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(E L A S T I C I T Y)

The "Second" Axial-symmetric Problem in Micropolar Elasticity

by

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Summary. This note is concerned with the so-called "second" axial-symmetric problem in micropolar elasticity, expressed by a displacement vector $\mathbf{u}=(0, u_\theta, 0)$ and the rotation vector $\varphi=(\varphi_r, 0, \varphi_z)$. The above-mentioned components of vectors $\mathbf{u}$ and $\boldsymbol{\varphi}$ should satisfy the system of Eqs. (1.4). Introduction of elastic potentials Φ and Ψ permits to reduce the solution of this system of Eqs. (1.4) to the solution of ordinary elliptic equations (2.4). Functions Φ and Ψ are not independent of each other; they have to fulfil additional relations (2.3). Practical application of elastic potentials is exemplified by means of the semi-infinite elastic space which is loaded by moments.

1. Introduction

The state of deformation of micropolar body is described in the form of the system of six equations. These equations having vectorial form are constructed as follows [1—4]

$$(1.1)\qquad \begin{aligned} &(\mu+a)\,\nabla^2\mathbf{u}+(\lambda+\mu-a)\,\mathrm{grad\,div}\,\mathbf{u}+2a\,\mathrm{rot}\,\boldsymbol{\varphi}+\mathbf{X}=0,\\ &(\gamma+\varepsilon)\,\nabla^2\boldsymbol{\varphi}-4a\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\varphi}+2a\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=0.\end{aligned}$$

Let $\mathbf{u}$ be here a displacement vector, $\boldsymbol{\varphi}$ a rotation vector. Let $\mathbf{X}$ stand for a vector of body forces and $\mathbf{Y}$ a body couple vector. Quantities $a, \beta, \gamma, \varepsilon, \mu, \lambda$ denote material constants.

After introducing Eqs. (1.1) into the system of cylindrical coordinates, and taking the problem as the axial-symmetric problem, we may derive from Eq. (1) two independent of each other systems of equations. In the first take place the following components of vector $\mathbf{u}, \boldsymbol{\varphi}, \mathbf{X}, \mathbf{Y}$:

$$(1.2)\qquad \mathbf{u}=(u_r, 0, u_z),\quad \boldsymbol{\varphi}=(0, \varphi_\theta, 0),\quad \mathbf{X}=(X_r, 0, X_z),\quad \mathbf{Y}=(0, Y_\theta, 0).$$

In the second system of equations the remaining components of vectors $\mathbf{u}, \boldsymbol{\varphi}, \mathbf{X}, \mathbf{Y}$ occur

$$(1.3)\qquad \mathbf{u}=(0, u_\theta, 0),\quad \boldsymbol{\varphi}=(\varphi_r, 0, \varphi_z),\quad \mathbf{X}=(0, X_\theta, 0),\quad \mathbf{Y}=(Y_r, 0, Y_z).$$

Equations expressing displacements and rotations related with this state of deformation take the form:

$$(1.4)\qquad \begin{cases} (\gamma+\varepsilon)\left(\nabla^2-\dfrac{1}{r^2}\right)\varphi_r-4a\varphi_r+(\beta+\gamma-\varepsilon)\dfrac{\partial\kappa}{\partial r}-2a\dfrac{\partial u_\theta}{\partial z}+Y_r=0, \\[2ex] (\gamma+\varepsilon)\,\nabla^2\,\varphi_z-4a\varphi_z+(\beta+\gamma-\varepsilon)\dfrac{\partial\kappa}{\partial z}+\dfrac{2a}{r}\,\dfrac{\partial}{\partial r}(ru_\theta)+Y_z=0, \\[2ex] (\mu+a)\left(\nabla^2-\dfrac{1}{r^2}\right)u_\theta+2a\left(\dfrac{\partial\varphi_r}{\partial z}-\dfrac{\partial\varphi_z}{\partial r}\right)+X_\theta=0. \end{cases}$$

Introduce here the notations:

$$\kappa=\frac{1}{r}\,\frac{\partial}{\partial r}(r\varphi_r)+\frac{\partial\varphi_z}{\partial z},\qquad \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

