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APPLIED MECHANICS
(THERMOELASTICITY)

Thermoelastic Distortion Problems

by

W. NOWACKI

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1. Introduction In the present Note we shall be concerned with a simply connected, homogeneous and isotropic elastic body into which distortions (we denote them by ε_{ij}^0) are introduced. These distortions will be considered as functions of position and time, continuous and differentiable.

The distortions (initial deformations) ε_{ij}^0 may arise in various ways. Thus, they may be plastic strains formed in the body under the effect of previous loading or else alterations due to plastic hot working or, finally, deformations resulting from errors in assembling (as, e. g., in plates or shells).

Would the distortions verify the equation of compatibility, there should be no reason for stresses to be induced in the body. If, on the contrary, the distortions do not comply with the conditions of compatibility, then in an elastic body the stresses, σ_{ij}, and temperature, θ will appear.

We shall attempt in this paper to establish a linear theory of thermoelasticity distortional effects being taken into account, a theory based on thermodynamics of irreversible processes. Thus, we shall derive constitutive relations, a formula for free energy and entropy, we shall give a complete set of differential equations of the distortion problem and, finally, formulate the variation principle and derive the theorem on reciprocity.

2. Considerations on thermodynamics. The starting point for our considerations will be:

a) equation of motion

$$\sigma_{ij,j}+X_i=\varrho\ddot{u}_i, \tag{2.1}$$

b) equation of energy

$$T\dot{S}=-q_{i,i}, \tag{2.2}$$

c) differential equation derived from the second law of thermodynamics

$$dU=\sigma_{ij}\,d\varepsilon_{ij}+TdS. \tag{2.3}$$

In the above equations following notations were used: σ_{ij} denotes the tensor of the state of stress, ε_{ij} — that of the state of strain, $\vec{X}$ — the vector of body forces, $\vec{q}$ — the vector of heat flux, $\vec{u}$ — the vector of displacement, U — internal energy, S — entropy, ϱ — density and T — absolute temperature. The point above the symbol of a function denotes its time derivative.

Eq. (2.2) may be presented in the form

$$\dot{S} = -\left(\frac{q_i}{T}\right)_{,i} + \sigma, \qquad \sigma = -\frac{q_i T_{,i}}{T^2}. \tag{2.4}$$

The first term on the right-hand side of Eq. (2.4) refers to the exchange of entropy between the body and its surroundings, while σ denotes the production of entropy due to heat conductivity. In conformity with the postulate of thermodynamics of irreversible processes there should be $\sigma \geqq 0$. This inequality — Onsager's theorem being applied — will be verified by the Fourier conduction law

$$q_i = -kT_{,i}. \tag{2.5}$$

Let us introduce the expression for the Helmholtz's free energy $F = U - ST$, where $F \equiv F(\varepsilon_{ij}, T)$ and expand it — in the vicinity of natural state (where $\varepsilon_{ij} = \varepsilon_{ij}^0$, $T = T_0$) — into Taylor series with respect to the powers $\varepsilon_{ij} - \varepsilon_{ij}^0$ and $T - T_0 = \theta$. Discarding the terms of third (and higher) order, we obtain

$$\begin{aligned} F(\varepsilon_{ij}, T) &= \mu_T(\varepsilon_{ij} - \varepsilon_{ij}^0)(\varepsilon_{ij} - \varepsilon_{ij}^0) + \frac{\lambda_T}{2}(e - e^0)^2 - \gamma_T(e - e^0)\,\theta - \frac{m}{2}\theta^2, \\ e &= \varepsilon_{kk} \qquad e^0 = \varepsilon_{kk}^0, \end{aligned} \tag{2.6}$$

whence

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} = 2\mu_T(\varepsilon_{ij} - \varepsilon_{ij}^0) + [\lambda_T(e - e^0) - \gamma_T\,\theta]\,\delta_{ij}. \tag{2.7}$$

Linear terms do not appear in formula (2.6), since the free energy, entropy and tensor of the state of stress should vanish in the natural state. The symbols μ_T and λ_T appearing in relation (2.6) denote Lamé constants; the quantities γ_T and m will be defined in subsequent paragraphs.

