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APPLIED MECHANICS

Three-dimensional Problem of Micropolar Theory of Elasticity

by

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Summary. Static and dynamic equations in displacements and rotations are transformed into equations comprising only displacements or rotations. The differential equations of the title problem derived in such a way are particularly convenient when we try to obtain singular solutions. The effect of the temperature field on deformation of the body in question is considered.

1. Static problem. The point of departure of our considerations are equilibrium equations in displacements and rotations [1, 2]

$$(\mu+\alpha)\,\nabla^2\,\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\varphi}+\mathbf{X}=0, \tag{1.1}$$

$$[(\gamma+\varepsilon)\,\nabla^2-4\alpha]\,\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\varphi}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=0 \tag{1.2}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varphi}$, the rotational vector, $\mathbf{X}$, the vector of body forces and $\mathbf{Y}$, the vector of body couples.

Quantities $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon$ are material constants of the micropolar medium.

Let us perform divergence operation on Eqs. (1.1) and (1.2). Thus we obtain two relations:

$$\nabla^2\,\mathrm{div}\,\mathbf{u}=-\frac{1}{\lambda+2\mu}\,\mathrm{div}\,\mathbf{X}, \tag{1.3}$$

$$H\,\mathrm{div}\,\boldsymbol{\varphi}=-\frac{1}{\beta+2\gamma}\,\mathrm{div}\,\mathbf{Y}, \tag{1.4}$$

where

$$H=\nabla^2-\frac{1}{\nu^2},\qquad \nu^2=\frac{\beta+2\gamma}{4\alpha}.$$

It should be noted that when the body force ($\mathbf{X}=0$) is missing the dilatation $e=\mathrm{div}\,\mathbf{v}$ is the harmonic function, and function $f=\mathrm{div}\,\boldsymbol{\varphi}$ satisfies the Helmholtz equation, for $\mathbf{Y}=0$.

Let us perform then a rotation operation on Eqs. (1.1) and (1.2). Hence we obtain two other relations, viz.

$$(\mu+\alpha)\,\nabla^2\,\mathrm{rot}\,\mathbf{u}+2\alpha\,\mathrm{rot\,rot}\,\boldsymbol{\varphi}=-\mathrm{rot}\,\mathbf{X}, \tag{1.5}$$

$$\big((\gamma+\varepsilon)\,\nabla^2-4\alpha\big)\,\mathrm{rot}\,\boldsymbol{\varphi}+2\alpha\,\mathrm{rot\,rot}\,\mathbf{u}=-\mathrm{rot}\,\mathbf{Y}. \tag{1.6}$$

Proceeding we carry out the operation $\nabla^2 \mathscr{D}$, where $\mathscr{D}=(\gamma+\varepsilon)\nabla^2-4\alpha$. Making use of the relations (1.3) and (1.6), we arrive after a circuit of transformations at the equation in which only the displacements take place

$$D\nabla^2\nabla^2\mathbf{u}=\frac{1}{2\alpha\mu l^2}\Big(2\alpha\nabla^2\operatorname{rot}\mathbf{Y}-(\gamma+\varepsilon)\nabla^2 G\mathbf{X}-\\ -\frac{4\alpha^2}{\lambda+2\mu}\operatorname{grad}\operatorname{div}\mathbf{X}+\frac{(\lambda+\mu-\alpha)(\gamma+\varepsilon)}{\lambda+2\mu}G\operatorname{grad}\operatorname{div}\mathbf{X}\Big). \tag{1.7}$$

We have introduced further notations, namely

$$D=\nabla^2-\frac{1}{l^2},\qquad G=\nabla^2-\frac{1}{\kappa^2},\qquad l^2=\frac{(\mu+\alpha)(\gamma+\varepsilon)}{4\alpha\mu},\qquad \kappa^2=\frac{\gamma+\varepsilon}{4\alpha}.$$

