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APPLIED MECHANICS

Some Theorems of Asymmetric Thermoelasticity

by

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1. Introduction

The present author has examined in his previous papers [1], [2] the basic relations as well as the nonstationary asymmetric linear thermoelasticity equations. In this Note we are going to deal with the steady-state problems. We intend, namely, to derive variational theorems and theorems on the minimum of potential energy, as well as on minimum of the complementary work, and to extend the Reissner variational theorem on the problems of asymmetric thermoelasticity.

Let us recall briefly the basic relations and equations derived in [1] and [2].

The free energy F referred to the volume unit has the following form

$$F = \mu\gamma_{(ij)}\gamma_{(ij)} + \alpha\gamma_{\langle ij\rangle}\gamma_{\langle ij\rangle} + \gamma\varkappa_{(ij)}\varkappa_{(ij)} + \varepsilon\varkappa_{\langle ij\rangle}\varkappa_{\langle ij\rangle} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} + \frac{\beta}{2}\varkappa_{kk}\varkappa_{nn} - \nu\gamma_{kk}\theta - \frac{c_\varepsilon}{2T_0}\theta^2, \tag{1.1}$$

where

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\,\omega_k, \quad \varkappa_{ji} = \omega_{i,j}, \tag{1.2}$$

γ_{ij} denotes the asymmetric strain tensor and $\varkappa_{ij}$ stands for the asymmetric torsion-flexure tensor. The symbols ( ) and $\langle\ \rangle$ refer to the symmetric and skew-symmetric parts of the tensor, respectively, u_i stands for the components of displacement vector, and ω_i — for those of rotation vector. In the relation (1.1) the notation $\theta = T - T_0$, is introduced, T denoting the absolute temperature, and T_0 — the temperature of the body in its natural state, μ, λ are Lamé's constants, while α, γ, ε, β are material constants related to the isothermal state. ϵ_{ijk} is the well known Cartesian alternator.

There is $\nu = 3K\alpha_t$, where K is the modulus of compressibility, and α_t is the coefficient of the linear thermal expansion. The free energy is the quadratic form of its own arguments, positive definite. Its differential is the exact differential. The material constants have to satisfy the following inequalities

$$\mu > 0, \quad \lambda > 0, \quad \mu - \alpha > 0, \quad \alpha < 0, \quad 2\gamma + 3\beta > 0, \quad \gamma > 0, \quad \varepsilon > 0 \tag{1.3}$$

Taking into account

$$\sigma_{ji} = \frac{\partial F}{\partial \gamma_{ji}}, \quad \mu_{ji} = \frac{\partial F}{\partial \varkappa_{ji}}, \quad S = -\frac{\partial F}{\partial T}$$

we obtain from (1.1) the following system of constitutive relations

$$\sigma_{ij} = 2\mu\gamma_{(ij)} + 2\alpha\gamma_{\langle ij\rangle} + (\lambda\gamma_{kk} - \nu\theta)\,\delta_{ij}, \tag{1.4}$$

$$\mu_{ij} = 2\mu\varkappa_{(ij)} + 2\varepsilon\varkappa_{\langle ij\rangle} + \beta\varkappa_{kk}\,\delta_{ij}, \tag{1.5}$$

$$S = \nu\gamma_{kk} + \frac{c_\varepsilon}{T_0}\,\theta. \tag{1.6}$$

Here σ_{ji} denotes the asymmetric stress tensor and μ_{ji} — the asymmetric couple--stresses tensor. S is the entropy referred to the volume unit.

Substituting σ_{ji} and μ_{ji} from (1.4) and (1.5) into the equilibrium equations

$$\sigma_{ji,j} + X_i = 0, \qquad \varepsilon_{ijk}\,\sigma_{jk} + \mu_{ji,j} + Y_i = 0 \tag{1.7}$$

and expressing the quantities γ_{ji} and $\varkappa_{ji}$ in terms of u_i, ω_i, we obtain a system of equations in vector form

$$\begin{aligned} &(\lambda+2\mu)\,\mathrm{grad\,div}\,\vec{u} - (\mu+\alpha)\,\mathrm{rot\,rot}\,\vec{u} + 2\alpha\,\mathrm{rot}\,\vec{\omega} + \vec{X} = \nu\,\mathrm{grad}\,\theta,\\ &(\beta+2\gamma)\,\mathrm{grad\,div}\,\vec{\omega} - (\gamma+\varepsilon)\,\mathrm{rot\,rot}\,\vec{\omega} + 2\alpha\,\mathrm{rot}\,\vec{u} - 4\alpha\vec{\omega} + \vec{Y} = 0. \end{aligned} \tag{1.8}$$