Solving the constitutive relations (2.7) with respect to strains, we get

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon_{ij}^0 + \frac{\gamma_T}{3K_T}\delta_{ij}\,\theta + 2\mu_T'\sigma_{ij} + \lambda_T'\sigma_{kk}\,\delta_{ij}, \qquad \mu_T' = \frac{1}{4\mu}, \\ \lambda_T' &= -\frac{\lambda_T}{6\mu_T K_T}, \qquad K_T = \lambda_T + \frac{2}{3}\mu_T. \end{aligned} \tag{2.8}$$

To determine the quantity γ_T we have to consider the free dilatation of a volume element assuming the distortion and stresses are none. It results from (2.8) that — for $\varepsilon_{ij}^0 = 0$, $\sigma_{ij} = 0$ — there is

$$\varepsilon_{ij}' = \frac{\gamma_T}{3K_T}\delta_{ij}\,\theta = \alpha_t\,\delta_{ij}\,\theta. \tag{2.9}$$

Here the symbol α_t stands for the coefficient of linear thermal dilatation and the relation (2.9) expresses the property of an isotropic body consisting in that with the rise of temperature only the elementary volume of body changes. Consequently, $\gamma_T = 3K_T \alpha_T$.

Let us now determine the entropy S. Introducing

$$dS = \left(\frac{\partial S}{\partial \varepsilon_{ij}}\right)_T d\varepsilon_{ij} + \left(\frac{\partial S}{\partial T}\right)_\varepsilon dT \tag{2.10}$$

into (2.3) and taking advantage of the condition that dU should be an exact differential, we obtain

$$\left(\frac{\partial S}{\partial \varepsilon_{ij}}\right)_T = -\left(\frac{\partial \sigma_{ij}}{\partial T}\right)_\varepsilon = \gamma_T \delta_{ij}. \tag{2.11}$$

Thus,

$$dS = \gamma_T de + \frac{\lambda_\varepsilon}{T} dT, \tag{2.12}$$

where $c_\varepsilon = T\left(\frac{\partial S}{\partial T}\right)_\varepsilon$ means the specific heat of the body under constant strain. Integrating (2.12) – assuming that for natural state $S = 0$ – we get

$$S = \gamma_T (e - e^0) + c_\varepsilon \log \frac{T}{T_0}. \tag{2.13}$$

Assuming $\left|\frac{\theta}{T_0}\right| \ll 1$ and expanding $\log T/T_0$ into a series (only its first term will be retained), we obtain

$$S \approx \gamma_T (e - e^0) + c_\varepsilon \frac{\theta}{T_0}. \tag{2.14}$$

Now, introducing (2.12) into (2.3) and performing integration, we obtain

$$U = L + T_0 \gamma_T (e - e^0) + c_\varepsilon \theta, \quad L = \mu_T (\varepsilon_{ij} - \varepsilon_{ij}^0)(\varepsilon_{ij} - \varepsilon_{ij}^0) + \frac{\lambda_T}{2}(e - e^0)^2. \tag{2.15}$$

The value of U being known, we obtain the formula determining the free energy. It reads

$$F = U - ST = L - \gamma_T (e - e^0)\,\theta + c_\varepsilon \theta - c_\varepsilon \log \frac{T}{T_0}. \tag{2.16}$$

Expanding $\log T/T_0$ into a series and retaining two first terms of this series, we obtain the following expression for the free energy

$$F = L - \gamma_T (e - e^0)\,\theta - \frac{c_\varepsilon}{2T_0} \theta^2. \tag{2.17}$$

Comparing formulae (2.6) and (2.17), we conclude that $m = \frac{c_\varepsilon}{T_0}$.

Combining Eqs. (2.3), (2.5) and (2.13), we obtain the equation for heat conductivity

$$kT_{,ji} = T\gamma_T(\dot{e}-\dot{e}^0)+c_\varepsilon\dot{T}, \tag{2.18}$$

or

$$\begin{gathered}\theta_{,jj} = \frac{1}{\varkappa}\dot{\theta} - \eta_T(\dot{e}-\dot{e}^0)\left(1+\frac{\theta}{T_0}\right) = 0,\\ \varkappa = \frac{k}{c_\varepsilon}, \quad \eta_T = \frac{\gamma_T T_0}{k}, \quad T = T_0+\theta.\end{gathered} \tag{2.19}$$

Eq. (2.19) may be linearized under the assumption $\left|\frac{\theta}{T_0}\right| \ll 1$. Taking into account the sources of heat we get finally

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta_T(\dot{e}-\dot{e}^0) = -\frac{Q}{\varkappa}, \quad Q = \frac{W}{k}. \tag{2.20}$$

The symbol W denotes the quantity of heat produced in time and volume unit.