Again, perform the operation $\nabla^2 H$ on Eq. (1.2), by help of the relations (1.4) and (1.5). In result of omitting function $\mathbf{u}$, we derive the differential equation for the rotation $\boldsymbol{\varphi}$:

$$\nabla^2 HD\boldsymbol{\varphi}=\frac{1}{4\alpha\mu l^2}\Big[2\alpha H\operatorname{rot}\mathbf{X}-(\mu+\alpha)H\nabla^2\mathbf{Y}-\\ -\frac{4\alpha^2}{\beta+2\gamma}\operatorname{grad}\operatorname{div}\mathbf{Y}+\frac{(\beta+\gamma-\varepsilon)(\mu+\alpha)}{\beta+2\gamma}\nabla^2\operatorname{grad}\operatorname{div}\mathbf{Y}\Big]. \tag{1.8}$$

Consider now two limit cases. For Hooke's body Eq. (1.7) takes the form

$$\nabla^2\nabla^2\mathbf{u}=-\nabla^2\mathbf{X}+\frac{\lambda+\mu}{\lambda+2\mu}\operatorname{grad}\operatorname{div}\mathbf{X}. \tag{1.9}$$

If body forces are not coming into play, then the displacements are harmonic functions.

The second limit case concerns a body in which only rotations may be expected. The deformations occurring in this body are described by Eq. (1.2), into which we have to insert $\alpha=0$. In virtue of (1.8) we get

$$(\gamma+\varepsilon)\nabla^2\nabla^2\boldsymbol{\varphi}=-\nabla^2\mathbf{Y}+\frac{\beta+\gamma-\varepsilon}{\beta+2\gamma}\operatorname{grad}\operatorname{div}\mathbf{Y}. \tag{1.10}$$

If the body couples are missing, then function $\boldsymbol{\varphi}$ becomes the biharmonic function.

Eqs. (1.7) and (1.8) may be suitable to determine singular solutions of the system of Eqs. (1.1) and (1.2), that is the solutions dependent on the distance between two points $\mathbf{x}$ and $\mathbf{x}'$.

We now consider the elementary case when $\mathbf{X}=0$. Then Eq. (1.7) reduces in the case of an infinite region which is considered in this note to the equation

$$\nabla^2 D\mathbf{u}=\frac{1}{2\mu l^2}\operatorname{rot}\mathbf{Y}. \tag{1.11}$$

This equation is expressed in terms of the components of the displacement u_i and the body couple Y_i. We obtain

$$\nabla^2 D u_i = \frac{1}{2\mu l^2} \epsilon_{ijk} Y_{k,j}. \tag{1.12}$$

Solution of the above-written equation is possible *via* the Fourier integral transform. It will take the form

$$u_i = \frac{\epsilon_{ijk}}{2\mu l^2 (2\pi)^{3/2}} \frac{\partial}{\partial x_j} \iiint_{-\infty}^{\infty} \frac{\tilde{Y}_k e^{-i x_s \xi_s}}{\zeta^2 \left(\zeta^2 + \frac{1}{l^2}\right)} d\xi_1 \, d\xi_2 \, d\xi_3, \tag{1.13}$$
$$\zeta^2 = \xi_1^2 + \xi_2^2 + \xi_3^2.$$

For a concentrated body couple, applied to the origin of the coordinate system and directed towards the x_p-axis, we obtain the following

$$Y_k = \delta(\mathbf{x}) \delta_{kp}, \qquad \tilde{Y}_k = \frac{1}{(2\pi)^{3/2}} \delta_{kp}. \tag{1.14}$$

Putting (1.14) into (1.13), we have, on carrying out the integration:

$$u_i^{(p)} = -\frac{1}{8\pi\mu} \varepsilon_{ijp} \frac{\partial}{\partial x_j} \left(\frac{e^{-R/l} - 1}{R} \right), \qquad R = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \tag{1.15}$$

in agreement with the result obtained in [3] by another method.

Analogically to the case $\mathbf{Y}=0$, $\mathbf{X} \neq 0$ is to be derived by the use of (1.8):

$$\nabla^2 D \boldsymbol{\varphi} = \frac{1}{2\mu l^2} \operatorname{rot} \mathbf{X}, \tag{1.16}$$

which, for $X_k = \delta(\mathbf{x}) \delta_{kp}$, yields

$$\varphi_i^{(p)} = -\frac{1}{8\pi\mu} \epsilon_{ijp} \frac{\partial}{\partial x_j} \left(\frac{e^{-R/l} - 1}{R} \right), \tag{1.17}$$

which is compatible with the theorem on the reciprocity of works.

