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APPLIED MECHANICS

(ELASTICITY)

The Plane Problem of Micropolar Thermoelasticity

by

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1. Introduction

In this paper we shall be concerned with the plane state of strain induced in an elastic micropolar medium (Cosserat's medium) by the action of temperature.

We confine ourselves to the problem of stationary flow of heat.

However, prior to discussion of the plane problem, we shall dwell briefly on the general state of stress in a micropolar body.

The action of temperature gives rise to the formation in the body of displacements $\mathbf{u}(\mathbf{x}, t)$ and of rotations $\boldsymbol{\varphi}(\mathbf{x}, t)$. The state of deformation of the body is described by two asymmetric tensors: The tensor of deformation γ_{ji} and the curvature-twist tensor κ_{ji}. Both these tensors are connected with the quantities $\mathbf{u}$ and $\boldsymbol{\varphi}$ by the following relations [1]—[3]

$$\gamma_{ji}=u_{i,j}-\varepsilon_{kji}\varphi_k, \qquad \kappa_{ji}=\varphi_{i,j}, \qquad i,j,k=1,2,3. \tag{1.1}$$

The state of stress is characterized by two asymmetric tensors: the tensor of force stresses σ_{ji} and that of couple stresses μ_{ji}. They are connected with the tensors γ_{ji}, κ_{ji} and σ_{ji}, κ_{ji} by the following constitutive equations [4]:

$$\begin{aligned} \sigma_{ji}&=(\mu+a)\,\gamma_{ji}+(\mu-a)\,\gamma_{ij}+(\lambda\gamma_{kk}-\nu\theta)\,\delta_{ij}, \\ \mu_{ji}&=(\gamma+\varepsilon)\,\kappa_{ji}+(\gamma-\varepsilon)\,\kappa_{ij}+\beta\kappa_{kk}\,\delta_{ij}, \qquad i,j,k=1,2,3. \end{aligned} \tag{1.2}$$

The above equations should be regarded as Duhamel—Neumann equations extended on a micropolar body. In the relations (1.2) the symbols μ and λ are Lamé's constants, while $a, \beta, \gamma, \varepsilon$ denote other material constants. There is $\nu=(3\lambda+2\mu)\,a_t$, where a_t stands for the coefficient of thermal expansion. Substituting Eqs. (1.2) and (1.1) into the equations of equilibrium

$$\sigma_{ji,j}=0, \qquad \varepsilon_{ijk}\,\sigma_{jk}+\mu_{ji,j}=0, \qquad i,j,k=1,2,3. \tag{1.3}$$

we obtain a system of equations in displacements and rotations expressed as follows

$$(1.4)\qquad \begin{aligned} &(\mu+a)\,\nabla^2\mathbf{u}+(\lambda+\mu-a)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+2a\,\mathrm{rot}\,\boldsymbol{\varphi}=\nu\,\mathrm{grad}\,\theta,\\ &(\gamma+\varepsilon)\,\nabla^2\boldsymbol{\varphi}-4a\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}+2a\,\mathrm{rot}\,\mathbf{u}=0, \qquad \nabla^2=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}.\end{aligned}$$

The term θ representing the increase of temperature (the increase with respect to the temperature of the body being in natural state) may be determined from the equation of heat conductivity

$$(1.5)\qquad \nabla^2\theta=-\frac{W}{\lambda_0}.$$

Here the symbol W denotes the quantity of heat produced per time and volume unit, while λ_0 is the coefficient of heat conductivity. Eqs. (1.4) and (1.5) should be supplemented with boundary conditions. They take the following form:

$$(1.6)\qquad \begin{aligned} &p_i=\sigma_{ji}\,n_j=0, \qquad m_i=\mu_{ji}\,n_j=0,\\ &\lambda_0\frac{\partial\theta}{\partial n}=\lambda_1(\vartheta-\theta), \qquad \mathbf{x}\in A.\end{aligned}$$

The first two conditions refer to the absence of loading (forces and moments) on the surface A bounding the body. The symbol ϑ denotes here the temperature of the medium surrounding the body considered, λ_0 and λ_1 denote, respectively, the coefficients of internal and external heat conductivity.

In the sequel we shall consider the plane state od strain.