3. Differential equations of thermoelasticity with distortions. Introducing constitutive equations (2.7) into the equation of motion (2.1), we obtain the following set of equations

$$\mu_T u_{i,jj}+(\lambda_T+\mu_T)u_{i,jj}+X_i = \varrho\ddot{u}_i+\gamma_T\theta_{,i}+\Lambda_i, \tag{3.1}$$

where

$$\Lambda_i = 2\mu\varepsilon^0_{ij,j}+\lambda_T e^0_{,i} \quad \text{or} \quad \Lambda_i = 2\mu_T e^0_{ij}+K_T e^0_{,i}, \tag{3.2}$$

provided we denote by e^0_{ij} the following relation

$$e^0_{ij} = \varepsilon^0_{ij} - \frac{1}{3}\delta_{ij}e^0.$$

Eqs. (3.1) together with the equation of heat conductivity

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta_T\dot{e} = -\frac{Q}{\varkappa} - \eta_T\dot{e}^0, \tag{3.3}$$

form a set of differential equations of thermoelasticity, the effect of distortion ε^0_{ij} being accounted for. For $\varepsilon^0_{ij} = 0$. Eqs. (3.1) and (3.3) reduce to the known equations of thermoelasticity [1].

Eqs. (3.1) and (3.3) forming this set may be separated if we introduce the vector function $\vec{\varphi}$ and the scalar one, ζ. These functions are connected with the displacements u_i and temperature θ by the following relations [2]

$$u_i = (\Omega\delta_{ij} - \Gamma\partial_i\partial_j)\varphi_j+\gamma_0\partial_i\zeta, \tag{3.4}$$

$$\theta = \eta_T\partial_i\partial_j\Box^2_2\varphi_j+(1+a)\Box^2_1\zeta, \tag{3.5}$$

where

$$\Omega = (1+a)\,\Box_1^2 D - \gamma_0\,\eta_T\,\partial_t\,\nabla^2, \quad \Gamma = aD - \gamma_0\,\eta_T\,\partial_t,$$

$$\Box_1^2 = \nabla^2 - \frac{1}{c_1^2}\,\partial_t^2, \quad \Box_2^2 = \nabla^2 - \frac{1}{c_2^2}\,\partial_t^2, \quad D = \nabla^2 - \frac{1}{\varkappa}\,\partial_t,$$

$$a = \frac{\mu_T+\lambda_T}{\mu_T}, \quad \gamma_0 = \frac{\gamma_T}{\mu_T}, \quad c_1 = \left(\frac{\lambda_T+2\mu_T}{\varrho}\right)^{1/2} \quad c_2 = \left(\frac{\mu_T}{\varrho}\right)^{1/2}.$$

Introducing Eqs. (3.4) and (3.5) into Eqs. (3.1) and (3.3), respectively, we obtain a system of wave equations as below:

$$\Box_2^2\,(\Box_1^2 D - m_T\,\eta_T\,\partial_t\,\nabla^2)\,\varphi_i + \frac{1}{c_2^2\,\varrho}\,(X_i - \Lambda_i) = 0, \tag{3.6}$$

$$(\Box_1^2 D - m_T\,\eta_T\,\partial_t\,\nabla^2)\,\zeta + \frac{\mu_T}{c_1^2\,\varrho}\,(Q - \eta_T\,\dot{e}^0) = 0, \quad m_T = \frac{\gamma_T}{c_1^2\,\varrho}. \tag{3.7}$$

The solution of Eqs. (3.1) and (3.3) become markedly simplified for the particular case $\varepsilon_{ij} = \frac{1}{3}\,\delta_{ij}\,e^0$.