The temperature θ appearing in the first equation of (1.8) is determined from the equation of the conduction of heat

$$\nabla^2\theta = -W/k. \tag{1.9}$$

Herein W is the source of heat, the quantity of heat generated per volume and time unit, and k — the coefficient of thermal conductivity

We have to supplement Eqs. (1.8) and (1.9) with boundary conditions. Let us assume namely that on a part of the surface A (denoted by A_u) displacements $\hat{u}_i$ and rotations $\hat{\omega}_i$ are prescribed. Moreover, on the remaining part of the surface, $A_\sigma = A - A_u$, the tensions $\hat{p}_i$ and moments $\hat{m}_i$ be prescribed. Thus we have

$$\begin{aligned} &u_i(x) = \hat{u}_i(x), \quad \omega_i(x) = \hat{\omega}_i(x), \quad x \in A_u,\\ &p_i(x) = \sigma_{ji}(x)\,n_j(x) = \hat{p}_i(x), \quad m_i(x) = \mu_{ji}(x)\,n_j(x) = \hat{m}_i(x), \quad x \in A_\sigma. \end{aligned} \tag{1.10}$$

Here n_i denotes the components of the unit normal vector to the surface A, its components being directed outwards. Boundary condition of the equation of heat conduction may take also a different form. Most frequently it is assumed that the temperature θ or the heat flux $-k'\dfrac{\partial\theta}{\partial n}$ are given on the surface A.

2. Theorem on minimum of the potential energy

Let the body be in the state of static equilibrium under the action of external forces and raising temperature. Let the components of the displacement vector $\vec{u}$ and of the rotation vector $\vec{\omega}$ be given on the surface A_u, and tensions p_i and moments m_i — on A_σ.

Let us assume that there exists a system of displacements u_i and rotations ω_i satisfying the equilibrium Eq. (1.10). We shall consider the displacements $u_i+\delta u_i$ and rotations $\omega_i+\delta\omega_i$ consistent with the constraints imposed on the body. Virtual displacements δu_i and rotations $\delta\omega_i$ ought to be the functions of the class $C^{(3)}$ taking zero values on A_u and arbitrary values on A_σ.

The virtual work principle takes now the form, [1], [2]

$$\int_V (X_i\,\delta u_i+Y_i\,\delta\omega_i)\,dV+\int_A (p_i\,\delta u_i+m_i\,\delta\omega_i)\,dA=\int_V (\sigma_{ji}\,\gamma_{ji}+\mu_{ji}\,\varkappa_{ji})\,dV. \tag{2.1}$$

This equation may be transformed — taking into account (1.4) and (1.5) — as follows:

$$\int_V (X_i\,\delta u_i+Y_i\,\delta\omega_i)\,dV+\int_A (p_i\,\delta u_i+m_i\,\delta\omega_i)\,dA=\delta H_\varepsilon-\nu\int_V \theta\delta\gamma_{kk}\,dV. \tag{2.2}$$

Here

$$\delta H_\varepsilon=\int_V [2\mu\gamma_{(ij)}\,\delta\gamma_{(ij)}+2\alpha\gamma_{\langle ij\rangle}\,\delta\gamma_{\langle ij\rangle}+2\gamma\varkappa_{(ij)}\,\delta\varkappa_{(ij)}+ +2\varepsilon\varkappa_{\langle ij\rangle}\,\delta\varkappa_{\langle ij\rangle}+\lambda\gamma_{kk}\,\delta\gamma_{nn}+\beta\varkappa_{kk}\,\delta\varkappa_{nn}]\,dV.$$

Since the body forces and the body couples as well as the tensions and moments of surface do not vary, we may write Eq. (2.2) in the following form

$$\delta\Gamma=0, \tag{2.3}$$

where

$$\begin{aligned}\Gamma&=H_\varepsilon-\int_V (X_i\,u_i+Y_i\,\omega_i)\,dV-\int_V (p_i\,u_i+m_i\,\omega_i)\,dA-\nu\int_V \theta\gamma_{kk}\,dV,\\ H_\varepsilon&=\int_V\left(F+\nu\gamma_{kk}\,\theta+\frac{c_\varepsilon}{2T_0}\,\theta^2\right)dV.\end{aligned} \tag{2.4}$$

The quantity Γ called the potential energy is extremum. Proceeding in an analogous way as for symmetric thermoelasticity we arrive at the conclusion that Γ is minimum. The theorem on potential energy states that from among all the displacements u_i and rotations ω_i which satisfy the given boundary conditions only those fulfilling at the same time the equilibrium equations load to the minimum of potential energy.