2. The plane state of strain. Differential equations and their solutions

In the plane state of strain all the causes and effects depend on two variables only. Assuming that the displacements and rotations do not depend on the variable x_3 we have:

$$(2.1)\qquad \mathbf{u}\equiv(u_1,u_2,0), \qquad \boldsymbol{\varphi}\equiv(0,0,\varphi_3),$$

where u_1, u_2, φ_3 are functions of the variables $x_1. x_2$. In accordance with definition (1.1) we obtain for the plane state of strain the following components of the tensors γ_{ji} and κ_{ji}

$$(2.2)\qquad \begin{aligned} &\gamma_{11}=\partial_1 u_1, \qquad \gamma_{22}=\partial_2 u_2, \qquad \gamma_{12}=\partial_1 u_2-\varphi_3,\\ &\gamma_{21}=\partial_2 u_1+\varphi_3, \qquad \kappa_{13}=\partial_1\varphi_3, \qquad \kappa_{23}=\partial_2\varphi_3.\end{aligned}$$

The remaining values γ_{ji} and κ_{ji} are equal to zero. From the relations (1.2) we get

$$(2.3)\qquad \begin{aligned} &\sigma_{ji}=(\mu+a)\,\gamma_{ji}+(\mu-a)\,\gamma_{ij}+(\lambda\gamma_{kk}-\nu\theta)\,\delta_{ji},\\ &\sigma_{33}=\gamma_{kk}\,\lambda-\nu\theta, \qquad \mu_{j3}=(\gamma+\varepsilon)\,\kappa_{j3}, \qquad \mu_{3j}=(\gamma-\varepsilon)\,\kappa_{j3}, \qquad j=1,2.\end{aligned}$$

Here $\gamma_{kk}=\gamma_{11}+\gamma_{22}$. The state of stress σ_{ji} and the state of couple-stress μ_{ji} are characterized by the following matrices

$$\sigma=\begin{vmatrix}\sigma_{11} & \sigma_{12} & 0\\ \sigma_{21} & \sigma_{22} & 0\\ 0 & 0 & \sigma_{33}\end{vmatrix}, \qquad \mu=\begin{vmatrix}0 & 0 & \mu_{13}\\ 0 & 0 & \mu_{23}\\ \mu_{31} & \mu_{32} & 0\end{vmatrix}. \tag{2.4}$$

The equations of equilibrium (1.3) for the plane state of strain are thus limited to the three following equations

$$\begin{aligned} &\partial_1 \sigma_{11}+\partial_2 \sigma_{21}=0,\\ &\partial_1 \sigma_{12}+\partial_2 \sigma_{22}=0,\\ &\sigma_{12}-\sigma_{21}+\partial_1 \mu_{13}+\partial_2 \mu_{23}=0. \end{aligned} \tag{2.5}$$

Eliminating the stress from Eqs. (2.5), we arrive, on taking into consideration Eqs. (2.2) and (2.3), at the following set of three equations

$$\begin{aligned} &(\mu+a)\,\nabla_1^2 u_1+(\mu+\lambda-a)\,\partial_1 e+2a\,\partial_2\varphi_3=\nu\partial_1\theta,\\ &(\mu+a)\,\nabla_1^2 u_2+(\mu+\lambda-a)\,\partial_2 e-2a\,\partial_1\varphi_3=\nu\partial_2\theta,\\ &(\gamma+\varepsilon)\,\nabla_1^2\varphi_3-4a\varphi_3+2a\,(\partial_1 u_2-\partial_2 u_1)=0. \end{aligned} \tag{2.6}$$

We have

$$\nabla_1^2=\partial_1^2+\partial_2^2, \qquad e=\partial_1 u_1+\partial_2 u_2.$$

Within the system of polar coordinates we have to deal with the following vectors of displacements and rotations

$$\mathbf{u}\equiv(u_r\, u_\vartheta, 0), \qquad \varphi\equiv(0, 0, \varphi_z). \tag{2.7}$$