Assuming

$$\vec{u} = \operatorname{grad}\Phi + \operatorname{rot}\vec{\psi}, \quad \vec{X} = \varrho\,(\operatorname{grad}\vartheta + \operatorname{rot}\vec{\chi}), \tag{3.8}$$

we reduce the set of Eqs. (3.1) and (3.3) to the following system of wave equations

$$(\Box_1^2 D - \eta_T\,m_T\,\partial_t\,\nabla^2)\,\Phi = -\frac{m_T\,Q}{\varkappa} + \beta_T\,De^0 - \eta_T\,m_T\,\dot{e}^0 - \frac{1}{c_1^2}\,\vartheta, \tag{3.9}$$

$$\Box_2^2\,\vec{\psi} = -\frac{1}{c_2^2}\,\vec{\chi}, \quad \beta_T = \frac{K_T}{c_1^2\,\varrho}. \tag{3.10}$$

It is evident that in an unbounded medium the longitudinal waves are caused by: sources of heat Q, distortions e^0 and body forces $\varrho\operatorname{grad}\vartheta$.

The temperature induced by the action of these causes may be determined from the following formula

$$\theta = \frac{1}{m_T}\left(\Box_1^2\,\Phi - \beta_T\,e^0 + \frac{1}{c_1^2}\,\vartheta\right). \tag{3.11}$$

From Eqs. (3.1) and (3.3) we may pass to the limit case, that is to elastokinetics, when no sources of heat exist, motion is assumed to proceed under conditions of adiabatic thermodynamic process ($\dot{S} = 0$, $S = \text{const}$).

It results from Eq. (21.4) that, for $\dot{S} = 0$, there is

$$\gamma_T\,(\dot{e} - \dot{e}^0) + \frac{c_\varepsilon}{T_0}\,\dot{\theta} = 0.$$

Integrating the above expression with respect to time under the assumption that for the initial state there is $\theta(x, 0) = 0$, $e(x, 0) = 0$, $e^0(x, 0) = 0$, we obtain the following relation

$$\theta = -\varkappa \eta_T (e - e^0). \tag{3.12}$$

Eq. (3.12) takes the place of equation of heat conductivity (3.3). Introducing Eq. (3.12) into Eq. (3.1), we obtain

$$\mu_s u_{i,jj} + (\lambda_s + \mu_s) u_{j,ji} + X_i = \varrho \ddot{u}_i + 2\mu_s \varepsilon^0_{ij,j} + \lambda_s e^0_{,i}. \tag{3.13}$$

The symbols $\mu_s = \mu_T$, $\lambda_s = \lambda_T + \varkappa \eta_T m_T$ stand for Lamé constants for the adiabatic state.

4. Variational theorem. The starting point for our considerations is the principle of virtual work accompanying the variation of displacements, [3]

$$\int_V (X_i - \varrho \ddot{u}_i)\, \delta u_i \, dV + \int_A p_i \, \delta u_i \, dA = \int_V \sigma_{ij}\, \delta \varepsilon_{ij}\, dV. \tag{4.1}$$

This principle holds true for all the relations between the state of stress and that of strain.

Introducing into (4.1) the constitutive relations (2.7)

$$\sigma_{ij} = 2\mu_T(\varepsilon_{ij} - \varepsilon^0_{ij}) + [\lambda_T(e - e^0) - \gamma_T \theta]\, \delta_{ij}, \tag{4.2}$$

we obtain

$$\int_V (X_i - \varrho \ddot{u}_i)\, \delta u_i \, dV + \int_A p_i \, \delta u_i \, dA + \gamma_T \int_V \theta \delta e \, dV + \int_V [2\mu_T \varepsilon^0_{ij}\, \delta\varepsilon_{ij} + \lambda_T e^0 \, \delta e]\, dV = \delta W. \tag{4.3}$$

Here

$$W = \mu_T \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda_T}{2} e^2.$$

Following Biot, [1], we introduce the vector function $\vec{H}$. This function is related to the vector of heat flux $\vec{q}$ and the entropy S by the following formulae

$$\vec{q} = T_0 \dot{\vec{H}} = -k \operatorname{grad} \theta, \quad S = -\operatorname{div}(\vec{H}). \tag{4.4}$$

