1, u_2, b_2:

$$\left[(\lambda + 2\mu)\partial_1^2 - \varrho\partial_t^2\right] u_1 - \frac{1}{\mu_e} B_2^0 \partial_1 b_2 = 0, \tag{9}$$

$$\left[\mu\partial_1^2 - \varrho\partial_t^2\right] u_2 + \frac{1}{\mu_e} B_1^0 \partial_1 b_2 = 0, \tag{10}$$

$$(\partial_1^2 - \beta\partial_t) b_2 - \beta\partial_1 \partial_t (B_2^0 u_1 - B_1^0 u_2) = 0. \tag{11}$$

For a monochromatic wave propagated in the direction of the x_1-axis we assume that

$$(u_1, u_2, b_2) = (u_1^0, u_2^0, b_2^0)\, e^{i(kx_1 - \omega t)} \tag{12}$$

Substituting (12) into Eqs. (9)–(11) we arrive at a system of three homogeneous equations. The condition of existence of a non-trivial solution is the frequency equation

$$\begin{vmatrix} k^2 - \sigma_1^2 & 0 & \frac{ik\epsilon_2}{B_2^0} \\ 0 & k^2 - \sigma_2^2 & -\frac{ik\epsilon_1}{B_1^0} \\ ikB_2^0 & -ikB_1^0 & ik^2\nu + 1 \end{vmatrix} = 0, \tag{13}$$

where we have introduced the notation

$$\sigma_1 = \frac{\omega}{c_1}, \qquad \sigma_2 = \frac{\omega}{c_2}, \qquad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2}, \qquad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2},$$

$$\epsilon_1 = \frac{(B_1^0)^2}{\varrho\mu_e c_2^2}, \qquad \epsilon_2 = \frac{(B_2^0)^2}{\varrho\mu_e c_1^2}, \qquad \nu = (\mu_e \sigma \omega)^{-1}. \tag{14}$$

Eq. (13) yields

$$(k^2 - \sigma_1^2)\left[(k^2 - \sigma_2^2)(ik^2\nu + 1) + k^2\epsilon_1\right] + k^2\epsilon_2(k^2 - \sigma_2^2) = 0. \tag{15}$$

This equation contains a number of particular cases which we now proceed to examine.

A) First we consider the particular case of a perfect conductor; then $\sigma = \infty$ $\beta = \infty$, $\nu = 0$ and Eq. (15) takes the simpler form

$$k^4(1 + \epsilon_1 + \epsilon_2) - k^2\left[\sigma_1^2(1 + \epsilon_1) + \sigma_2^2(1 + \epsilon_2)\right] + \sigma_1^2\sigma_2^2 = 0. \tag{16}$$

a) If the initial electromagnetic field is absent, e.e. when $B_1^0 = B_2^0$ and $\epsilon_1 = \epsilon_2 = 0$ then Eq. (16) yielus

$$(k^2 - \sigma_1^2)(k^2 - \sigma_2^2) = 0. \tag{17}$$

We are now faced with elastic waves u_1, u_2 unperturbed by the electromagnetic field. The longitudinal wave u_1 is propagated with the velocity c_1 and the transverse wave with the velocity c_2.

b) In the presence of the initial field $B_1^0 = 0$. $B_2^0 \neq 0$ Eq. (16) takes the simpler form

$$(k^2 - \sigma_2^2)\left[k^2(1 + \epsilon_2) - \sigma_1^2\right] = 0 \tag{18}$$

Now we have an unperturbed transverse wave u_2 propagated with the phase velocity $v = c_2$ and a perturbed longitudinal wave u_1.