Let us go back to Eq. (2.2) and transform the last integral appearing in the right-hand part of this equation to the form

$$\nu\int_V \theta\delta\gamma_{kk}\,dV=\nu\int_V \theta\,\delta u_{k,k}\,dV=\nu\int_A \theta n_k\,\delta u_k\,dA-\nu\int_V \theta_{,k}\,\delta u_k\,dV. \tag{2.5}$$

After substituting (2.5) into (2.2) we obtain

$$\delta H_\varepsilon=\int_V [(X_i-\gamma\theta_{,i})\,\delta u_i+Y_i\,\delta\omega_i]\,dV+\int_{A_\sigma} [(p_i+\nu\theta n_i)\,\delta u_i+m_i\,\delta\omega_i]\,dA. \tag{2.6}$$

Now, we shall consider an identical body (i.e., of the same form and material), but be placed under isothermal conditions. Let the body forces X_i^* and the body couples Y_i^* act on the body. The tensions p_i^* and moments m_i^* are assumed to be given on the surface A_σ, while displacements u_i^* and rotations ω_i^* — on A_u. We ask the following question: What should be the quantities X_i^*, Y_i^* — expressing forces and couples acting inside the body — and, on the other hand, the quantities p_i^* and m_i^* — expressing the tensions and moments acting on the surface A_σ — with identical boundary conditions for A_u in order to obtain the same field of displacements u_i and rotations ω_i in both viz., thermoelastic and isothermal problems. To get the answer, we shall compare (2.6) with the virtual work equation

$$\delta H_\varepsilon = \int_V (X_i^* \delta u_i + Y_i^* \delta \omega_i)\, dV + \int_A (p_i^* \delta u_i + m_i^* \delta \omega_i)\, dA. \tag{2.7}$$

In view of the identity of u_i and ω_i fields, the left-hand parts of Eqs. (2.6) and (2.7) are identical, too. Thus, we obtain the following relations

$$\begin{aligned} X_i^* &= X_i - \nu\theta_{,i}, & Y_i^* &= Y_i, & x &\in V, \\ p_i^* &= p_i + \nu\theta n_i, & m_i^* &= m_i, & x &\in A_\sigma, \\ u_i^* &= u_i, & \omega_i^* &= \omega_i, & x &\in A_u. \end{aligned} \tag{2.8}$$

Relations (2.8) represent the body forces analogy by means of which each steady-state problem can be reduced to the isothermal problem of the theory of asymmetric thermoelasticity.

3. Theorem on minimum of complementary energy

Let us solve Eq. (1.4) with respect to γ_{ij} and Eq. (1.5) with respect to $\varkappa_{ij}$. We have

$$\gamma_{ij} = 2\mu' \sigma_{(ij)} + 2\alpha' \sigma_{\langle ij \rangle} + \lambda' \delta_{ij} \sigma_{kk} + \alpha_t \theta \delta_{ij}, \tag{3.1}$$

$$\varkappa_{ij} = 2\gamma' \mu_{(ij)} + 2\varepsilon' \mu_{\langle ij \rangle} + \beta' \delta_{ij} \mu_{kk}. \tag{3.2}$$

We introduced here the following notations

$$2\mu' = \frac{1}{2\mu}, \quad 2\alpha' = \frac{1}{2\alpha}, \quad 2\gamma' = \frac{1}{2\gamma}, \quad 2\varepsilon' = \frac{1}{2\varepsilon},$$

$$\lambda' = -\frac{\lambda}{6\mu K}, \ \beta' = -\frac{\beta}{6\Omega}, \ K = \lambda + \frac{2}{3}\mu, \ \Omega = \gamma + \frac{2}{3}\beta.$$

