BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES TECHNIQUES

Volume XIV, Numéro 3

VARSOVIE 1966

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume XIV, No. 3 — 1966

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APPLIED MECHANICS

Couple-stresses in the Theory of Thermoelasticity. II.

by

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Presented on November 5, 1965

1. Introduction

In [1] we derived constitutive equations of thermoelasticity for the Cossérat medium and gave basic displacement differential equations as well as an extended equation of heat conduction. Finally, we gave a system of wave equations obtained from separation of basic equations of thermoelasticity. In the present paper we give the virtual woik principle, extended to the pioblem of thermoelasticity for the Cossérat medium as well as the energetic theorem, uniqueness of solutions, and the theorem on reciprocity and its consequences. We shall use here the notations of [1] and will add some new ones.

2. Virtual work principle

The virtual work principle for isothermic and static problem was given by W. T. Koiter [2]. For dynamic problems considered here this principle has the form:

$$(2.1)\qquad \int_V [(X_i - \varrho\ddot{u}_i)\,\delta u_i + Y_i\,\delta\omega_i]\,dV + \int_A (p_i\,\delta u_i + g_i\,\delta\omega_i)\,dA = \int_V (s_{ij}\,\delta\gamma_{ij} + m_{ji}\,\delta\varkappa_{ij})\,dV.$$

The left-hand side of this equation denotes variation of work of external forces, whereas the right — the variation of internal forces.

Inserting into (2.1) the constitutional relations

$$(2.2)\qquad s_{ij} = 2\mu\,\gamma_{ij} + (\lambda\gamma_{kk} - \beta\theta)\,\delta_{ij},$$

$$(2.3)\qquad m_{ji} = 4\mu l^2\,(\varkappa_{ij} + \eta\varkappa_{ji}),$$

we express the right-hand side of (2.1) by the components of strain tensor γ_{ij}, components of torsion-flexure tensor $\varkappa_{ij}$ and temperature θ:

$$(2.4)\qquad \int_V [(X_i - \varrho\ddot{u}_i)\,\delta u_i + Y_i\,\delta\omega_i]\,dV + \int_A (p_i\,\delta u_i + g_i\,\delta\omega_i)\,dA = \delta W - \beta\int_V \theta\delta\gamma_{kk}\,dV,$$

where

$$\delta W = \int_V [2\mu\gamma_{ij}\,\delta\gamma_{ij} + \lambda\gamma_{kk}\,\delta\gamma_{kk} + 4\mu l^2 (\varkappa_{ij}\,\delta\varkappa_{ij} + \eta\varkappa_{ji}\,\delta\varkappa_{ji})]\,dV.$$

Eq. (2.4) should be supplemented by further equation, since only four causes, namely X_i, Y_i, p_i, g_i appear in this equation in explicit form.

We have adjoined to Eq. (2.4) the relation

$$(2.5)\qquad -\beta\int_V \theta\delta\gamma_{kk}\,dV = \int_A \theta n_i\,\delta H_i\,dA + \frac{c_\varepsilon}{T_0}\int_V \theta\delta\theta\,dV + \frac{T_0}{k}\int_V \dot{H}_i\,\delta H_i\,dV,$$

derived from the equation of heat conduction by M. A. Biot [3]. The vector $\vec{H}$ in (2.5) is connected with the vector of heat flow $\vec{q}$ and the entropy S by following relations

$$(2.6)\qquad \vec{q} = T_0\,\dot{\vec{H}}, \quad S = -\operatorname{div}(\vec{H}).$$

Taking into account the Fourier law of heat conduction

$$(2.7)\qquad \vec{q} = -k\operatorname{grad}\theta$$

and the relation for the entropy rate [3]

$$(2.8)\qquad -\operatorname{div}\vec{q} = T_0\,\dot{S} = \beta\dot{\gamma}_{kk}T_0 + c_\varepsilon\,\dot{\theta}$$

we obtain the following connection of the vector $\vec{H}$ with temperature θ and dilatation γ_{kk}

$$(2.9)\qquad \dot{H}_i = -\frac{k}{T_0}\theta_{,i}, \quad H_{i,i} = -\frac{c_\varepsilon}{T_0}\theta - \frac{\beta}{T_0}\gamma_{kk}.$$