Comparing formulae (2.14) and (4.4), we get

$$T_0 \dot{S} = -\dot{H}_{i,i} T_0 = T_0 \gamma_T \dot{e} + c_\varepsilon \dot{\theta}, \quad \delta H_{i,i} = \gamma_T \delta e + \frac{c_\varepsilon}{T_0} \delta\theta. \tag{4.5}$$

Let us now multiply the first of the equation group (4.4) by δH_i and integrate it over the region V. Taking advantage of the second of the equation set (4.5), we obtain

$$\gamma_T \int_V \theta \delta e \, dV + \frac{c_\varepsilon}{T_0} \int_V \theta \delta\theta \, dV + \int_A \theta \delta H_n \, dA + \frac{T_0}{k} \int_V \dot{H}_i \, \delta H_i \, dV = 0. \tag{4.6}$$

After eliminating from Eqs. (4.3) and (4.6) the integral $\int_V \theta \delta e \, dV$ and introducing the function of dissipation and the heat potential

$$D = \frac{T_0}{2k} \int_V (\dot{H}_i)^2 \, dV, \qquad \delta D = \frac{T_0}{k} \int_V \dot{H}_i \, \delta H_i \, dV, \qquad P = \frac{c_\varepsilon}{T_0} \int_V \theta^2 \, dV,$$

we obtain the following two forms of the variational theorem

$$(4.7') \qquad \delta(W+P+D) = \int_V (X_i - \varrho \ddot{u}_i) \, \delta u_i \, dV + \int_A p_i \, \delta u_i \, dA + \int_V \varepsilon_{ij}^0 \, \delta \hat{\sigma}_{ij} \, dV - \int_A \theta \delta H_n \, dA,$$

or

$$(4.7'') \qquad \delta(W+P+D) = \int_V (X_i - \varrho \ddot{u}_i) \, \delta u_i \, dV + \int_A p_i \, \delta u_i \, dA + \int_V \Gamma_{ij}^0 \, \delta \varepsilon_{ij} \, dV - \int_A \theta \delta H_n \, dA,$$

where

$$\delta \hat{\sigma}_{ij} = 2\mu_T \, \delta \varepsilon_{ij} + \lambda_T \, \delta_{ij} \, \delta e, \qquad \Gamma_{ij}^0 = 2\mu_T \, \varepsilon_{ij}^0 + \lambda_T \, \delta_{ij} \, e^0.$$

From the variational principle (4.7′) and (4.7″) we can derive the theorem on energy, putting $u_i + \delta u_i = u_i + \frac{\partial u_i}{\partial t} dt$, and considering the actual motion of the body at the same point after a time lag dt. Introducing into (4.7″) the following notations:

$$\delta u_i = \frac{\partial u_i}{\partial t} dt = v_i \, dt, \qquad \delta \theta = \dot{\theta} dt, \qquad \delta H_i = \dot{H}_i \, dt = -\frac{k}{T_0} \theta_{,i} \, dt, \qquad \text{and so on,}$$

we obtain the equation

$$(4.8) \qquad \frac{d}{dt}(K+P+W) + \chi_0 = \int_V X_i \, v_i \, dV + \int_A p_i \, v_i \, dA + \int_V \Gamma_{ij}^0 \, \dot{\varepsilon}_{ij} \, dV + \frac{k}{T_0} \int_A \theta \theta_{,n} \, dA.$$

where

$$K = \frac{1}{2} \varrho \int_V v_i \, v_i \, dV, \qquad \chi_0 = \frac{k}{T_0} \int_V \theta_{,i} \, \theta_{,i} \, dV > 0,$$

Proceeding similarly to the methods described in [4] we may derive from the equation of energy the theorem on uniqueness of the solution of differential equations of thermoelasticity. We shall now consider a particular case, namely the transition from Eqs. (4.7′) and (4.7″) to classical elastokinetics. Assuming that the sources of heat are lacking and bearing in mind the basic assumption of elastokinetics, i.e., that the motion proceeds in adiabatic conditions ($\dot{S} = 0$, $S = \text{const}$), we obtain from (4.5):

$$\theta = -\varkappa\eta_T(e-e^0), \qquad \eta_T = \frac{\gamma_T T_0}{k}. \tag{4.9}$$