In turn, consider the case when body forces and body couples are missing, $\mathbf{X}=0$, $\mathbf{Y}=0$. In this case we can obtain the homogeneous equations

$$\nabla^2 \nabla^2 D \mathbf{u} = 0, \qquad \nabla^2 D H \boldsymbol{\varphi} = 0. \tag{1.18}$$

Solution of these equations may, according to Boggio's theorems, be expressed in the form of the sum of partial solutions

$$\mathbf{u} = \mathbf{u}' + \mathbf{u}'', \qquad \boldsymbol{\varphi} = \boldsymbol{\varphi}' + \boldsymbol{\varphi}'' + \boldsymbol{\varphi}''', \tag{1.19}$$

where now

$$\begin{gathered} \nabla^2 \nabla^2 \mathbf{u}' = 0, \qquad D \mathbf{u}'' = 0, \\ \nabla^2 \boldsymbol{\varphi}' = 0, \qquad D \boldsymbol{\varphi}'' = 0, \qquad H \boldsymbol{\varphi}''' = 0. \end{gathered} \tag{1.20}$$

These equations find practical application for solving the boundary problems, For instance, they may be used in a case considering the elastic half-space $x_3 \geqslant 0$, and when Fourier's exponential integral transform is applied

$$(1.21)\qquad \begin{cases} f(x_1, x_2, x_3) = \dfrac{1}{2\pi} \displaystyle\iint_{-\infty}^{\infty} \tilde{f}(\xi_1, \xi_2, x_3)\, e^{-i(\xi_1 x_1 + \xi_2 x_2)}\, d\xi_1\, d\xi_2, \\ \tilde{f}(\xi_1, \xi_2, x_3) = \dfrac{1}{2\pi} \displaystyle\iint_{-\infty}^{\infty} f(x_1, x_2, x_3)\, e^{i(\xi_1 x_1 + \xi_2 x_2)}\, dx_1\, dx_2, \end{cases}$$

Eq. (1.20) may be read as follows

$$(1.22)\qquad \begin{gathered} (\partial_3^2 - \zeta^2)^2\, \tilde{\mathbf{u}}' - = 0, \quad (\partial_3^2 - \eta^2)\, \tilde{\mathbf{u}}'' = 0, \\ (\partial_3^2 - \zeta^2)\, \tilde{\boldsymbol{\varphi}}' = 0 \quad (\partial_3^2 - \eta^2)\, \tilde{\boldsymbol{\varphi}}'' = 0, \quad (\partial_3^2 - \tau^2)\, \tilde{\boldsymbol{\varphi}}''' = 0, \\ \zeta^2 = \zeta_1^2 + \zeta_2^2, \quad \eta^2 = \zeta^2 + \frac{1}{l^2}, \quad \tau^2 = \zeta^2 + \frac{1}{\nu^2}. \end{gathered}$$

Eqs. (1.19) being set in the terms of Fourier's transform will take the form

$$(1.23)\qquad \begin{gathered} \tilde{\mathbf{u}} = \mathbf{A} e^{-\xi x_3} + \mathbf{B} x_3\, \zeta\, e^{-\xi x_3} + \mathbf{C}\, e^{-\eta x_3}, \\ \tilde{\boldsymbol{\varphi}} = \mathbf{D} e^{-\xi x_3} + \mathbf{E} e^{-\eta x_3} + \mathbf{F} e^{-\tau x_3}. \end{gathered}$$

We have obtained here eighteen integration constants. Twelve additional relations can be obtained if we substitute functions $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\varphi}}$, on which we applied the Fourier transform before, into Eqs. (1.1), (1.2). By equating the relevant coefficients at the same exponential function we get the sought-for relations occurring among the coefficients **A**, ..., **F**.