Introducing (2.5) into Eq. (2.4) we have

$$(2.10)\qquad \delta(W+P+D) = \int_V [(X_i - \varrho\ddot{u}_i)\,\delta u_i + Y_i\,\delta\omega_i]\,dV + \int_A (p_i\,\delta u_i + g_i\,\delta\omega_i)\,dA - \int_A \theta n_i\,\delta H_i\,dA.$$

We have applied here the heat potential P and the dissipation function D, introduced already by M. A. Biot [3]

$$(2.11)\qquad P = \frac{c_\varepsilon}{2T_0}\int_V \theta^2\,dV, \quad D = \frac{T_0}{2k}\int_V (\dot{H})^2\,dV, \quad \delta D = \frac{T_0}{k}\int_V \dot{H}_i\,\delta H_i\,dV.$$

For $Y_i = 0$, $g_i = 0$, $\varkappa_{ij} = 0$ Eq. (2.10) reduces to the variational equation of coupled thermoelastic medium without couple-stresses. The variational principle — Eq. (2.10) — may serve to derive the energetic theorem, if we compare the functions u_i, ω_i, θ in the point x at the moment t with those actually appearing in the same point after a time lapse dt. Thus, introducing into Eq. (2.10)

$$(2.12)\qquad \begin{aligned} &\delta u_i = \frac{\partial u_i}{\partial t}dt = v_i\,dt, \quad \delta\omega_i = \frac{\partial\omega_i}{\partial t}dt = w_i\,dt, \\ &\delta\theta = \dot{\theta}dt, \quad \delta H_i = \dot{H}_i\,d_t = -\frac{k}{T_0}\theta_{,i}, \quad \delta W = \dot{W}dt \end{aligned}$$

and so on.

We obtain the following formula

$$(2.13)\qquad \frac{d}{dt}(K+W+P)+\chi_0=\int\limits_V (X_i v_i+Y_i w_i)\,dV+ +\int\limits_A (p_i v_i+g_i w_i)\,dA+\frac{k}{T_0}\int\limits_A \theta\theta_{,i} n_i\,dA,$$

where

$$K=\frac{1}{2}\varrho\int\limits_V v_i v_i\,dV,\qquad \chi_0=\frac{dD}{dt}=\frac{k}{T_0}\int\limits_V (\theta_{,i})^2\,dV>0,$$

Here K denotes kinetic energy, and χ_0 is proportional to the source of entropy which is always a positive quantity.

3. Uniqueness theorem

The demonstration of the uniqueness of solutions of thermoelasticity equations may be derived basing on the energetic theorem, Eq. (2.13). Assume that there are two solutions, one characterized by the functions u_i', θ' and the other – by u_i'', θ''. Denoting the difference between these two solutions by

$$(3.1)\qquad u_i^*=u_i'-u_i'',\qquad \theta^*=\theta'-\theta''$$

it is easily seen that the functions u_i^*, θ^* satisfy homogeneous differential equations, homogeneous boundary conditions and initial conditions. Thus, the solutions u_i^*, θ^* refer to a body wherein body forces, body couples and heat sources inside the body are lacking. At the same time, there are no force vectors and couple vectors transmitted through the surface A.

Thus, it remains to prove that inside the body the values for the stresses σ_{ij}^*, deformations γ_{ij}^*, $\varkappa_{ij}^*$ and temperature θ^* are zero-values.

The entropy equation (3.13) for the functions u_i^*, θ^* will assume the following form

$$(3.2)\qquad \frac{d}{dt}(K^*+W^*+P^*)=-\chi_\theta^*\leqq 0,$$

or

$$(3.3)\qquad \frac{d}{dt}\int\limits_V\left[\frac{1}{2}\varrho v_i^* v_i^*+\mu\gamma_{ij}^*\gamma_{ij}^*+\frac{\lambda}{2}(\gamma_{kk}^*)^2+2\mu l^2(\varkappa_{ij}^*\varkappa_{ij}^*+\eta\varkappa_{ij}^*\varkappa_{ji}^*)+ +\frac{\beta}{2\eta\varkappa}(\theta^*)^2\right]dV\leqq 0.$$

Consequently, we infer from the inequality (3.3) that the integral cannot rise for $t>0$. Since the initial conditions for the integrand functions are homogeneous, the integral itself at the initial moment should be equal to zero. Its values cannot be negative as the integrand expression is a sum of squares with positive coefficients. Thus, the value of the integrand appearing in (3.3) is necessarily zero for $t\geqslant 0$. This leads to the following equalities

$$(3.4)\qquad v_i^*=0,\quad \gamma_{ij}^*=0,\quad \varkappa_{ij}^*=0,\quad \theta^*=0$$

or

$$v_i' = v_i'', \quad \gamma_{ij}' = \gamma_{ij}'', \quad \varkappa_{ij}' = \varkappa_{ij}'', \quad \theta' = \theta'' \quad \text{for} \quad t \geqslant 0 \tag{3.5}$$

in the region V.