In polar coordinates Eqs. (2.6) obtain the following forms:

$$\begin{aligned} &(\mu+a)\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\vartheta}{\partial\vartheta}\right)+(\lambda+\mu-a)\frac{\partial e}{\partial r}+\frac{2a}{r}\frac{\partial\varphi_z}{\partial\vartheta}=\nu\frac{\partial\theta}{\partial r},\\ &(\mu+a)\left(\nabla^2 u_\vartheta-\frac{u_\vartheta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\vartheta}\right)+(\lambda+\mu-a)\frac{\partial e}{r\partial\vartheta}-2a\frac{\partial\varphi_z}{\partial r}=\nu\frac{\partial\theta}{r\partial\vartheta},\\ &[(\gamma+\varepsilon)\,\nabla^2-4a]\,\varphi_z+\frac{2a}{r}\left(\frac{\partial}{\partial r}(ru_\vartheta)-\frac{\partial u_r}{\partial\vartheta}\right)=0. \end{aligned} \tag{2.8}$$

In the above equations there is

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}, \qquad e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\vartheta}{\partial r}.$$

For the uni-dimensional problem, referred to the space, half-space and an elastic layer, i.e. $u_1=u_1(x_1)$, $u_2=0$ what remains from the set of equations (2.6) is the equation:

$$(\lambda+2\mu)\,\partial_1^2 u_1=\nu\partial_1\theta, \qquad u_2=0, \qquad \varphi_3=0. \tag{2.9}$$

In the case of axi-symmetric deformations, i.e. for $\mathbf{u} \equiv (u_r, 0, 0)$ the system of Eqs. (2.8) reduces to the form

$$(2.10) \qquad (\lambda+2\mu)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)u_r=\nu\frac{\partial\theta}{\partial r}.$$

As is seen, Eqs. (2.9) and (2.10) derived above coincide with the equations of classical thermoelasticity for the case of uni-dimensional problems. The stress tensor σ_{ji} is symmetric, the couple stress tensor is equal to zero. Let us return now to Eqs. (2.6). We shall introduce therein the vector

$$(2.11) \qquad \begin{gathered} \boldsymbol{\zeta}=\tfrac{1}{2}\operatorname{rot}\mathbf{u}-\boldsymbol{\varphi} \\ \zeta_1=0, \quad \zeta_2=0, \quad \zeta_3=\tfrac{1}{2}(\partial_1 u_2-\partial_2 u_1)-\varphi_3. \end{gathered}$$

Eqs. (2.6) will now take the form

$$(2.12) \qquad \begin{gathered} \mu\nabla_1^2 u_1+(\lambda+\mu)\partial_1 e-2a\,\partial_2\zeta_3=\nu\partial_1\theta, \\ \mu\nabla_1^2 u_2+(\lambda+\mu)\partial_2 e+2a\,\partial_1\zeta_3=\nu\partial_2\theta, \\ [(\gamma+\varepsilon)\nabla_1^2-4a]\zeta_3-\tfrac{1}{2}(\gamma+\varepsilon)\nabla_1^2(\partial_1 u_2-\partial_2 u_1)=0. \end{gathered}$$

The solution of this system of equations will consists of the two parts

$$(2.13) \qquad \begin{gathered} u_1=u_1'+u_1'', \quad u_2=u_2'+u_2'', \\ \zeta_3=\zeta_3'+\zeta_3'', \quad \zeta_3'=0. \end{gathered}$$

The functions marked with a single "prime" should be particular integrals of non--homogeneous equations (2.12), while the functions with two "primes" should stand for the general solutions of homogeneous equations (2.12). Introducing (2.13) into (2.12), we get

$$(2.14) \qquad \begin{gathered} \mu\nabla_1^2 u_1'+(\lambda+\mu)\partial_1 e'=\nu\partial_1\theta, \\ \mu\nabla_1^2 u_2'+(\lambda+\mu)\partial_2 e'=\nu\partial_2\theta, \\ \nabla_1^2(\partial_1 u_2'-\partial_2 u_1')=0, \quad \zeta_3'=0. \end{gathered}$$

and

$$(2.15) \qquad \begin{gathered} \mu\nabla_1^2 u_1''+(\lambda+\mu)\partial_1 e''-2a\,\partial_2\zeta_3''=0, \\ \mu\nabla_1^2 u_2+(\lambda+\mu)\partial_2 e''+2a\,\partial_1\zeta_3''=0, \\ [(\gamma+\varepsilon)\nabla_1^2-4a]\zeta_3''-\tfrac{1}{2}(\gamma+\varepsilon)\nabla_1^2(\partial_1 u_2''-\partial_2 u_1'')=0. \end{gathered}$$