The equation

$$k^2(1 + \epsilon_2) - \sigma_1^2 = 0$$

implies that

$$v = c_1(1 + \epsilon_2)^{1/2} \tag{19}$$

Thus, the velocity of the longitudinal wave is increased, for $\epsilon_2 > 0$.

c) If $B_1 \neq 0$ and $B_2 = 0$ for $\epsilon_2 = 0$ Eq. (16) leads to the relation

$$(k^2 - \sigma_1^2)(k^2(1 + \epsilon_1) - \sigma_2^2) = 0 \tag{20}$$

In this case the longitudinal wave u_1 is unperturbed and propagated with the phase velocity $v = c_1$. The transverse wave u_2 is perturbed by the electromagnetic field. The equation

$$k^2(1 + \epsilon_1) - \sigma_2^2) = 0$$

leads to the formula for the phase velocity of the transverse wave

$$v = c_2(1 + \epsilon_1)^{1/2} \tag{21}$$

d) In the case $\epsilon_1 \neq 0, \epsilon_2 \neq 0$ both waves, u_1 and u_2, are perturbed by the electromagnetic field. Eq. (16) has the solutions

$$\left.\begin{matrix} k_1^2 \\ k_2^2 \end{matrix}\right\} = \frac{1}{2(1 + \epsilon_1 + \epsilon_2)}\left[\sigma_1^2(1 + \epsilon_1) + \sigma_2^2(1 + \epsilon_2) \pm \sqrt{\Delta}\right], \tag{22}$$

where

$$\begin{aligned} \Delta &= \left[\sigma_1^2(1 + \epsilon_1) + \sigma_2^2(1 + \epsilon_2)\right]^2 - 4\sigma_1^2\sigma_2^2(1 + \epsilon_1\epsilon_2) \\ &= \left[\sigma_1^2(1 + \epsilon_1) - \sigma_2^2(1 + \epsilon_2)\right]^2 + 4\sigma_1^2\sigma_2^2(\epsilon_1 + \epsilon_2) > 0 \end{aligned}$$

It is evident that $k_1^2 > 0$, $k_2^2 > 0$ and that the solutions of the biquadratic equation are real. The solution of Eqs. (9)–(11) for $\sigma = \infty$ has the form

$$u_1 = Ae^{-i\omega(t - \frac{x_1}{v_1})} + Be^{-i\omega(t + \frac{x_1}{v_1})} + Ce^{-i\omega(t - \frac{x_1}{v_2})} + De^{-i\omega(t + \frac{x_1}{v_2})}, \tag{23}$$

where

$$v_1 = \omega/k_1, \qquad v_2 = \omega/k_2.$$

Observe that the quantities k_1, k_2 depend on the parameter ω. Thus, we are faced with dispersed waves. Let us now return to the frequency equation (15) and consider again some particular cases of wave propagation.

e) If the initial electromagnetic field is absent, ($B^{\circ} = B_2^{\circ} = 0$, $\epsilon_1 = \epsilon_2 = 0$), Eq. (15) takes the form

$$(k^2 - \sigma_1^2)(k^2 - \sigma_2^2)(ik^2\nu + 1) = 0 \tag{24}$$

We now have three independent waves. The longitudinal wave u_1 is propagated with the velocity c_1 and the transverse wave u_2 with the velocity c_2 . The case

$$ik^2\nu + 1 = 0 , \tag{25}$$

represents quasistatic oscillations of the electromagnetic field, which are not coupled with the displacement $\underline{u}$.

f) Consider the case $B_1^{\circ} = 0$, $B_2^{\circ} \neq 0$. Then the characteristic equation (15) takes the simpler form

$$(k^2 - \sigma_2^2)\left[\nu k^4 - k^2(\nu\sigma_1^2 + i(1 + \epsilon_2)) + i\sigma_1^2\right] = 0 . \tag{26}$$

It is evident that the transverse wave is unperturbed by the electromagnetic field, while the longitudinal wave u_1 and the wave b_2 are propagated with the velocity $v = \omega/k$. The quantity k satisfies the equation

$$\nu k^4 - k^2\left[\nu\sigma_1^2 + i(1 + \epsilon_2)\right] + i\sigma_1^2 = 0 \tag{27}$$