Making use of relations (2.2) and (2.3), we see that

$$s_{ij}' = s_{ij}'', \quad m_{ji}' = m_{ji}'', \quad r_{ij}' = r_{ij}'',$$

too.

Thus we may conclude on the uniqueness of the solutions of thermoelasticity as regards the deformations, stresses and temperature. For the displacements we obtain

$$u_i' = u_0'' + \text{linear term}. \tag{3.6}$$

The linear term describes here the rotation and translation of the body considered as perfectly rigid. In the case of displacements prescribed on A, the linear term vanishes.

4. Theorem on reciprocity

Consider two sets of causes and effects acting upon an elastic body contained in the region V and bounded by the surface A. We number among causes the body forces X_i, body couples Y_i, heat sources Q, loads p_i and g_i on the surface A and heating of the surface A (given the temperature θ or flow of heat $-k\theta_{,n}$ on A). The effects are the displacements u_i, rotations ω_i and temperature θ. The second set of causes and effects will be distinguished from the first by adding the sign "prime".

Consider the expression

$$I = \int_V (\overline{X}_i \overline{u}_i' - \overline{X}_i' \overline{u}_i)\, dV + \int_V (\overline{Y}_i \overline{\omega}_i' - \overline{Y}_i' \overline{\omega}_i)\, dV + \int_A (\overline{p}_i \overline{u}_i' - \overline{p}_i' \overline{u}_i)\, dA + \int_A (\overline{g}_i \overline{\omega}_i' - \overline{g}_i' \overline{\omega}_i)\, dA. \tag{4.1}$$

Here the functions

$$\begin{aligned} \overline{X}_i(x, p) &= \mathcal{L}[X_i(x, t)] = \int_0^\infty X_i(x, t)\, e^{-pt}\, dt, \\ \overline{u}_i(x, p) &= \mathcal{L}[u_i(x, t) = \int_0^\infty u_i(x, t)\, e^{-pt}\, dt, \quad \text{and so on,} \end{aligned} \tag{4.2}$$

are Laplace transforms of the function X_i, u_i ..., etc. Transform the surface integrals appearing in the expression (4.1) making use of the known dependencies

$$p_i = \sigma_{ji} n_j, \quad g_i = m_{ji} n_j \tag{4.3}$$

and applying the divergence theorem. Thus we obtain the expression

$$I = \int_V [(\overline{X}_i + \overline{\sigma}_{ji,j}) \overline{u}_i' - (\overline{X}_i' + \overline{\sigma}_{ji,j}') \overline{u}_i + (\overline{Y}_i + \overline{m}_{ji,j}) \overline{\omega}_i' - (\overline{Y}_i' + m_{ji,j}') \overline{\omega}_i + \overline{\sigma}_{ji} \overline{u}_{i,j}' - \overline{\sigma}_{ji}' \overline{u}_{i,j} + \overline{m}_{ji} \overline{\omega}_{i,j}' - \overline{m}_{ji}' \overline{\omega}_{i,j}]\, dV. \tag{4.4}$$

Exploit the equations of motion for both sets of causes and effects. Applying to them Laplace transforms and assuming that the initial conditions are homogeneous, we have:

$$\bar{\sigma}_{ij,j}+\bar{X}_i-p^2\bar{u}_i=0,\quad \bar{\sigma}'_{ij,j}+\bar{X}'_i-p^2\bar{u}'_i=0. \tag{4.5}$$

It is evident that, taking into account Eqs. (4.5), the first two terms under the integral in expression (4.4) vanish. Further, we decompose the force-stress tensor σ_{ij} into symmetric part s_{ij} and asymmetric part r_{ij} and exploit the formulae for r_{ij}:

$$\bar{r}_{mn}=-\tfrac{1}{2}\varepsilon_{imn}(\bar{m}_{ji,j}+\bar{Y}_i),\quad \bar{r}'_{mn}=-\tfrac{1}{2}\varepsilon_{imn}(\bar{m}'_{ji,j}+\bar{Y}'_i). \tag{4.6}$$