Introducing (4.9) into Eq. (4.3), we get

$$\int_V (X_i - \varrho\ddot{u}_i)\,\delta u_i\,dV + \int_A p_i\,du_i\,dA + \int_V \varepsilon_{ij}^0\,\delta\sigma_{ij}^*\,dV = \delta W^*. \tag{4.10}$$

Following notations have been introduced in (4.10):

$$\delta\sigma_{ij}^* = 2\mu_s\,\delta\varepsilon_{ij} + \lambda_s\,\delta_{ij}\,\delta e, \qquad \delta W^* = \int_V (2\mu_s\,\varepsilon_{ij}\,\delta\varepsilon_{ij} + \lambda_s\,e\delta e)\,dV,$$

$\mu_s = \mu_T$, $\lambda_s = \lambda_T + \varkappa\eta_T m_T$ being Lamé constants for adiabatic conditions. Now, the basic equation of energy for elastokinetics will take the form:

$$\frac{d}{dt}(K+W^*) = \int_V X_i v_i\,dV + \int_A p_i v_i\,dA + \int_V \varepsilon_{ij}^0\,\dot{\sigma}_{ij}^*\,dV, \tag{4.11'}$$

or

$$\frac{d}{dt}(K+W^*) = \int_V X_i v_i\,dV + \int_A p_i v_i\,dA + \int_V \Gamma_{ij}^{0*}\,\varepsilon_{ij}\,dV, \tag{4.11''}$$

where

$$\Gamma_{ij}^{0*} = 2\mu_s\,\varepsilon_{ij}^0 + \lambda_s\,\delta_{ij}\,e^0.$$

The displacements u_i appearing in equations and relations (4.9)–(4.11″) have a different meaning than in formulae (4.1)–(4.8). We determine them solving the following set of equations:

$$\mu_s\,u_{i,jj} + (\lambda_s+\mu_s)\,u_{j,ji} + X_i = \varrho\ddot{u}_i + \Lambda_i^*, \tag{4.12}$$

where

$$\Lambda_i^* = 2\mu_s\,\varepsilon_{ij}^0 + \lambda_s\,e_{,i}^0.$$

5. Theorem on reciprocity. In this paragraph we shall consider two systems of causes and effects. As causes provoking the motion we consider body forces, surface forces, sources of heat, surface heating and distortions, while displacements u_i and temperature θ will be considered as effects. The second system of causes and effects will be distinguished from the first by primes.

As a starting point for our considerations we take the equations of motion and of heat conductivity after the inverse Laplace transformation has been performed on them. Initial conditions are assumed to be homogeneous.

We have then

$$\bar{\sigma}_{ij,j} + \bar{X}_i = p^2\,\bar{u}_i, \qquad \bar{\sigma}'_{ij,j} + X'_i = p^2\,\bar{u}_i, \tag{5.1}$$

$$\begin{aligned} &\bar{\theta}_{,jj} - \frac{p}{\varkappa}\bar{\theta} - \eta_T p\bar{e} = -\frac{1}{\varkappa}\bar{Q} - \eta_T p\bar{e}^0, \\ &\bar{\theta}'_{,jj} - \frac{p}{\varkappa}\bar{\theta}' - \eta_T p\bar{e}' = -\frac{1}{\varkappa}\bar{Q}' - \eta_T p\bar{e}'^0. \end{aligned} \tag{5.2}$$

where

$$\bar{\sigma}_{ij}(x,p) = \mathcal{L}[\sigma_{ij}(x,t)] = \int_0^\infty \sigma_{ij}(x,t)\,e^{-pt}\,dt, \qquad \text{and so on.}$$

We multiply the first of Eqs. (5.1) by $\bar{u}'_i$, and the second — by $\bar{u}_i$.