It is easy to check that

$$\gamma_{ji} = \frac{\partial F}{\partial \sigma_{ji}}, \quad \varkappa_{ji} = \frac{\partial F}{\partial \mu_{ji}}, \tag{3.3}$$

if F is expressed as the function of stresses σ_{ji}, couple-stresses μ_{ji} and temperature θ. We introduce the notation

$$W_\sigma = \mu' \sigma_{(ij)} \sigma_{(ij)} + \alpha' \sigma_{\langle ij \rangle} \sigma_{\langle ij \rangle} + \gamma' \mu_{(ij)} \mu_{(ij)} + \varepsilon' \mu_{\langle ij \rangle} \mu_{\langle ij \rangle} + \frac{\lambda'}{2} \sigma_{kk} \sigma_{nn} + \frac{\beta'}{2} \mu_{kk} \mu_{nn}. \tag{3.4}$$

Then

$$(3.5)\qquad \gamma_{ij} = \frac{\partial W_\sigma}{\partial \sigma_{ji}} + \alpha_t\,\theta\,\delta_{ij}, \quad \varkappa_{ij} = \frac{\partial W_\sigma}{\partial \mu_{ji}}.$$

We shall consider the integral

$$(3.6)\qquad I = \int_V (\gamma_{ji}\,\delta\sigma_{ji} + \varkappa_{ji}\,\delta\mu_{ji})\,dV.$$

In this expression $\delta\sigma_{ji}$, $\delta\mu_{ji}$ denote the virtual increments of stresses and couple-stresses. These increments are regarded as functions of class $C^{(2)}$, as very small and arbitrary quantities. Taking into consideration (3.5) we have

$$(3.7)\qquad \int_V (\gamma_{ji}\,\delta\sigma_{ji} + \varkappa_{ji}\,\delta\mu_{ji})\,dV = \delta H_\sigma + \alpha_t \int_V \theta\,\delta\sigma_{kk}\,dV.$$

where

$$\delta H_\sigma = \int_V \left(\frac{\partial W_\sigma}{\partial \sigma_{ji}}\,\delta\sigma_{ji} + \frac{\partial W_\sigma}{\partial \mu_{ji}}\,\delta\mu_{ji} \right) dV.$$

Transforming the left-hand side of Eq. (3.7), taking into account the relation (1.2) and introducing notations $\delta p_i = \delta\sigma_{ji}\,n_j$, $\delta m_i = \delta\mu_{ji}\,n_j$, we obtain

$$(3.8)\qquad \int_A (u_i\,\delta p_i + \omega_i\,\delta m_i)\,dA - \int_V \left[u_i\,\delta\sigma_{ji,j} + \omega_i\,[\epsilon_{ijk}\,\delta\sigma_{kj} + \delta\mu_{ji,j}]\right] dV = \delta H_\sigma + \alpha_t \int_V \theta\delta\sigma_{kk}\,dV.$$

We require the stresses $\sigma_{ji}+\delta\sigma_{ji}$ and couple-stresses $\mu_{ji}+\delta\mu_{ji}$ to be statically possible. It means that the equilibrium conditions

$$(3.9)\qquad \sigma_{ji,j} + \delta\sigma_{ji,j} + X_i + \delta X_i = 0,$$

$$(3.10)\qquad \epsilon_{ijk}(\sigma_{jk} + \delta\sigma_{jk}) + \mu_{ji,j} + \delta\mu_{ji,j} + Y_i + \delta Y_i = 0$$

have to be satisfied inside the volume V and the boundary conditions

$$(3.11)\qquad p_i + \delta p_i = (\sigma_{ji} + \delta\sigma_{ji})\,n_j, \quad m_i + \delta m_i = (\mu_{ji} + \delta\mu_{ji})\,n_j$$

on the surface A_σ.