Thus we get Eqs. $(2.14)_{1,2}$ identical with equations of classical thermoelasticity [5]. The condition $\zeta_3'=0$ leads to the relation $\varphi_3'=\frac{1}{2}(\partial_1 u_2'-\partial_2 u_1')$ which holds true for the classical theory of thermoelasticity. The condition $(2.14)_3$ will be satisfied if we assume the displacements u_1', u_2' in the form

$$(2.16) \qquad u_1'=\partial_1\Phi, \quad u_2'=\partial_2\Phi.$$

Substituting (2.16) into Eqs. $(2.14)_{1,2}$, we obtain — after integration — the Poisson's equation for the function Φ:

$$\nabla_1^2 \Phi = m\theta, \qquad m = \frac{\nu}{\lambda+2\mu}. \tag{2.17}$$

The function Φ is the particular integral of the system of Eqs. (2.14) and, by the same, it is the particular integral of differential equations (2.12).

The stresses and strain marked with primes will be expressed by means of the function Φ as follows

$$\begin{gathered} \gamma'_{ji} = \Phi_{,ij}, \qquad \kappa'_{ji} = 0, \\ \sigma'_{ji} = 2\mu(\Phi_{,ji} - \delta_{ji}\nabla_1^2 \Phi), \qquad \mu'_{ji} = 0. \end{gathered} \tag{2.18}$$

In the case of an infinite region the function Φ will be expressed by the formula [5]:

$$\Phi(\xi_1, \xi_2) = -\frac{m}{4\pi}\int_A \frac{\theta(x_1, x_2)\, dx_1\, dx_2}{R(x_1, x_2; \zeta_1, \xi_2)}, \tag{2.19}$$

where

$$R = [(x_1-\xi_1)^2 + (x_2-\xi_2)^2]^{1/2}.$$

For a bounded region we have to solve Eq. (2.17) under the boundary condition $\Phi=0$.

Thus we have to solve only the system of Eqs. (2.15) which refers to the isothermic problem ($\theta=0$). This is a typical boundary problem of the theory of elasticity of micropolar medium. If we assume the boundary to be free of loading, we may write the boundary conditions for the system of equations in the following form

$$(\sigma'_{ji} + \sigma''_{ji})\, n_j = 0, \qquad \mu''_{ji}\, n_j = 0, \qquad i, j = 1, 2. \tag{2.20}$$

In this way, the displacements u''_1, u''_2 and the rotation φ''_3 being known, we may determine the stresses σ''_{ji} and μ''_{j3} from Eqs. (2.3) (obviously we have to put therein $\theta=0$).

In the next paragraph we shall advance a somewhat different procedure of determining the stresses σ''_{ji} and μ''_{ji} using the functions of stresses, a procedure which seems particularly convenient in cases, where the boundary of the body should be free of loadings.

3. The functions of stresses of the thermoelastic problem

Let us return to formulae (2.2). It is easily seen that the quantities appearing in these formulae are connected to each other by the following relations

$$\begin{gathered} \partial_1 \gamma_{21} - \partial_2 \gamma_{11} - \kappa_{13} = 0, \qquad \partial_1 \gamma_{22} - \partial_2 \gamma_{12} - \kappa_{23} = 0, \\ \partial_1 \kappa_{23} - \partial_2 \kappa_{13} = 0. \end{gathered} \tag{3.1}$$

These relations may be written also as below

$$\begin{gathered} \partial_1^2 \gamma_{22} + \partial_2^2 \gamma_{11} = \partial_1 \partial_2 (\gamma_{12} + \gamma_{21}), \\ \partial_2^2 \gamma_{12} - \partial_1^2 \gamma_{21} = \partial_1 \partial_2 (\gamma_{22} - \gamma_{11}) - (\partial_1 \kappa_{13} + \partial_2 \kappa_{23}), \\ \partial_1 \kappa_{23} - \partial_2 \kappa_{13} = 0. \end{gathered} \tag{3.2}$$