The solutions $k_{1,2}$ are complex. Therefore the longitudinal wave u_1 and the wave u_2 are dispersed and damped. The phase velocity v_α and the damping coefficients ϑ_α are determined from the formulae

$$v_\alpha = \frac{\omega}{\mathrm{Re}(k_\alpha)} , \qquad \vartheta_\alpha = \mathrm{Jm}(k_\alpha), \qquad \alpha = 1,2 \tag{28}$$

The solution of the considered equations for the waves u_1 and b_2 has the form

$$\begin{aligned} u_1 = {} & A\exp\left[-i\omega(t - \frac{x_1}{v_1}) - \vartheta_1 x_1\right] + B\exp\left[-i\omega(t + \frac{x_1}{v_1}) + \vartheta_1 x_1\right] + \\ & + C\exp\left[-i\omega(t - \frac{x_1}{v_2}) - \vartheta_2 x_1\right] + D\exp\left[-i\omega(t + \frac{x_1}{v_2}) + \vartheta_2 x_1\right] . \end{aligned} \tag{29}$$

and

$$b_2 = \hat{A}\exp\left[-i\omega(t-\frac{x_1}{v_1})-\vartheta_1 x_1\right] + \hat{B}\exp\left[-i\omega(t+\frac{x_1}{v_1})+\vartheta_1 x_1\right] + \\ + \hat{C}\exp\left[-i\omega(t-\frac{x_1}{v_2})-\vartheta_2 x_1\right] + \hat{D}\exp\left[-i\omega(t+\frac{x_1}{v_2})+\vartheta_2 x_1\right] . \tag{30}$$

The relations between the constants $A, B, \ldots$ and $\hat{A}, \hat{B}, \ldots$ is deduced from Eq. (9).

g) The case $B_1 \neq 0$, $B_2 = 0$ leads to the characteristic equation

$$(k^2 - \sigma_1^2)\left[(k^2 - \sigma_2^2)(ik^2\nu + 1) + \epsilon_1 k^2\right] = 0 \tag{31}$$

We find therefore that the longitudinal wave u_1 is unperturbed by the electromagnetic field, while the waves u_2 and b_2 are coupled. The phase velocity of these waves and the damping coefficient are determined from the formulae (28) and the quantities k are found from the equation

$$\nu k^4 - k^2(\nu\sigma_2^2 + i(1+\epsilon_1) + i\sigma_2^2 = 0 \tag{32}$$

The solution of (32) are complex : consequently, the waves u_2 and b_2 are damped and since the $k's$ depend on the frequency ω , the waves are dispersed. The form of the waves u_2 and b_2 is the same as (29) as (30).

h) In the most general case $B_1 \neq 0$, $B_2 \neq 0$ we are faced with coupled waves u_1 , u_2 , b_2 . The quantity k is determined form Eq. (15). Since the solutions are complex and depend on ω the waves are both damped and dispersed.

There is no difficulty in generalizing the considered solution to thermo magnetoelastic media with a finite electric conductivity. A detailed exposition of the problem is presented in a paper by A.J. Wilson[1].

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1) A.J. Wilson. The propagation of magnetothermoelastic plane waves. Proc. Camb. Phil. Soc. 59 (1963), p. 483.

2.5. Two-dimensional problems of magnetothermoelasticity

Consider first an elastic medium with a perfect electric conductivity. The motion of this medium is described by the system of differential equations

$$\mu\nabla^2\underline{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\underline{u}+\frac{1}{\mu_e}(\mathrm{rot}\,\underline{b})\times\underline{B}^0+\underline{X}=\varrho\ddot{\underline{u}}+\gamma\,\mathrm{grad}\,\theta \tag{1}$$

$$(\nabla^2-\frac{1}{\varkappa}\partial_t)\theta-\eta\,\mathrm{div}\,\dot{\underline{u}}=-\frac{Q}{\varkappa}, \tag{2}$$

$$\underline{b}=\mathrm{rot}(\underline{u}\times\underline{B}^0). \tag{3}$$

We assume that the vector of magnetic induction $\underline{B}$ is parallel to the x_3-axis, i.e.$(\underline{B}=(0,0,B_3))$. Then, taking into account the relation (3), Eqs. (1) take the form