After simple transformations the expression I assumes the form

$$I=\int_V(\bar{s}_{ji}\bar{\gamma}'_{ij}-\bar{s}'_{ij}\bar{\gamma}_{ij}+\bar{m}_{ji}\bar{\varkappa}'_{ij}-\bar{m}'_{ji}\bar{\varkappa}_{ij})\,dV. \tag{4.7}$$

Finally, making use of relations (2.2) and (2.3), we get

$$I=\beta\int_V(\bar{\theta}'\bar{\gamma}_{kk}-\bar{\theta}\bar{\gamma}'_{kk})\,dV. \tag{4.8}$$

Comparing the expressions (4.1) and (4.8), we obtain the first part of the theorem on reciprocity

$$\int_V(\bar{X}_i\bar{u}'_i-\bar{X}'_i\bar{u}_i+\bar{Y}_i\bar{\omega}'_i-\bar{Y}'_i\bar{\omega}_i)\,dV+\int_A(\bar{p}_i\bar{u}'_i-\bar{p}'_i\bar{u}_i+ \\ +\bar{g}_i\bar{\omega}'_i-\bar{g}'_i\bar{\omega}_i)\,dA+\beta\int_V(\bar{\theta}\bar{\gamma}'_{kk}-\bar{\theta}'\bar{\gamma}_{kk})\,dV=0. \tag{4.9}$$

The second part of the theorem on reciprocity is obtained by taking account of the heat conductivity equations for both sets. We apply to these equations Laplace transform assuming that the initial conditions for the temperature are homogeneous.

$$\bar{\theta}_{,jj}-\frac{p}{\varkappa}\bar{\theta}-\eta_0 p\bar{\gamma}_{kk}=-\frac{\bar{Q}}{\varkappa},\quad \bar{\theta}'_{,jj}-\frac{p}{\varkappa}\bar{\theta}'-\eta_0 p\bar{\gamma}'_{kk}=-\frac{\bar{Q}'}{\varkappa}. \tag{4.10}$$

Multiplying the first of Eqs. (4.10) by $\bar{\theta}'$, the second by $\bar{\theta}$, subtracting one from another, integrating over the region V and making use of Green's transformation, we get

$$p\eta_0\int_V(\bar{\gamma}_{kk}\bar{\theta}'-\bar{\gamma}'_{kk}\bar{\theta})\,dV+\frac{1}{\varkappa}\int_V(\bar{Q}'\bar{\theta}-\bar{Q}\bar{\theta}')\,dV- \\ -\int_A(\bar{\theta}'\bar{\theta}_{,n}-\bar{\theta}'_{,n}\bar{\theta})\,dA=0. \tag{4.11}$$

Eliminating from (4.9) and (4.11) the common term, we obtain the final form of the theorem on reciprocity for the Cossérat medium

$$(4.12)\qquad \frac{\eta_0 \varkappa p}{\beta}\Big[\int_V (\overline{X}_i \overline{u}_i' - \overline{X}_i' \overline{u}_i + \overline{Y}_i \overline{\omega}_i' - \overline{Y}_i' \overline{\omega}_i)\, dV + \int_A (\overline{p}_i \overline{u}_i' - \overline{p}_i' \overline{u}_i + \overline{g}_i \overline{\omega}_i' - \overline{g}_i' \overline{\omega}_i)\, dA + \varkappa \int_A (\overline{\theta}\,\overline{\theta}'_{,n} - \overline{\theta}_{,n}\overline{\theta}')\, dA + \int_V (\overline{Q}'\,\overline{\theta} - \overline{Q}\,\overline{\theta}')\, dV = 0.$$

It is obvious that in this relation appear all the causes and effects. Applying the inverse Laplace transformation to Eq. (4.12) we have