We subtract the second from the first and integrate the resulting formula over the volume of the body. After simple transformations we obtain

$$(5.3)\qquad \int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\,dV + \int_A (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\,dA = \int_V (\bar{\sigma}_{ij}\bar{\varepsilon}'_{ij} - \bar{\sigma}'_{ij}\bar{\varepsilon}_{ij})\,dV.$$

Making use of constitution equations (2.7),we reduce (5.3) to the form

$$(5.4)\qquad \int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\,dV + \int_A (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\,dA + \gamma_T \int_V (\bar{\theta}\bar{e}' - \bar{\theta}'\bar{e})\,dV +$$
$$+ \int_V [2\mu_T(\bar{\varepsilon}^0_{ij}\bar{\varepsilon}'_{ij} - \bar{\varepsilon}'^0_{ij}\bar{\varepsilon}_{ij}) + \lambda_T(\bar{e}^0\bar{e}' - \bar{e}'^0\bar{e})]\,dV = 0.$$

Similar operations will be performed on the group of equations (5.2): we multiply the first equation by θ' and the second — by $\bar{\theta}$. We subtract then the second from the first and integrate the resulting formula over the volume of the body V. After simple transformations we get

$$(5.5)\qquad \int_A (\bar{\theta}'\bar{\theta}_{,n} - \bar{\theta}\bar{\theta}'_{,n})\,dA - p\eta_T \int_V (\bar{e}\bar{\theta}' - \bar{e}'\bar{\theta})\,dV +$$
$$+ p\eta_T \int_V (\bar{e}^0\bar{\theta}' - \bar{e}'^0\bar{\theta})\,dV + \frac{1}{\varkappa}\int_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV = 0.$$

Eliminating from (5.4) and (5.5) the term common to both, we obtain the following equation

$$(5.6)\qquad \eta_T \varkappa p \left\{ \int_V (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i)\,dV + \int_A (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\,dA + \int_V (\bar{\varepsilon}^0_{ij}\bar{\sigma}'_{ij} - \bar{\varepsilon}^0_{ij}\bar{\sigma}_{ij})\,dV \right\} =$$
$$= \varkappa\gamma_T \int_A (\bar{\theta}'\bar{\theta}_{,n} - \bar{\theta}\bar{\theta}'_{,n})\,dA + \gamma_T \int_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV.$$

Here:

$$\bar{\sigma}_{ij} = 2\mu_T\bar{\varepsilon}_{ij} + (\lambda_T\bar{e} - \gamma_T\bar{\theta})\,\delta_{ij}, \qquad \bar{\sigma}'_{ij} = 2\mu_T\bar{\varepsilon}'_{ij} + (\lambda_T\bar{e}' - \gamma_T\bar{\theta}')\,\delta_{ij}.$$

Performing on (5.6) the inverse Laplace transformation, we obtain the theorem on reciprocity in its final form, the effect of distortion being taken into account.

$$(5.7)\qquad \eta_T\varkappa \left\{ \int_V dV(x) \int_0^t d\tau \left[X_i(x,\tau)\frac{\partial u'_i(x,t-\tau)}{\partial\tau} - X'_i(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau} \right] + \right.$$
$$+ \int_A dA(x) \int_0^t d\tau \left[p_i(x,\tau)\frac{\partial u'_i(x,t-\tau)}{\partial\tau} - p'_i(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau} \right] +$$

$$+\int_V dV(x)\int_0^t d\tau\left[\varepsilon_{ij}^0(x,\tau)\frac{\partial\sigma'_{ij}(x,t-\tau)}{\partial\tau}-\varepsilon_{ij}'^0(x,t-\tau)\frac{\partial\sigma_{ij}(x,\tau)}{\partial\tau}\right]\Bigg\}=$$

$$=\varkappa\gamma_T\int_A dA(x)\int_0^t d\tau\,[\theta'(x,t-\tau)\,\theta_{,n}(x,\tau)-\theta'_{,n}(x,t-\tau)\,\theta(x,\tau)]+$$

$$+\gamma_T\int_V dV(x)\int_0^t d\tau\,[Q(x,\tau)\,\theta'(x,t-\tau)-Q'(x,t-\tau)\,\theta(x,\tau)].$$

For $\varepsilon_{ij}^0=0$, $\varepsilon_{ij}'^0=0$ the above theorem reduces to the theorem given by V. Jonescu--Cazimir in [5].

In a particular case of an unbounded region Eq. (5.7) becomes markedly simplified. If, namely, the body forces, sources of heat and distortions act in a bounded region, the surface integrals in (5.7) vanish.