They are equations of compatibility for the bi-dimensional problem in a micropolar medium. Solving the relations (2.3) with respect to the quantities γ_{ji} and κ_{j3} $(i,j=1,2)$, and substituting the results into Eqs. (3.2), we obtain three equations in stresses [6]

$$(3.3)\qquad \begin{aligned} &\partial_2^2\,\sigma_{11}+\partial_1^2\,\sigma_{22}-\frac{\lambda}{2(\lambda+\mu)}\nabla_1^2(\sigma_{11}+\sigma_{22})+\frac{\mu\nu}{\lambda+\mu}\nabla_1^2\,\theta=\partial_1\,\partial_2(\sigma_{12}+\sigma_{21}),\\ &(\partial_2^2-\partial_1^2)(\sigma_{12}+\sigma_{21})+\frac{\mu}{a}\nabla_1^2(\sigma_{12}-\sigma_{21})=\\ &\qquad=2\partial_1\,\partial_2(\sigma_{22}-\sigma_{11})-\frac{4\mu}{\gamma+\varepsilon}(\partial_1\,\mu_{13}+\partial_2\,\mu_{23}),\\ &\partial_1\,\mu_{23}-\partial_2\,\mu_{13}=0.\end{aligned}$$

We shall now introduce the function of stresses F and Ψ and connect them with the stresses by the following relations [7]

$$(3.4)\qquad \begin{aligned} &\sigma_{11}=\partial_2^2\,F-\partial_1\,\partial_2\,\Psi, \qquad \sigma_{22}=\partial_1^2\,F+\partial_1\,\partial_2\,\Psi,\\ &\sigma_{12}=-\partial_1\,\partial_2\,F-\partial_2^2\,\Psi, \qquad \sigma_{21}=-\partial_1\,\partial_2\,F+\partial_1^2\,\Psi,\\ &\mu_{13}=\partial_1\,\Psi, \qquad \mu_{23}=\partial_2\,\Psi.\end{aligned}$$

Substituting the relations (3.4) into Eqs. (2.5), we see that the equations are identically satisfied. Substituting, in turn, (3.4) into the equations of compactness $(3.3)_{1,2}$, we get the following equations

$$(3.5)\qquad \begin{aligned} &\nabla_1^2\,\nabla_1^2\,F+\kappa\nabla_1^2\,\theta=0,\\ &\nabla_1^2(1-l^2\,\nabla_1^2)\,\Psi=0,\end{aligned}$$

where

$$l^2=\frac{(\gamma+\varepsilon)(a+\mu)}{4a\mu}, \qquad \kappa=\frac{2\mu\nu}{\lambda+2\mu}.$$

The functions F and Ψ are not mutually independent. They are connected by the relations $(3.2)_{1,2}$. Consequently, we obtain

$$(3.6)\qquad \begin{aligned} &-\partial_1(1-l^2\,\nabla_1^2)\,\Psi=\mathrm{A}\partial_2\,\nabla_1^2\,F+B\partial_2\,\theta,\\ &\partial_2(1-l^2\,\nabla_1^2)\,\Psi=A\partial_1\,\nabla_1^2\,F+B\partial_1\,\theta.\\ &A=\frac{(\lambda+2\mu)(\gamma+\varepsilon)}{4\mu(\lambda+\mu)}, \qquad B=\frac{\nu(\gamma+\varepsilon)}{\lambda+\mu}.\end{aligned}$$

We have still to set the boundary conditions for Eqs. (3.5).

Let us assume that on the boundary s are prescribed the loadings $\mathbf{p}\equiv(p_1.\,p_2,0)$ and moments $\mathbf{m}\equiv(0,0,m_3)$. Expressing the boundary conditions

$$(3.7)\qquad p_1=\sigma_{ji}\,n_j, \qquad m_3=\mu_{j3}\,n_j, \qquad i,j=1,2$$

by means of the functions F and Ψ, we obtain the following equations

$$\frac{d}{ds}\left(\frac{\partial F}{\partial x_2}-\frac{\partial \Psi}{\partial x_1}\right)=p_1, \qquad \frac{d}{ds}\left(\frac{\partial F}{\partial x_1}+\frac{\partial \Psi}{\partial x_2}\right)=-p_2,$$
(3.8)
$$n_1\frac{\partial \Psi}{\partial x_1}+n_2\frac{\partial \Psi}{\partial x_2}=m_3.$$