The quantities $\delta\sigma_{ji}$ and $\delta\mu_{ji}$ on A_u may be arbitrary. In view of the equilibrium equations (1.7) and boundary conditions (1.10) we have

$$\delta\sigma_{ji,j} + \delta X_i = 0, \quad \epsilon_{ijk}\,\delta\sigma_{jk} + \delta\mu_{ji,j} + \delta Y_i = 0, \quad x \in V,$$

and

$$\delta p_i = \delta\sigma_{ji}\,n_j, \quad \delta m_i = \delta\mu_{ji}\,n_j, \quad x \in A_\sigma.$$

As we want to compare all the fields of stresses and couple-stresses satisfying the equilibrium equations, but not necessarily the comptability equation, it should be assumed that $\delta X_i = 0$, $\delta Y_i = 0$ inside the volume V, and $\delta p_i = 0$, $\delta m_i = 0$ on the surface A_σ, leaving the increments δp_i, δm_i on the A_u surface arbitrary. Under these restrictions Eq. (3.8) takes the form

$$\int_{A_u} (u_i\,\delta p_i+\omega_i\,\delta m_i)\,dA = \delta H_\sigma+\alpha_t\int_V \theta\delta\sigma_{kk}\,dV. \tag{3.12}$$

Because displacements u_i, rotations ω_i and temperature θ do not vary, we have

$$\delta\Gamma^* = 0, \tag{3.13}$$

where

$$\Gamma^* = H_\sigma+\alpha_t\int_V \sigma_{kk}\,dV - \int_{A_u} (p_i\,u_i+m_i\,\omega_i)\,dA, \quad H_\sigma = \int_V W_\sigma\,dV. \tag{3.14}$$

The expression Γ^* is said to represent the complementary work. Similarly as in the theory of symmetric thermoelasticity, it can be proved here that Γ^* becomes minimum. Eq. (3.13) is the theorem on minimum of the complementary work extended to the problem of the theory of asymmetric thermoelasticity. This theorem says that from among all the tensor fields σ_{ji}, μ_{ji} satisfying the equilibrium equations and the boundary conditions given by the tensions p_i and moments m_i only those actually occur which reduce the functional Γ^* to minimum.

4. The extended Reissner's theorem

Thus, the Reissner's variational theorem [3] formulated in most general terms can be easily extended so as to include the problems of the theory of asymmetric thermoelasticity.

Let us consider now the following functional $I \equiv I(\gamma_{ji}, \varkappa_{ji}, u_i, \omega_i, \sigma_{ji}, \mu_{ji})$

$$\begin{aligned} I = \int_V \{W_\varepsilon - \nu\theta\gamma_{kk} - X_i\,u_i - Y_i\,\omega_i - \sigma_{ji}\,[\gamma_{ji}-(u_{i,j}-\epsilon_{kji}\,\omega_k)] - \\ - \mu_{ji}(\varkappa_{ji}-\omega_{i,j})\}\,dV - \int_{A_\sigma} (\hat{p}_i\,u_i+\hat{m}_i\,\omega_i)\,dA - \\ - \int_{A_u} [p_i\,(u_i-\hat{u}_i)+m_i\,(\omega_i-\hat{\omega}_i)]\,dA, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} W_\varepsilon = \mu\gamma_{(ij)}\,\gamma_{(ij)}+\alpha\gamma_{\langle ij\rangle}\,\gamma_{\langle ij\rangle}+\gamma\varkappa_{(ij)}\,\varkappa_{(ij)}+\varepsilon\varkappa_{\langle ij\rangle}\,\varkappa_{\langle ij\rangle}+ \\ +\frac{\lambda}{2}\gamma_{kk}\,\gamma_{nn}+\frac{\beta}{2}\varkappa_{kk}\,\varkappa_{nn}. \end{aligned} \tag{4.2}$$

Here $\hat{p}_i$ and $\hat{m}_i$ are forces and moments given on A_σ, $\hat{u}_i$, $\hat{\omega}_i$ — components of the displacement vector $\vec{u}$ and of the rotation vector on $\vec{\omega}$, respectively. Let us seek for the conditions necessary for I to be stationary. Equalling the first variation I to zero and taking into account that functions γ_{ij}, $\varkappa_{ji}$, u_i, ω_i, σ_{ji}, μ_{ji} show virtual increments inside the volume V, while the virtual increments of functions u_i, ω_i

can be arbitrary on A_σ, and the virtual increments of functions p_i, m_i — arbitrary on A_u we obtain