2. Dynamic problem. We adopt here similar reasoning as before and employ the same procedure with reference to the dynamic problem. Equations of motion in displacements and rotations have the form:

$$(2.1)\qquad \Box_2\, \mathbf{u} + (\lambda + \mu - \alpha)\, \mathrm{grad\,div}\, \mathbf{u} + 2\alpha\, \mathrm{rot}\, \boldsymbol{\varphi} + \mathbf{X} = 0,$$

$$(2.2)\qquad \Box_4\, \boldsymbol{\varphi} + (\beta + \gamma - \varepsilon)\, \mathrm{grad\,div}\, \boldsymbol{\varphi} + 2\alpha\, \mathrm{rot}\, \mathbf{u} + \mathbf{Y} = 0.$$

Here we introduce the following notation:

$$\Box_2 = (\mu + \alpha)\, \nabla^2 - \rho \partial_t^2, \qquad \Box_4 = (\gamma + \varepsilon)\, \nabla^2 - 4\alpha - I \partial_t^2$$

in which ρ denotes density, while I is rotational inertia. Perform now the divergence operation on Eqs. (2.1) and (2.2). Hence we obtain

$$(2.3)\qquad \Box_1\, \mathrm{div}\, \mathbf{u} = -\mathrm{div}\, \mathbf{X},$$

$$(2.4)\qquad \Box_3\, \mathrm{div}\, \boldsymbol{\varphi} = -\mathrm{div}\, \mathbf{Y}.$$

We have set up here the wave operators:

$$\Box_1 = (\lambda + 2\mu)\, \nabla^2 - \rho \partial_t^2, \qquad \Box_3 = (\beta + 2\gamma)\, \nabla^2 - 4\alpha - I \partial_t^2.$$

Further, perform the rotation operation on Eqs. (2.1) and (2.2). Thereupon we obtain two other relations, viz.

$$\Box_2 \operatorname{rot} \mathbf{u} + 2\alpha \operatorname{rot} \operatorname{rot} \boldsymbol{\varphi} = -\operatorname{rot} \mathbf{X}, \tag{2.5}$$

$$\Box_4 \operatorname{rot} \boldsymbol{\varphi} + 2\alpha \operatorname{rot} \operatorname{rot} \mathbf{u} = -\operatorname{rot} \mathbf{Y}. \tag{2.6}$$

Carrying out the operation $\Box_1 \Box_4$ on (2.1), and bearing in mind the relations (2.3) and (2.6), we arrive at the equation which contains the displacement only

$$\Box_1 (\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \mathbf{u} = 2\alpha \Box_1 \operatorname{rot} \mathbf{Y} - \Box_1 \Box_4 \mathbf{X} - 4\alpha^2 \operatorname{grad} \operatorname{div} \mathbf{X} + \\ + (\lambda + \mu - \alpha) \Box_4 \operatorname{grad} \operatorname{div} \mathbf{X}. \tag{2.7}$$

Eq. (2.2) is acted on analogically with the operator $\Box_3 \Box_4$, and the relations (2.4) and (2.5) are taken into account; this operation leads to the equation in which only the rotations can occur

$$\Box_3 (\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \boldsymbol{\varphi} = 2\alpha \Box_3 \operatorname{rot} \mathbf{X} - \Box_2 \Box_3 \mathbf{Y} - 4\alpha^2 \operatorname{grad} \operatorname{div} \mathbf{Y} + \\ + (\beta + \gamma - \varepsilon) \Box_2 \operatorname{grad} \operatorname{div} \mathbf{Y}. \tag{2.8}$$

Consider now some special cases. For Hooke's body ($\alpha = 0$) we have

$$\Box_1 \mathring{\Box}_2 \mathbf{u} = -\Box_1 \mathbf{X} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{X}, \quad \mathring{\Box}_2 = \mu \nabla^2 - \rho \partial_t^2. \tag{2.9}$$

When body forces are absent we can obtain the well-known wave equation (cf. [5])

$$\left(\nabla^2 - \frac{1}{c_1^2} \partial_t^2\right)\left(\nabla^2 - \frac{1}{c_2^2} \partial_t^2\right) \mathbf{u} = 0, \qquad c_1 = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}, \qquad c_2 = \mu \rho^{1/2}. \tag{2.10}$$

The other special case in question is the wave motion in a medium in which only rotations are possible. Assuming $\alpha = 0$ in Eq. (2.8), we derive the equations which become

$$\mathring{\Box}_3 \mathring{\Box}_4 \boldsymbol{\varphi} = -\mathring{\Box}_3 Y + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \mathbf{Y}. \tag{2.11}$$