In the "second" axial-symmetric problem the following components of stress tensor take place

$$(1.5)\qquad \boldsymbol{\sigma}=\begin{pmatrix} 0 & \sigma_{r\theta} & 0 \\ \sigma_{\theta r} & 0 & \sigma_{\theta z} \\ 0 & \sigma_{z\theta} & 0 \end{pmatrix},\qquad \boldsymbol{\mu}=\begin{pmatrix} \mu_{rr} & 0 & \mu_{rz} \\ 0 & \mu_{\theta\theta} & 0 \\ \mu_{zr} & 0 & \mu_{zz} \end{pmatrix}.$$

These stresses are given by the following equations [4]:

$$(1.6)\qquad \begin{cases} \sigma_{r\theta}=\mu\left(\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right)+\dfrac{a}{r}\,\dfrac{\partial}{\partial r}(ru_\theta)-2a\varphi_z, \\[2ex] \sigma_{\theta r}=\mu\left(\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right)-\dfrac{a}{r}\,\dfrac{\partial}{\partial r}(ru_\theta)+2a\varphi_z, \\[2ex] \sigma_{\theta z}=(\mu-a)\,\dfrac{\partial u_\theta}{\partial z}-2a\varphi_r, \\[2ex] \sigma_{z\theta}=(\mu+a)\,\dfrac{\partial u_\theta}{\partial z}+2a\varphi_r, \end{cases}$$

as well as

$$(1.7)\qquad \begin{cases} \mu_{rr}=2\gamma\,\dfrac{\partial\varphi_r}{\partial r}+\beta\kappa,\qquad \mu_{\theta\theta}=2\gamma\,\dfrac{\varphi_r}{r}+\beta\kappa, \\[2ex] \mu_{zz}=2\gamma\,\dfrac{\partial\varphi_z}{\partial z}+\beta\kappa, \\[2ex] \mu_{rz}=\gamma\left(\dfrac{\partial\varphi_z}{\partial r}+\dfrac{\partial\varphi_r}{\partial z}\right)-\varepsilon\left(\dfrac{\partial\varphi_r}{\partial z}-\dfrac{\partial\varphi_z}{\partial r}\right), \\[2ex] \mu_{zr}=\gamma\left(\dfrac{\partial\varphi_z}{\partial r}+\dfrac{\partial\varphi_r}{\partial z}\right)+\varepsilon\left(\dfrac{\partial\varphi_r}{\partial z}-\dfrac{\partial\varphi_z}{\partial r}\right). \end{cases}$$

In turn, let us consider a homogeneous system of Eqs. (1.4).

2. Elastic potentials

Two functions Φ, Ψ related to ratotions φ_1, φ_2 and displacement u_3 are now introduced in the following manner:

$$\text{(2.1)}\quad \varphi_r=\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Psi}{\partial r\,\partial z},\qquad \varphi_z=\frac{\partial\Phi}{\partial z}-\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\Psi,\qquad u_\theta=-\frac{2a}{\mu+a}\,\frac{\partial\Psi}{\partial r}.$$

Substituting (2.1) into homogeneous equations into displacement and rotations (1.4) and using of the identity

$$\text{(2.2)}\qquad \left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial\Phi}{\partial r}=\frac{\partial}{\partial r}\nabla^2\Phi$$

permit for derivation, from Eqs. $(1.4)_{1,2}$, the relations

$$\text{(2.3)}\qquad \begin{aligned}(\nu^2\nabla^2-1)\,\Phi&=-\frac{\mu}{\mu+a}\,\frac{\partial}{\partial z}(l^2\nabla^2-1)\,\Psi,\\ \frac{\partial}{\partial z}(\nu^2\nabla^2-1)\,\Phi&=\frac{\mu}{\mu+a}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)(l^2\nabla^2-1)\,\Psi.\end{aligned}$$

Further, we introduce the notations:

$$\nu^2=\frac{2\gamma+\beta}{4a},\qquad l^2=\frac{(\mu+a)(\gamma+\varepsilon)}{4\mu a}.$$

Substitution of relations (2.1) into Eqs. $(1.4)_3$ fulfils this equation in identical way.