$$(4.13)\qquad \frac{\eta_0 \varkappa}{\beta}\Big\{\int_V dV(x) \int_0^t \Big[X_i(x,\tau)\frac{\partial u_i'(x,t-\tau)}{\partial\tau} - X_i'(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau} + Y_i(x,\tau)\frac{\partial \omega_i'(x,t-\tau)}{\partial\tau} - Y_i'(x,t-\tau)\frac{\partial \omega_i(x,\tau)}{\partial\tau}\Big] d\tau + \int_A dA(x)\int_0^t \Big[p_i(x,\tau)\frac{\partial u_i'(x,t-\tau)}{\partial\tau} - p_i'(x,t-\tau)\frac{\partial u_i(x,\tau)}{\partial\tau} + g_i(x,\tau)\frac{\partial \omega_i'(x,t-\tau)}{\partial\tau} - g_i'(x,t-\tau)\frac{\partial \omega_i(x,\tau)}{\partial\tau} + \varkappa\big(\theta(x,\tau)\,\theta'_{,n}(x,t-\tau) - \theta'(x,t-\tau)\,\theta_{,n}(x,\tau)\big)\Big] d\tau + \int_V dV(x)\int_0^t [\theta(x,\tau)\,Q'(x,t-\tau) - \theta'(x,t-\tau)\,Q(x,\tau)]\Big\} d\tau = 0.$$

The theorem on reciprocity (4.13), at $Y_i = Y_i' = 0$, $g_i = g_i' = 0$ passes into the theorem on reciprocity for an elastic medium without couple-stresses, given by V. Cazimir-Ionescu [4].

For static loads and for stationary heat flow we get the system of equations

$$(4.14)\qquad \int_V (X_i u_i' - X_i' u_i + Y_i \omega_i' - Y_i' \omega_i)\, dV + \int_A (p_i u_i' - p_i' u_i + g_i \omega_i' - g_i' \omega_i)\, dA + \beta \int_V (\theta \gamma_{kk}' - \theta' \gamma_{kk})\, dV = 0,$$

$$(4.15)\qquad \int_V (Q'\theta - Q\theta')\, dV + \varkappa \int_V (\theta\theta'_{,n} - \theta'\theta_{,n})\, dA = 0.$$

In the reciprocity equation (4.14) the temperature θ and θ' is treated as known functions, obtained from solution of the heat conductivity equations

$$(4.16)\qquad \theta_{,jj} = -\frac{Q}{\varkappa}, \quad \theta'_{,jj} = -\frac{Q'}{\varkappa}.$$

Solution (4.15) may be treated as a theorem on reciprocity for the problem of heat conductivity.

5. Conclusions from the theorem on reciprocity

Let us consider first an infinite Cossérat medium. Let a concentrated and instantaneous force $X_i = \delta(x-\xi)\,\delta(t)\,\delta_{ik}$ directed along the axis x_k be acting in the point ξ of this medium. Denote by $U_i^{(k)}(x, \xi, t)$ the displacement caused by this force. Further, let a concentrated and instantaneous force $X_i' = \delta(x-\eta)\,\delta(t)\,\delta_{ij}$, directed along the axis x_j, be acting at the point η. Denote the displacement caused by this force by $U_i^{(j)}(x, \eta, t)$. From the theorem on reciprocity (4.13) formulated for an infinite region we have

$$(5.1)\qquad \int_V dV(x) \int_0^t d\tau \left[\delta(x-\xi)\,\delta(\tau)\,\delta_{ik} \frac{\partial U_i^{(j)}(x,\eta,t-\tau)}{\partial \tau} - \right. \\ \left. - \delta(x-\eta)\,\delta(t-\tau)\,\delta_{ij} \frac{\partial U_i^{(k)}(x,\xi,\tau)}{\partial \tau} \right] = 0,$$

hence

$$\dot{U}_k^{(j)}(\xi, \eta, t) = \dot{U}_j^{(k)}(\eta, \xi, t).$$

After integration with respect to time, we finally get

$$(5.2)\qquad U_k^{(j)}(\xi, \eta, t) = U_j^{(k)}(\eta, \xi, t).$$

Let a concentrated and instantaneous force $X_i = \delta(x-\xi)\,\delta(t)\,\delta_{ik}$ be acting at the point ξ of infinite medium, and at the point η — concentrated and instantaneous source of heat $Q' = \delta(x-\eta)\,\delta(t)$. Denote by $\Theta^{(k)}(x, \xi, t)$ the temperature caused by the action of force X_i, and by $U_i(x, \eta, t)$ the displacement caused by the action of the source Q'. From Eq. (4.13) we obtain the following relation

$$(5.3)\qquad \int_V dV(x) \int_0^t d\tau \left[\delta(x-\eta)\,\delta(t-\tau) \frac{\partial \Theta(x,\xi,\tau)}{\partial \tau} + \right. \\ \left. + \frac{\eta_0 \varkappa}{\beta} \delta(x-\xi)\,\delta(\tau)\,\delta_{ik} \frac{\partial U_i(x,\eta,t-\tau)}{\partial \tau} \right] = 0,$$

wherefrom

$$(5.4)\qquad \Theta^{(k)}(\eta, \xi, t) = -\frac{\eta_0 \varkappa}{\beta} \frac{\partial U_k(\xi, \eta, t)}{\partial t}.$$