Integrating over the boundary of the cylinder cross-section we get

$$\frac{\partial F}{\partial x_2}-\frac{\partial \Psi}{\partial x_1}=f_2, \qquad \frac{\partial F}{\partial x_1}+\frac{\partial \Psi}{\partial x_2}=f_1, \qquad \frac{\partial \Psi}{\partial n}=m_3. \tag{3.9}$$

In the above equations the following notations have been introduced

$$f_1=-\int_{s_0}^{s}p_2(s)\,ds, \qquad f_2=\int_{s_0}^{s}p_1(s)\,ds.$$

The relations (3.9) are equivalent to the equations

$$\frac{\partial F}{\partial n}+\frac{\partial \Psi}{\partial s}=f_1 n_1+f_2 n_2, \qquad \frac{\partial F}{\partial s}-\frac{\partial \Psi}{\partial n}=f_2 n_1-f_1 n_2, \qquad \frac{\partial \Psi}{\partial n}=m_3. \tag{3.10}$$

Let us now consider a simply connected cylinder, the boundary of which is free of loadings. The temperature field satisfies the equation

$$\nabla^2\theta=-\frac{W}{\lambda_0}, \tag{3.11}$$

with the boundary condition (1.6)

$$\lambda_0\frac{\partial \theta}{\partial n}=\lambda_1(\theta_0-\theta). \tag{3.12}$$

In order to determine the stresses, we have to solve the following system of equations

$$\nabla_1^2\nabla_1^2 F=\frac{\kappa W}{\lambda_0}, \qquad \nabla_1^2(l^2\nabla_1^2-1)\,\Psi=0, \tag{3.13}$$

with homogeneous boundary conditions

$$\frac{\partial F}{\partial n}+\frac{\partial \Psi}{\partial s}=0, \qquad \frac{\partial F}{\partial s}-\frac{\partial \Psi}{\partial n}=0, \qquad \frac{\partial \Psi}{\partial n}=0. \tag{3.14}$$

Moreover, the following relations should be verified

$$-\partial_1(1-l^2\nabla_1^2)\,\Psi=A\partial_2\nabla_1^2 F+B\partial_2\theta,$$
(3.15)
$$\partial_2(1-l^2\nabla_1^2)\,\Psi=A\partial_1\nabla_1^2 F+B\partial_1\theta.$$

Let us consider a particular case, where the heat sources are absent ($W=0$), the temperature being constant ($\theta=$const). In this case Eqs. (3.13) become homogeneous, and in relations (3.15) the terms expressing the temperature — vanish.

If we assume, however, that within the region of the cylinder cross-section there is $F\equiv 0$ and $\Psi=0$, then the homogeneous system of Eqs. $(3.13)_{1,2}$ with homogeneous boundary conditions (3.14) as well as the relations (3.15) will be satisfied. From (3.4) we have that the stresses σ_{ij} μ_{j3} are equal to zero. Only the stress σ_{33} as given by the formula (2.3) differs from zero. There is

$$\sigma_{33} = -\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\,a_t\,\theta. \tag{3.16}$$

Thus, it is seen that in the particular case θ=const an infinite cylinder undergoes deformation with no stresses in the x_1, x_2 plane. Let us remind that in the classical thermoelasticity such a state appears — in conformity with the known theorem of Muskhelishvili [8] — for the temperature field $\theta(x_1, x_2)$, verifying the equation $\nabla^2\theta=0$ under the boundary condition (3.12). In a cylinder made of micropolar material the temperature field $\theta(x_1, x_2)$ verifying the homogeneous Eq. (3.11) leads to stresses σ_{ji} and μ_{j3} $(i, j=1, 2)$ differing from zero.

A more ample discussion of bi-dimensional stationary problems of thermoelasticity with some examples will be published in "Archiwum Mechaniki Stosowanej".

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В. НОВАЦКИЙ, ПЛОСКАЯ ПРОБЛЕМА МИКРОПОЛЯРНОЙ ТЕРМОУПРУГОСТИ

В настоящей работе приводятся соотношения и дифференциальные уравнения для плоской задачи термоупругости в микрополярной среде (в среде Коссератов). Выведены также дифференциальные уравнения в перемещениях и вращениях равно как и дифференциальные уравнения для функций напряжений.