Relations (2.3) yield, for functions Φ and Ψ, the following equation

$$\text{(2.4)}\qquad \left\{\begin{aligned}&\nabla^2(\nu^2\nabla^2-1)\,\Phi=0,\\ &\nabla^2(l^2\nabla^2-1)\,\Psi=0.\end{aligned}\right.$$

Insertion of (2.1) into the equations for stresses enables us to express these stresses by functions Φ and Ψ:

$$\text{(2.5)}\qquad \left\{\begin{aligned}&\sigma_{r\theta}=\frac{4a\mu}{\mu+a}\,\frac{1}{r}\,\frac{\partial\Psi}{\partial r}-2a\frac{\partial\Phi}{\partial z},\qquad \sigma_{\theta r}=-\frac{4a\mu}{\mu+a}\,\frac{\partial^2\Psi}{\partial r^2}+2a\frac{\partial\Phi}{\partial z},\\ &\sigma_{\theta z}=-\frac{4a\mu}{\mu+a}\,\frac{\partial^2\Psi}{\partial z\,\partial r}-2a\frac{\partial\Phi}{\partial r},\qquad \sigma_{z\theta}=2a\,\frac{\partial\Phi}{\partial r},\end{aligned}\right.$$

and

$$\text{(2.6)}\qquad \left\{\begin{aligned}&\mu_{rr}=2\gamma\frac{\partial^2}{\partial r^2}\left(\Phi+\frac{\partial\Psi}{\partial z}\right)+\beta\nabla^2\Phi,\qquad \mu_{\theta\theta}=2\gamma\frac{1}{r}\,\frac{\partial}{\partial r}\left(\Phi+\frac{\partial\Psi}{\partial z}\right)+\beta\nabla^2\Phi,\\ &\mu_{zz}=2\gamma\left[\frac{\partial^2\Phi}{\partial z^2}-\frac{\partial}{\partial z}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\Psi\right]+\beta\nabla^2\Phi,\\ &\mu_{zr}=\frac{\partial}{\partial r}\left[2\gamma\left(\frac{\partial\Phi}{\partial z}+\frac{\partial^2\Psi}{\partial z^2}\right)-(\gamma-\varepsilon)\,\nabla^2\Psi\right],\\ &\mu_{rz}=\frac{\partial}{\partial r}\left[2\gamma\left(\frac{\partial\Phi}{\partial z}+\frac{\partial^2\Psi}{\partial z^2}\right)-(\gamma+\varepsilon)\,\nabla^2\Psi\right].\end{aligned}\right.$$

Observe that

$$\mu_{rr}+\mu_{\theta\theta}+\mu_{zz}=(2\gamma+3\beta)\nabla^2\Phi, \quad \mu_{zr}-\mu_{rz}=2\varepsilon\nabla^2\Psi. \tag{2.7}$$

The procedure for solving the "second" axial-symmetric problem will be investigated by virtue of exampled a semi-infinite elastic body, which is loaded by moments $m(r)$. In this case we are dealing with the following conditions

$$\sigma_{z\theta}(r,0)=0, \quad \mu_{zr}(r,0)=0, \quad \mu_{zz}(r,0)=-m(r). \tag{2.8}$$

Here the moment vector m is oriented in the direction of the positive z-axis. Suppose now that, for $|r^2+z^2|\to\infty$ both the displacements, the rotations and the stresses should disappear.

We start from differential equations (2.4). Performing on these equations Hankel's transformation, which is defined by the relations:

$$\begin{aligned} \bar{f}(\zeta,z)&=\int_0^\infty rf(r,z)J_0(\zeta r)\,dr,\\ f(r,z)&=\int_0^\infty \zeta\bar{f}(\zeta,z)J_0(\zeta r)\,d\zeta, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \tilde{f}(\zeta,z)&=\int_0^\infty rf(r,z)J_1(\zeta r)\,dr,\\ f(r,z)&=\int_0^\infty \zeta\tilde{f}(\zeta,r)J_1(\zeta r)\,d\zeta, \end{aligned} \tag{2.10}$$

we arrive *via* (2.4) at the ordinary differential equations

$$\begin{aligned} &\left(\frac{d^2}{dz^2}-\zeta^2\right)\left(\frac{d^2}{dz^2}-\eta^2\right)\bar{\Phi}=0,\\ &\left(\frac{d^2}{dz^2}-\zeta^2\right)\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\Psi}=0. \end{aligned} \tag{2.11}$$