Here

$$\mathring{\Box}_3 = (\beta + 2\gamma) \nabla^2 - I \partial_t^2, \quad \mathring{\Box}_4 = (\gamma + \varepsilon) \nabla^2 - I \partial_t^2.$$

If the body couples are missing, Eq. (2.11) becomes the biwave equation

$$\left(\nabla^2 - \frac{1}{c_3^2} \partial_t^2\right)\left(\nabla^2 - \frac{1}{c_4^2} \partial_t^2\right) \boldsymbol{\varphi} = 0, \quad c_3 = \left(\frac{\beta + 2\gamma}{I}\right)^{1/2}, \quad c_4 = \left(\frac{\gamma + \varepsilon}{I}\right)^{1/2}, \tag{2.12}$$

of the same character of propagation of elastic waves as Eq. (2.10).

Eqs. (2.7) and (2.8) may readily be adapted to determine the required singular solutions. Consider now an elementary case, i.e. the solution of Eq. (2.7), assuming that $\mathbf{X} = 0$. It remains merely to consider the equation:

$$(\Box_2 \Box_4 + 4\alpha^2 \nabla^2) \mathbf{u} = 2\alpha \operatorname{rot} \mathbf{Y}. \tag{2.13}$$

In the case of occurrence of forced harmonic vibrations, when $Y = \mathbf{Y}(\mathbf{x})\, e^{i\omega t}$ and $\mathbf{u} = \mathbf{u}^{+}(x)\, e^{i\omega t}$, Eq. (2.13) becomes the elliptic differential equation

$$(2.14)\qquad (\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\, \mathbf{u}^* = \frac{2\alpha}{(\mu+\alpha)(\gamma+\varepsilon)} \operatorname{rot} \mathbf{Y}^*.$$

Here k_1^2, k_2^2 are the roots of the equation:

$$(2.15)\qquad k^4 - k^2\left(\sigma_2^2 + \sigma_4^2 + p(s-2)\right) + \sigma_2^2(\sigma_4^2 - 2p) = 0,$$

where

$$\sigma_2 = \frac{\omega}{c_2}, \qquad \sigma_4 = \frac{\omega}{c_4}, \qquad s = \frac{2\alpha}{\mu+\alpha}, \qquad p = \frac{2\alpha}{\gamma+\varepsilon}.$$

The roots

$$\left.\begin{matrix} k_1^2 \\ k_2^2 \end{matrix}\right\} = \frac{1}{2}\left(\sigma_2^2 + \sigma_4^2 + p(s-2) \pm \sqrt{\left(\sigma_4^2 - \sigma_2^2 + p(s-2)\right)^2 + 4ps\sigma_2^2}\right)$$

are the real values, because the discriminant is positive.

The integral Fourier transform is applied to Eq. (2.14); we obtain thus the following amplitude

$$(2.16)\qquad u_i^* = -\frac{2\alpha p}{\rho c_2^2 (2\pi)^{3/2}}\, \epsilon_{ijk} \int\!\!\int\!\!\int_{-\infty}^{\infty} \frac{i\xi_j\, \tilde{Y}_k\, e^{-ix_s\xi_s}}{(\zeta^2 - k_1^2)(\zeta^2 - k_2^2)}\, d\zeta_1\, d\zeta_2\, d\zeta_3 .$$

$$\zeta^2 = \zeta_1^2 + \zeta_2^2 + \xi_3^2 .$$

For the concentrated moment, placed at the origin of the coordinate system and directed towards the x_l-axis, we obtain

$$(2.17)\qquad u_j^{(l)} = \frac{p\epsilon_{ljk}}{4\pi c_2^2 \rho\, (k_1^2 - k_2^2)}\, \frac{\partial}{\partial x_k}\left(\frac{e^{ik_1 R} - e^{ik_2 R}}{R}\right),$$

in conformity to the well-known result (cf. [5]).