$$(4.3)\qquad \delta I = 0 = \int_V \Big\{ \frac{\partial W_\varepsilon}{\partial \gamma_{ji}}\, \delta\gamma_{ji} + \frac{\partial W_\varepsilon}{\partial \varkappa_{ji}}\, \delta\varkappa_{ji} - \nu\theta\delta_{ij}\, \delta\gamma_{ji} - X_i\, \delta u_i - Y_i\, \delta\omega_i - \\ - \delta\sigma_{j,i}\, [\gamma_{ji} - (u_{,j} - \epsilon_{ijk}\, \omega_i)] - \sigma_{ji}\, [\delta\gamma_{ji} - (\delta u_{i,j} - \epsilon_{ijk}\, \omega_i)] - \\ - \delta\mu_{ji}\, [\varkappa_{ji} - \omega_{i,j}] - \mu_{ji}\, (\delta\varkappa_{ji} - \delta\omega_{i,j}) \Big\}\, dV - \int_{A\sigma} (\hat{p}_i\, \delta u_i + \hat{m}_i\, \delta\omega_i)\, dA - \\ - \int_{A_u} [(u_i - \hat{u}_i)\, \delta p_i + (\omega_i - \hat{\omega}_i)\, \delta m_i]\, dA.$$

Integrating by parts, making use of Gauss' transformation and arranging the results in groups, we obtain

$$(4.4)\qquad \int \Big[\Big(\frac{\partial W_\varepsilon}{\partial \gamma_{ji}} - \sigma_{ij} - \delta_{ij}\, \nu\theta \Big) \delta\gamma_{ji} + \Big(\frac{\partial W_\varepsilon}{\partial \varkappa_{ji}} - \mu_{ji} \Big) \delta\varkappa_{ji} - (X_i + \sigma_{ji,j})\, \delta u_i - \\ - (\epsilon_{ijk}\, \sigma_{jk} + \mu_{ji,j} + Y_i)\, \delta\omega_i + (\gamma_{ji} - u_{i,j} + \epsilon_{kji}\, \omega_k)\, \delta\sigma_{ji} + \\ + (\varkappa_{ji} - \omega_{i,j})\, \delta\mu_{ji} \Big]\, dV - \int_A [(p_i - \hat{p}_i)\, \delta u_i + (m_i - \hat{m}_i)\, \delta\omega_i]\, dA - \\ - \int_{A_u} [(u_i - \hat{u}_i)\, \delta p_i + (\omega_i - \hat{\omega}_i)\, \delta m_i]\, dA = 0.$$

As result of independence of particular increments $\delta\gamma_{ji}$, $\delta\varkappa_{ji}$, δu_i, $\delta\omega_i$, $\delta\sigma_{ji}$, $\delta\mu_{ji}$ from each other, we obtain from Eq. (4.4) the following system of Euler equations of variational problem

$$(4.5)\qquad \begin{aligned} &\sigma_{ji,j} + X_i = 0, \quad \ni_{ijk}\, \sigma_{jk} + \mu_{ji,j} + Y_i = 0, \quad x \in V, \\ &\gamma_{ji} = u_{i,j} - \epsilon_{ijk}\, \omega_i, \quad \varkappa_{ji} = \omega_{i,j}, \quad x \in V, \\ &\frac{\partial W_\varepsilon}{\partial \gamma_{ji}} = \sigma_{ji} + \delta_{ij}\, \nu\theta, \quad \frac{\partial W_\varepsilon}{\partial \varkappa_{ji}} = \mu_{ji}, \quad x \in V, \end{aligned}$$

$$(4.6)\qquad \begin{aligned} &p_i = \hat{p}_i, \quad m_i = \hat{m}_i, \quad x \in A_\sigma, \\ &u_i = \hat{u}_i, \quad \omega_i = \hat{\omega}_i, \quad x \in A_u. \end{aligned}$$

This is the basic system of equations of the theory of asymmetric thermoelasticity. The theorem of E. Reissner extended to the problems of asymmetric thermoelasticity states that from among all the stress states σ_{ji}, couple-stresses states μ_{ji}, displacement states u_i and rotation states ω_i, satisfying the boundary conditions (4.6) and equilibrium equations (4.5) — only those actually appear which reduce the functional I to a minimum.

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В. НОВАЦКИЙ, **НЕКОТОРЫЕ ТЕОРЕМЫ, КАСАЮЩИЕСЯ НЕСИММЕТРИЧЕСКОЙ ТЕРМОУПРУГОСТИ**

В настоящей работе выведены следующие вариационные теоремы несимметрической термоупругости, а именно: теорема о минимуме потенциальной энергии, теорема о минимуме дополнительной работы, а также расширенная теорема Е. Рейсснера.