$$\begin{aligned}
&\mu\nabla^2u_1+(\lambda+\mu+a_0^2\varrho)\,\partial_1e+a_0^2\varrho\partial_3(\partial_3u_1-\partial_1u_3)+X_1=\gamma\,\partial_1\theta+\varrho\ddot{u}_1,\\
&\mu\nabla^2u_2+(\lambda+\mu+a_0^2\varrho)\,\partial_2e+a_0^2\varrho\partial_3(\partial_3u_2-\partial_2u_3)+X_2=\gamma\,\partial_2\theta+\varrho\ddot{u}_2\\
&\mu\nabla^2u_3+(\lambda+\mu)\,\partial_3e+\qquad\qquad+X_3=\varrho\ddot{u}_3
\end{aligned} \tag{4}$$

where we have introduced the notation

$$e=\partial_1u_1+\partial_2u_2+\partial_3u_3,\qquad a_0^2=\frac{B_3^2}{\varrho\mu_e}=c_1^2\epsilon_3,\qquad \nabla_1^2=\partial_1^2+\partial_2^2+\partial_3^2,$$

Eqs. (4) should be completed by the equation of heat conduction (2). The system of equations (14) is symmetric with respect to the diagonal and its structure is the same as that of the system of equations for an anisotropic body with transverse isotropy. This anisotropy vanishes when $a_0\to 0$.

In what follows we confine ourselves to the two-dimensional problem, assuming that all functions are independent of the variable x_3 . Thus, our system of equations is decomposed into two independent systems

$$\begin{aligned}
&\mu\nabla_1^2u_1+(\lambda+\mu+a_0^2\varrho)\,\partial_1e+X_1=\gamma\,\partial_1\theta+\varrho\ddot{u}_1,\\
&\mu\nabla_1^2u_2+(\lambda+\mu+a_0^2\varrho)\,\partial_2e+X_2=\gamma\,\partial_2\theta+\varrho\ddot{u}_2,
\end{aligned} \tag{5}$$

$$(\nabla_1^2 - \frac{1}{\varkappa}\partial_t)\theta - \eta\dot{e} = -\frac{Q}{\varkappa},$$

and

$$\mu\nabla_1^2 u_3 + X_3 = \varrho\ddot{u}_3 . \tag{6}$$

The last equation is unperturbed by the electromagnetic field and therefore we shall not investigate it here.

Eqs. (5) describe the plane state of strain $\underline{u} = (u_1, u_2, 0)$ and as $a_0 \to 0$ they become the coupled thermoelasticity equations.

Let us differentiate Eq. $(5)_1$ with respect to x_1, Eq. $(5)_2$ with respect to x_2 and add the result. Thus, we arrive at the wave equation for the dilatation e :

$$\hat{\Box}_1^2 e + \frac{1}{\varrho\hat{c}^2}(\partial_1 X_1 + \partial_2 X_2) - m_1\nabla_1^2\theta = 0, \tag{7}$$

Here

$$\Box_1^2 = \nabla_1^2 - \frac{1}{\hat{c}_1^2}\partial_t , \quad \hat{c}_1^2 = c_1^{\prime 2}(1+\epsilon_3), \quad m_1 = \gamma/\varrho\hat{c}_1^2 .$$

Eq. (7) is coupled with the heat conduction equation $(5)_3$; eliminating the temperature from Eqs. $(5)_3$ and (7) we arrive at a complicated wave equation, namely

$$(D\hat{\Box}_1^2 - m_1\eta\partial_t\nabla_1^2)e = -\frac{1}{\varrho\hat{c}_1^2}D(\partial_1 X_1 + \partial_2 X_2) - \frac{m_1}{\varkappa}\nabla_1^2 Q, \qquad D = \nabla_1^2 - \frac{1}{\varkappa}\partial_t \tag{8}$$