We have introduced here the notations:

$$\eta=\zeta^2+\frac{1}{\nu^2}, \quad \rho^2=\zeta^2+\frac{1}{l^2}.$$

The functions quoted below constitute the solutions of Eqs. (2.11)

$$\bar{\Psi}=Ae^{-\zeta z}+Be^{-\rho z}, \quad \bar{\Phi}=Ce^{-\zeta z}+De^{-\eta z}. \tag{2.12}$$

Functions Ψ and Φ being not independent functions of each other, they have to fulfil relations (2.3). Carrying out of Hankel's transformations on these relations brings us to

$$\begin{aligned} \left(\frac{d^2}{dz^2}-\eta^2\right)\bar{\Phi}&=-\frac{\mu}{\mu+a}\,\frac{l^2}{\nu^2}\,\frac{d}{dz}\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\Psi},\\ \frac{d}{dz}\left(\frac{d^2}{dz^2}-\eta^2\right)\bar{\Phi}&=-\frac{\mu}{\mu+a}\,\frac{l^2}{\nu^2}\,\zeta^2\left(\frac{d^2}{dz^2}-\rho^2\right)\bar{\Psi}. \end{aligned} \tag{2.13}$$

Both the relations may be satisfied by means of the solutions (2.12) only then when

$$C=\frac{\mu}{\mu+a}\,\zeta A\,. \tag{2.14}$$

Proceeding we have to determine still three remaining constants of integration. We can obtain them on making use of boundary conditions (2.8).

By taking advantage of relations (2.5), (2.6), expressing boundary conditions by functions Φ and Ψ, and, finally, drawing on the boundary conditions the Hankel transformation, we get three equations

$$\left\{\begin{aligned} &\tilde{\sigma}_{z\theta}(\zeta,0)=[-2a\zeta\bar{\Phi}]_{z=0}=0,\\ &\tilde{\mu}_{zr}(\zeta,0)=\left\{-\zeta\left[2\gamma\left(\frac{\partial\bar{\Phi}}{\partial z}+\frac{\partial^2\bar{\Psi}}{\partial z^2}\right)-(\gamma-\varepsilon)\left(\frac{d^2}{dz^2}-\zeta^2\right)\bar{\Psi}\right]\right\}_{z=0}=0,\\ &\bar{\mu}_{zz}(\zeta,0)=\left[2\gamma\left(\frac{\partial^2\bar{\Phi}}{\partial z^2}+\zeta^2\frac{d\bar{\Psi}}{dz}\right)+\beta\left(\frac{d^2}{dz^2}-\zeta^2\right)\bar{\Phi}\right]_{z=0}=-\bar{m}(\zeta). \end{aligned}\right. \tag{2.15}$$

From Eqs. (2.15) follow the cons tants of integration

$$\left\{\begin{aligned} &A=\frac{\bar{m}(\zeta)(\zeta^2+a_0)}{2\gamma\zeta\Delta_0(\zeta)},\\ &B=-\frac{\zeta\bar{m}(\zeta)}{2\gamma\Delta_0(\zeta)}\left[1+\frac{\mu}{\mu+a}\left(\frac{\eta}{\zeta}-1\right)\right],\\ &D=-\frac{\mu}{\mu+a}\,\zeta A,\qquad C=\frac{\mu}{\mu+a}\,\zeta A. \end{aligned}\right. \tag{2.16}$$

Now, we introduce the notations

$$a_0=\frac{2a\mu}{\gamma(\mu+a)},\qquad \Delta_0(\zeta)=(\zeta^2+a_0)^2-\zeta\rho\zeta^2\left[1+\frac{\mu}{\mu+a}\left(\frac{\eta}{\zeta}-1\right)\right].$$

In turn, let us perform, on relations (2.1) the Hankel transformation. Accordingly, we obtain here:

$$\tilde{\varphi}_r=-\zeta\left(\bar{\Phi}+\frac{d\bar{\Psi}}{dz}\right),\qquad \bar{\varphi}_z=\frac{d\bar{\Phi}}{dz}+\zeta^2\bar{\Psi},\qquad \tilde{u}_\theta=\frac{2a}{\mu+a}\,\zeta^2\bar{\Psi}. \tag{2.17}$$