Consider, finally, the case of homogeneous equations (2.7) and (2.8). We have

$$(2.18)\qquad \Box_1 (\Box_2 \Box_4 + 4\alpha^2 \nabla^2)\, \mathbf{u} = 0,$$

$$(2.19)\qquad \Box_3 (\Box_2 \Box_4 + 4\alpha^2 \nabla^2)\, \boldsymbol{\varphi} = 0.$$

Following the Boggie theorem, the solution of these wave equations will be arranged as a pair, viz.

$$(2.20)\qquad \mathbf{u} = \mathbf{u}' + \mathbf{u}'', \qquad \boldsymbol{\varphi} = \boldsymbol{\varphi}' + \boldsymbol{\varphi}'',$$

where the components of these solutions satisfy the wave equations

$$(2.21)\qquad \Box_1 \mathbf{u}' = 0, \qquad (\Box_2 \Box_4 + 4\alpha^2 \nabla^2)\, \mathbf{u}'' = 0,$$

as well as

$$(2.22)\qquad \Box_3 \boldsymbol{\varphi}' = 0, \qquad (\Box_2 \Box_4 + 4\alpha^2 \nabla^2)\, \boldsymbol{\varphi}'' = 0.$$

In the case of occurrence of harmonic vibrations, Eqs. (2.18), (2.19) take the form

$$(\nabla^2+k_1^2)(\nabla^2+k_2^2)(\nabla^2+\sigma^2)\,\mathbf{u}^*=0, \tag{2.23}$$

$$(\nabla^2+k_1^2)(\nabla^2+k_2^2)(\nabla^2+k_3^2)\,\boldsymbol{\varphi}^*=0, \tag{2.24}$$

in which

$$\sigma=\frac{\omega}{c_1},\qquad k_3=\left(\frac{\omega^2-\omega_0^2}{c_3^2}\right)^{1/2},\qquad \omega_0^2=\frac{4\alpha}{I},\qquad c_3=\left(\frac{\beta+2\gamma}{I}\right)^{1/2}.$$

These give

$$\mathbf{u}^*=\mathbf{u}'^*+\mathbf{u}''^*+\mathbf{u}'''^*,\qquad \boldsymbol{\varphi}^*=\boldsymbol{\varphi}'^*+\boldsymbol{\varphi}''^*+\boldsymbol{\varphi}'''^*, \tag{2.25}$$

where

$$(\nabla^2+k_1^2)\,\mathbf{u}'^*=0,\qquad (\nabla^2+k_2^2)\,\mathbf{u}''^*=0,\qquad (\nabla^2+\sigma^2)\,\mathbf{u}'''^*=0, \tag{2.26}$$

and

$$(\nabla^2+k_1^2)\,\boldsymbol{\varphi}'^*=0,\qquad (\nabla^2+k_2^2)\,\boldsymbol{\varphi}''^*=0,\qquad (\nabla^2+k_3^2)\,\boldsymbol{\varphi}'''^*=0. \tag{2.27}$$

Solution of such a type is convenient for us if we want to investigate the problem of the elastic half-space $x_3 \geqslant 0$. In agreement with (2.25) we assume

$$\tilde{\mathbf{u}}^*=\mathbf{A}e^{-\eta x_3}+\mathbf{B}e^{-\chi x_3}+\mathbf{C}e^{-\tau x_3}, \tag{2.28}$$

$$\tilde{\boldsymbol{\varphi}}^*=\mathbf{D}e^{-\beta x_3}+\mathbf{E}e^{-\chi x_3}+\mathbf{F}e^{-\tau x_3}, \tag{2.29}$$

where

$$\eta=(\zeta^2-\sigma^2)^{1/2},\qquad \chi=(\zeta^2-k_1^2)^{1/2},\qquad \tau=(\zeta^2-k_2^2)^{1/2},$$
$$\beta=(\zeta^2-k_3^2)^{1/2},\qquad \zeta^2=\xi_1^2+\xi_2^2.$$

In these solutions, written up in the Fourier transforms, eighteen constants occur. We eliminate them by help of Eqs. (2.1) and (2.2). Finally, we obtain six independent constants which correspond to as many boundary conditions.

3. Problem of thermoelasticity. Let us consider the effect of temperature upon deformation process of the body in question. Equations governing the thermoelasticity problems for the stationary flow of heat have the following form [2]

$$(\mu+\alpha)\,\nabla^2\,\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\varphi}=\eta\,\mathrm{grad}\,\theta, \tag{3.1}$$

$$[(\gamma+\varepsilon)\,\nabla^2-4\alpha]\,\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\varphi}+2\alpha\mathrm{rot}\,\mathbf{u}=0. \tag{3.2}$$

Here $\theta=T-T_0$ denotes temperature increment; T, absolute temperature at the point $\mathbf{x}$ and instant t; T_0, temperature of the natural state of the body. The thermal term η grad θ appears only in the first equation.