The coupling between the dilatation **e** and the electromagnetic field is due to the presence of the qunatity ϵ_3 Making use of Eq. $(5)_{1,2}$ it is readily observed that

$$\Box_2^2\Omega = -\frac{1}{\varrho c_2^2}(\partial_1 X_2 - \partial_2 X_1), \tag{9}$$

where

$$\Omega = \partial_1 u_2 - \partial_2 u_1 ,$$
$$\Box_2^2 = \nabla_1^2 - \frac{1}{c_2^2}\partial_1^2 , \qquad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}. \tag{10}$$

The function Ω describes rotation about axis x_3. It is evident that the propagation of the torsional wave Ω is unperturbed by the temperature and electromagnetic fields.

Let us decompose the vectors $u = (u_1, u_2, 0)$ and $X = (X_1, X_2, 0)$ into the potential and solenoidal parts

$$\left.\begin{aligned} u_1 &= \partial_1\Phi - \partial_2\Psi, \quad u_2 = \partial_2\Phi + \partial_1\Psi, \\ X_1 &= \varrho(\partial_1\vartheta - \partial_2\chi), \quad X_2 = \varrho(\partial_2\vartheta + \partial_1\chi). \end{aligned}\right\} \tag{11}$$

Substituting (11) into the system of equations (5) we obtain a system of three wave equations, two of which are coupled, namely

$$\left.\begin{aligned} \hat{\Box}_1^2\Phi - m_1\theta &= -\frac{1}{\hat{c}_1^2}\vartheta, \\ D\theta - \eta\partial_t\nabla_1^2\Phi &= -\frac{Q}{\varkappa}, \end{aligned}\right\} \tag{12}$$

$$\Box_2^2\Psi = -\frac{1}{c_2^2}\chi. \tag{13}$$

Eliminating the temperature from Eqs. (12) we arrive at the wave equation

$$(\hat{\Box}_1^2 D - \eta m_1\partial_t\nabla_1^2)\Phi = -\frac{m_1}{\varkappa}Q - \frac{1}{\hat{c}_1^2}D\vartheta, \tag{14}$$

describing the propagation of a longitudinal wave, while Eq. (13) describes the transverse wave. The longitudinal wave is perturbed by both temperature and electromagnetic fields, whereas the transverse wave is not. Observe that the form of the wave equations (14) is analogous to that of the wave equation of thermoelasticity. The thermoelasticity equation is obtained when $a_0 \to 0$.

The knowledge of the functions Φ, Ψ makes it possible to calculate the remaining electromagnetic and thermal quantities. The temperature is obtained from Eq. $(12)_1$:

$$\theta = \frac{1}{m_1}(\hat{\Box}_1^2\Phi + \frac{1}{\hat{c}_1^2}\vartheta), \tag{15}$$

and the quantities $\underline{b}, \underline{j}, \underline{E}$ from the formulae

$$\underline{b} = \text{rot}(\underline{u} \times \underline{B}^0), \qquad \mu_e\underline{j} = \text{rot}\,\underline{b}, \qquad \underline{E} = -\underline{\dot{u}} \times \underline{B}^0. \tag{16}$$

Thus, we have

$$\underline{b} = (0,0,-B_3\nabla_1^2\Phi), \qquad \mu_e\underline{j} = (-B_3\partial_2\nabla_1^2\Phi,\ B_3\partial_1\nabla_1^2\Phi, 0)$$
$$\underline{E} = \left(-B_3(\partial_2\dot{\Phi} + \partial_1\dot{\Psi}),\ B_3(\partial_1\dot{\Phi} - \partial_2\dot{\Psi}), 0\right) \tag{17}$$