Let us consider now the transition from the coupled problem of thermoelasticity with distortions to the elastokinetic problem. The adiabacity of the process being assumed following relations hold true

$$\theta=-\varkappa\eta_T(e-e^0),\qquad \theta'=-\varkappa\eta_T(e'-e'^0). \tag{5.8}$$

They take place of the equations of heat conductivity. We perform on (5.8) the inverse Laplace transformation and then introduce the formula obtained into Eq. (5.4). After simple transformations, we obtain the theorem on reciprocity for elastokinetics

$$\int_V(\bar X_i\bar u_i'-\bar X_i'\bar u_i)\,dV+\int_A(\bar p_i\bar u_i'-\bar p_i'\bar u_i)\,dA+\int_V(\bar\varepsilon_{ij}^0\bar\sigma_{ij}'-\bar\varepsilon_{ij}'^0\bar\sigma_{ij})\,dV=0. \tag{5.9}$$

The functions $\bar u_i$, $\bar u_i'$, $\bar\sigma_{ij}$, $\bar\sigma_{ij}'$ appearing in (5.9) have a different meaning than those in Eq. (5.6). The displacements $\bar u_i$ and $\bar u_i'$ may be determined from the differential equations

$$\begin{aligned}\mu_s\bar u_{i,jj}+(\lambda_s+\mu_s)\bar u_{j,ji}+\bar X_i&=p^2\bar u_i+\bar\Lambda_i,\\ \mu_s\bar u'_{i,jj}+(\lambda_s+\mu_s)\bar u'_{j,ji}+\bar X_i'&=p^2\bar u_i'+\bar\Lambda_i',\end{aligned} \tag{5.10}$$

where

$$\bar\Lambda_i=2\mu_s\bar\varepsilon_{ij,j}^0+\lambda_s\bar e_{,i}^0,\qquad \bar\Lambda_i'=2\mu^s\bar\varepsilon_{ij,j}'^0+\lambda_s\bar e_{,i}'^0,$$

while the stresses $\bar\sigma_{ij}$ and $\bar\sigma_{ij}'$ from the relations

$$\bar\sigma_{ij}=2\mu_s\bar\varepsilon_{ij}+\lambda_s\delta_{ij}\bar e,\qquad \bar\sigma_{ij}'=2\mu_s\bar\varepsilon_{ij}'+\lambda_s\delta_{ij}\bar e'. \tag{5.11}$$

$\mu_s=\mu_T$, $\lambda_s=\lambda_T+\varkappa\eta_T m_T$ are here Lamé constants for adiabatic state. Performing on (5.9) the inverse Laplace transformation, we obtain

$$\begin{aligned}&\int_V dV(x)\int_0^t d\tau\,[X_i(x,\tau)\,u_i'(x,t-\tau)-X_i'(x,t-\tau)\,u_i(x,\tau)]+\\ &\quad+\int_A dA(x)\int_0^t d\tau\,[p_i(x,\tau)\,u_i'(x,t-\tau)-p_i'(x,t-\tau)\,u_i(x,\tau)]+\\ &\quad+\int_V dV(x)\int_0^t d\tau\,[\varepsilon_{ij}^0(x,\tau)\,\sigma_{ij}'(x,t-\tau)-\varepsilon_{ij}'^0(x,t-\tau)\,\sigma_{ij}(x,\tau)]=0.\end{aligned} \tag{5.12}$$

From the theorem on reciprocity (5.7),we can derive formulae similar to the known formulae of Somigliano and Green for elastostatics [3], [3], as well as formulae similar to those of Meysel [6] in the theory of thermal stresses.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA, INSTITUTE OF FUNDAMENTAL TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES
(ZAKŁAD MECHANIKI OŚRODKÓW CIĄGŁYCH, INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI, PAN)

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В. НОВАЦКИЙ, ДИСТОРСИОННЫЕ ЗАДАЧИ ТЕРМОУПРУГОСТИ.

В работе выведены конститутивные уравнения и основные дифференциальные уравнения. Сформулирован вариационный принцип и выведена теорема взаимности для дисторсионной задачи термоупругости.