Performation of the inverse transformation and introduction of constants of integration lead to equations in the form

$$\left\{\begin{aligned} \varphi_r=\frac{1}{2\gamma}\int_0^\infty \zeta^2\frac{\bar{m}}{\Delta_0}\left\{\frac{a}{\mu+a}(\zeta^2+a_0)\left[e^{-\zeta z}+\frac{\mu}{a}e^{-\eta z}\right]+\right.\\ \left.-\rho\zeta\left[1+\frac{\mu}{\mu+a}\left(\frac{\eta}{\zeta}-1\right)\right]e^{-\rho z}\right\}J_1(\zeta)\,d\zeta, \end{aligned}\right. \tag{2.18}$$

$$
(2.18)\quad \left\{
\begin{aligned}
\varphi_z &= \frac{1}{2\gamma}\int_0^\infty \zeta^2 \frac{\bar m}{\Delta_0}\left\{\frac{a}{\mu+a}(\zeta^2+a_0)\left[e^{-\zeta z}+\frac{\mu}{a}e^{-\eta z}\right]+\right.\\
&\qquad\left. -\zeta^2\left[1+\frac{\mu}{\mu+a}\left(\frac{\eta}{\zeta}-1\right)\right]e^{-\rho z}\right\}J_0(\zeta)\,d\zeta,\\
u_\theta &= \frac{a_0}{2\mu}\int_0^\infty \zeta\frac{\bar m}{\Delta_0}\left\{(\zeta^2+a_0)e^{-\zeta z}-\zeta^2\left[1+\frac{\mu}{\mu+a}\left(\frac{\eta}{\zeta}-1\right)\right]e^{-\rho z}\right\}J_1(\zeta r)\,d\zeta.
\end{aligned}
\right.
$$

The stresses sought for will be determined on taking into account (2.5), (2.6). Furthermore, the stresses occurring in the boundary conditions (2.8) are given:

$$
(2.19)\quad \left\{
\begin{aligned}
\sigma_{z\theta} &= -\frac{a_0}{2}\int_0^\infty \zeta^2\frac{\bar m}{\Delta_0}(\zeta^2+a_0)(e^{-\zeta z}-e^{-\eta z})J_1(\zeta r)\,d\zeta,\\
\mu_{zr} &= -\int_0^\infty \zeta^2\frac{\zeta\bar m}{\Delta_0}(\zeta^2+a_0)\left\{\frac{a}{\mu+a}\left[e^{-\zeta z}+\frac{\mu}{a}e^{-\eta z}\right]+\right.\\
&\qquad\left. -\left[1+\frac{\mu}{\mu+a}\left(\frac{\eta}{\zeta}-1\right)\right]e^{-\rho z}\right\}J_1(\zeta r)\,d\zeta,\\
\mu_{zz} &= \int_0^\infty \zeta\frac{\bar m}{\Delta_0}\left\{-\frac{a}{\mu+a}(\zeta^2+a_0)\left[\zeta^2e^{-\zeta z}+\frac{\mu}{a}\left(\frac{2a}{\gamma}+\zeta^2\right)e^{-\eta z}\right]+\right.\\
&\qquad\left. +\zeta\rho\zeta^2\left[1+\frac{\mu}{\mu+a}\left(\frac{\eta}{\zeta}-1\right)\right]e^{-\rho z}\right\}J_0(\zeta r)\,d\zeta.
\end{aligned}
\right.
$$

It is easy to prove from (2.18) that the boundary conditions (2.8) are satisfied.

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В. Новацки, **Вторая осе-симметрическая задача в теории микрополярной упругости**

Содержание. В настоящей работе рассматривается так называемая „вторая" осе-симметрическая задача микрополярной упругости характеризованная вектором смещения $\mathbf{u}=(0, u_\theta, 0)$ и вектором вращения $\boldsymbol{\varphi}=(\varphi_r, 0, \varphi_z)$. Вышеприведенные составляющие векторов $\mathbf{u}$ и $\boldsymbol{\varphi}$ должны удовлетворять систему уравнений (1.4). Введение упругих потенциалов Φ, Ψ позволяет свести решение системы уравнений (1.4) к решению простых эллиптических уравнений (1.4). Функции Φ и Ψ не являются независимыми друг от друга и должны выполнять дополнительные связи (2.3). Способ использования упругих потенциалов объясняется на примере упругого полупространства нагруженного моментами.