Consider a particularly simple example of an action of a linear heat source in the infinite elastic plane. Assume that there acts along the x_3 -axis the heat source $Q(r,t) = Q_0 e^{-i\omega t}\delta(r)/2\pi r$. The considered problem is axisymmetric. Eq. (14) takes the form

$$\left[(\nabla_r^2 - \frac{1}{\hat{c}_1^2}\partial_t^2)(\nabla_r^2 - \frac{1}{\varkappa}\partial_t) - m_1\eta\partial_t\nabla_r^2\right]\Phi = -\frac{m_1}{\varkappa}Q_0\frac{\delta(r)}{2\pi r}e^{-i\omega t}, \tag{18}$$

where

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Applying the Hankel integral transform we obtain a closed solution of Eq. (18)

$$\Phi(r,t) = \frac{Q_0 m_1 e^{-i\omega t}}{4\varkappa(k_2^2 - k_1^2)}\left[H_0^{(1)}(k_1 r) - H_0^{(1)}(k_2 r)\right]. \tag{19}$$

Here k_1 . k_2 are the solutions of the equation

$$k^4 - k^2(\hat{\sigma}_1^2 + q(1+\epsilon_r)) + q\hat{\sigma}_1^2 = 0\,, \tag{20}$$

where

$$\hat{\sigma}_1^2 = \frac{c_1^2}{\hat{c}_1^2}, \qquad \epsilon_r = \eta m_1\varkappa, \qquad q = \frac{i\omega}{\varkappa}\,.$$

We are interested in the real part of the solution (19). The temperature is determined on the basis of Eq. (15) (for $\vartheta = 0$):

$$\theta = \frac{i\,Q_0 e^{-i\omega t}}{4\varkappa(k_2^2 - k_1^2)}\left[(\hat{\sigma}_1^2 - k_1^2)H_0^{(1)}(k_1 r) - (\hat{\sigma}_1^2 - k_2^2)H_0^{(1)}(k_2 r)\right]. \tag{21}$$

Consider now the propagation of waves in an infinite elastic medium with a finite electric conductivity ($\sigma \neq 0$). The motion in this medium obeys the system of equations

$$\begin{aligned} &\mu \nabla^2 \underline{u} + (\lambda + \mu)\, \mathrm{grad\,div}\, \underline{u} + \frac{1}{\mu_e} (\mathrm{rot}\, \underline{b}) \times \underline{B}^0 + \underline{X} = \varrho \underline{\ddot{u}} + \gamma\, \mathrm{grad}\, \theta \;, \\ &(\nabla^2 - \beta \partial_t) \underline{b} = -\beta\, \mathrm{rot}(\underline{\dot{u}} \times \underline{B}^0) \;, \\ &(\nabla^2 - \frac{1}{\varkappa} \partial_t) \theta - \eta\, \mathrm{div}\, \underline{\dot{u}} = -\frac{Q}{\varkappa} \;. \end{aligned} \tag{22}$$

In the case of the presence of an initial magnetic induction $\underline{B} = (0, 0, B_3)$ and in the plane state of strain $\underline{u} = (u_1, u_2, 0)$ Eqs. (22) take the form

$$\begin{aligned} &\mu \nabla_1^2 u_1 + (\lambda + \mu) \partial_1 e - \gamma \partial_1 \theta - \frac{B_3}{\mu_e} \partial_1 b_3 + X_1 = \varrho \ddot{u}_1 \;, \\ &\mu \nabla_1^2 u_2 + (\lambda + \mu) \partial_2 e - \gamma \partial_2 \theta_2 - \frac{B_3}{\mu_e} \partial_2 b_3 + X_2 = \varrho \ddot{u}_2 \\ &(\nabla_1^2 - \beta \partial_t) b_3 = \beta B_3 \dot{e} \;, \qquad b_1 = b_2 = 0 \;, \\ &(\nabla^2 - \frac{1}{\varkappa} \partial_t) \theta - \eta \dot{e} = -\frac{Q}{\varkappa} \;. \end{aligned} \tag{23}$$