We may use here of the equations derived at the point 1, for $\mathbf{u}$ and $\boldsymbol{\varphi}$, taking advantage of the analogy of forces and body couples. This gives

$$\mathbf{X}=-\eta\,\mathrm{grad}\,\theta,\qquad \mathbf{Y}=0. \tag{3.3}$$

Introducing the latter into Eq. (1.7), we get

$$\nabla^2 \mathbf{u} = \frac{\eta}{\lambda+2\mu} \operatorname{grad} \theta, \quad \boldsymbol{\varphi} = 0. \tag{3.4}$$

This equation, of course, holds for the infinite elastic space, for only in this case are we allowed to shorten bilaterally the differential operators.

If we introduce the elastic potential of displacement $\mathbf{u} = \operatorname{grad} \Phi$, then Eq. (3.4) becomes the Poisson equation for the functions:

$$\nabla^2 \Phi = m\theta, \quad \boldsymbol{\varphi} = 0, \tag{3.5}$$

$$m = \frac{\eta}{\lambda+2\mu}.$$

It is interesting to take note of the fact that the temperature field does not induce any rotations in the infinite elastic region.

In the case of the dynamic problem we have to make use of the system of equations given in [2]:

$$\Box_2 \mathbf{u} + (\lambda+\mu-\alpha) \operatorname{grad} \operatorname{div} \mathbf{u} + 2\alpha \operatorname{rot} \boldsymbol{\varphi} = \eta \operatorname{grad} \theta, \tag{3.6}$$

$$\Box_4 \boldsymbol{\varphi} + (\beta+\gamma-\varepsilon) \operatorname{grad} \operatorname{div} \boldsymbol{\varphi} + 2\alpha \operatorname{rot} \mathbf{u} = 0, \tag{3.7}$$

and the equation of thermal conductivity

$$\left(\nabla^2 - \frac{1}{\kappa} \partial_t\right) \theta - \chi \operatorname{div} \mathbf{u} = 0. \tag{3.8}$$

Here $\kappa = \lambda_0 / c_\varepsilon$, where λ_0 is the coefficient of thermal conductivity, while c_ε — specific heat at constant deformation. After introduction of the following equation

$$\mathbf{X} = -\eta \operatorname{grad} \theta, \quad \mathbf{Y} = 0, \tag{3.9}$$

into Eqs. (2.7) and (2.8), we may write

$$\Box_1 \mathbf{u} = \eta \operatorname{grad} \theta, \tag{3.10}$$

$$\mathscr{H} \theta = \chi \operatorname{div} \dot{\mathbf{u}}, \quad \boldsymbol{\varphi} = 0, \quad \mathscr{H} = \nabla^2 - \frac{1}{\kappa} \partial_t. \tag{3.11}$$

Elimination of temperature from Eqs. (3.10) and (3.11) gives rise to the equation

$$(\Box_1 \mathscr{H} - \eta\chi \partial_t \nabla^2) \mathbf{u} = \eta \operatorname{grad} \theta. \tag{3.12}$$

If we set up, as in the stationary problem, the potential of thermoelastic displacement $\mathbf{u} = \operatorname{grad} \Phi$, we can re-write Eq. (3.12) in the form

$$(\Box_1 \mathscr{H} - \eta\chi\partial_t \nabla^2) \Phi = \eta\theta, \quad \boldsymbol{\varphi} = 0. \tag{3.13}$$

It can readily be seen that the shape of this equation coincides with that for Hooke's body.

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В. Новацки, **Трехмерная задача микрополярной теории упругости**

Содержание. Целью работы является такое преобразование дифференциальных уравнений микрополярной упругости, чтобы получить уравнения, в которых выступают только перемещения либо только повороты (вращения). Такие решения могут оказаться полезными при определении особых решений в бесконечном упругом пространстве, а также при решении краевых задач. Аналогичные уравнения для перемещений и поворотов (вращений) представлены также по отношению к задаче термоупругости в микрополярной среде.