Introducing into Eqs. (23) the representation (11) we arrive at a system of four equations, three of which are coupled

$$\begin{aligned} &\Box_1^2 \Phi - m\theta - \frac{\epsilon_3}{B_3} b_3 = -\frac{1}{c_1^2} \vartheta \;, \\ &D_1 b_3 - \beta B_3 \partial_t \nabla_1 \Phi = 0 \;, \\ &D\theta - \eta \partial_t \nabla_1^2 \Phi = -\frac{Q}{\varkappa} \;, \qquad D_1 = \nabla_1^2 - \beta \partial_t, \qquad D = \nabla^2 - \frac{1}{\varkappa} \partial_t, \qquad \Box_1^2 = \nabla_1^2 - \frac{1}{c_1^2} \partial_t^2 \end{aligned} \tag{24}$$

and an independent equation

$$\Box_2^2 \Psi = -\frac{1}{c_2^2} \chi \;. \tag{25}$$

Eliminating from Eqs. (24) the functions b_3 and θ we have the following equation for the potential Φ :

$$(\Box_1^2 D_1 D - \frac{1}{\varkappa}\partial_t \nabla_1^2(\epsilon D_1 + \hat{\epsilon}_3 D))\Phi = -\frac{1}{c_1^2} D D_1 \vartheta - \frac{m}{\varkappa} D_1 Q \tag{26}$$

Here

$$\epsilon = \eta m \varkappa , \qquad \hat{\epsilon}_3 = \beta \varkappa t_3$$

The knowledge of the function Φ makes it possible to determine the functions b_3 and θ from Eqs. $(24)_{2,3}$.

Observe that these functions satisfy equations of the type (20) with a different right-hand side. Thus, we have the equations

$$(\Box_1^2 D_1 D - \frac{1}{\varkappa}\partial_t \nabla_1^2(\epsilon D_1 + \hat{\epsilon}_3 D))\theta = -\frac{1}{\varkappa}(D_1 \Box_1^2 - \alpha\beta\partial_t) Q - \frac{\eta}{c_1^2}\partial_t \nabla_1^2 D_1 \vartheta \tag{27}$$

$$(\Box_1^2 D_1 D - \frac{1}{\varkappa}\partial_t \nabla_1^2(\epsilon D_1 + \hat{\epsilon}_3 D)) b_3 = \frac{m\beta B_3}{\varkappa}\partial_t \nabla_1^2 Q - \frac{\beta}{c_1^2} B_3 \partial_t \nabla_1^2 \vartheta . \tag{28}$$

The equations are considerably simplified in the absence of heat sources and for adiabatic processes. In this case the temperature is determined from the equation

$$\theta = -\eta\varkappa e = -\eta\varkappa\nabla_1^2\Phi \tag{29}$$

In view of (29) we obtain from (24) a system of coupled equations

$$\begin{aligned} &\bar{\Box}_1^2\Phi - \bar{\epsilon}_3 \frac{b_3}{B_3} = -\frac{1}{\bar{c}_1^2}\vartheta , \\ &D_1 b_3 - \beta B_3 \partial_t \nabla_1^2 \Phi = 0 , \end{aligned} \tag{30}$$

where

$$\bar{c}_1^2 = c_1^2(1+\epsilon) , \qquad \hat{\epsilon}_3 = \frac{B_3^2}{\varrho\mu_e\bar{c}_1^2} , \qquad \bar{\Box}_1^2 = \nabla_1^2 - \frac{1}{\bar{c}_1^2}\partial_t .$$

Eliminating from $(30)_{1,2}$ the function b_3 we have

$$(\bar{\Box}_1^2 D_1 - \beta\bar{\epsilon}_3\partial_t\nabla_1^2)\Phi = -\frac{1}{\bar{c}_1^2} D_1 \vartheta . \tag{31}$$

Similarly, eliminating from $(30)_{1,2}$ the function Φ we are led to the equation

$$(\Box_1^2 D_1 - \beta \bar{\epsilon}_3 \partial_t \nabla_1^2)\, b_3 = -\frac{\beta B_3}{\bar{c}_1^2} \partial_t \nabla_1^2 \vartheta \,. \tag{32}$$

Both the longitudinal wave Φ and the wave b_3 are damped and dispersed.