Let a concentrated and instantaneous force $X_i = \delta(x-\xi)\,\delta(t)\,\delta_{ik}$ be acting at the point ξ of infinite medium, and at the point η the concentrated and instantaneous body couple $Y_i' = \delta(x-\eta)\,\delta(t)\,\delta_{ij}$. Denote by $\Omega_i^{(k)}(x, \xi, t)$ the angular vector caused by the action of force X_i, and by $V_i^{(j)}(x, \eta, t)$ the displacement caused by the body couple Y_i. From Eq. (4.12) we get

$$(5.5)\qquad V_k^{(j)}(\xi, \eta, t) = \Omega^{(k)}(\eta, \xi, t).$$

Finally, let a body couple $Y_i = \delta(x-\xi)\,\delta(t)\,\delta_{ik}$ be acting at the point ξ, and a source of heat $Q' = \delta(x-\eta)\,\delta(t)$ at the point η. Denote the temperature caused by the action of body couple by $\vartheta^{(k)}(x, \xi, t)$ and the angular vector caused by the action of the source Q' by $\Omega_i(x, \eta, t)$. From the theorem on reciprocity (4.13) we obtain the following relation

$$\vartheta^{(k)}(\eta, \xi, t) = -\frac{\eta_0 \varkappa}{\beta} \frac{\partial \Omega_k(\xi, \eta, t)}{\partial t}. \tag{5.6}$$

It can be shown that the relations (5.2), (5.4)—(5.6) hold for a finite body at homogeneous boundary conditions. Let us consider a finite body V and assume that the causes which set the medium in motion are defined by the boundary conditions. We shall seek to find the expression for the displacements u_i, angular vectors ω_i and temperature θ at an internal point $x \in V$ by means of integrals on the surface A which bounds the region V. These functions should satisfy the equations of motion, the extended equation of heat conductivity and the boundary conditions.

When deriving the formulae for the functions $u_i(x, t)$ $\omega_i(x, t)$, $\theta(x, t)$ we shall use the theorem on reciprocity (4.13). Assume, first, that quantities marked with primes refer to displacements $u_i' = U_i^{(k)}(x, \xi, t)$, angular vector $\omega_i' = \Omega_i^{(k)}(x, \xi, t)$ and temperature $\theta = \Theta^{(k)}(x, \xi, t)$ caused in an infinite medium by a concentrated and instantaneous force $X_i' = \delta(x - \xi)\,\delta(t)\,\delta_{ik}$, applied at the point ξ and directed along the axis x_k. Assuming non-existence of the body forces ($X_i = 0$), body couples ($Y_i = Y_i' = 0$) and heat sources ($Q' = Q = 0$) we obtain from (4.13) the following expression for the function $\dot{u}_k\ (x, t)$

$$\begin{aligned}\dot{u}_k(x, t) = \int_A dA(\xi) \int_0^t d\tau \Big\{ & p_i(\xi, \tau) \frac{\partial U_i^{(k)}(\xi, t-\tau)}{\partial \tau} - \\ & - p_i^{(k)}(\xi, t-\tau) \frac{\partial u_i(\xi, \tau)}{\partial \tau} + g_i(\xi, \tau) \frac{\partial \Omega_i^{(k)}(\xi, x, t-\tau)}{\partial \tau} - \\ & - g_i^{(k)}(\xi, x, t-\tau) \frac{\partial \omega_i(\xi, \tau)}{\partial \tau} + \frac{\beta}{\eta_0} [\theta(\xi, \tau)\, \Theta_{,n}^{(n)}(\xi, x, t-\tau) - \\ & - \Theta^{(k)}(\xi, x, t-\tau)\, \theta_{,n}(\xi, \tau)] \Big\}, \qquad x \in V, \quad \xi \in A.\end{aligned} \tag{5.7}$$

Here we have introduced the following notations:

$$p_i^{(k)} = \sigma_{ij}^{(k)}(x, \xi, t)\, n_j(x), \qquad g_i^{(k)} = m_{ji}^{(k)}(x, \xi, t)\, n_j(x),$$

where by $\sigma_{ij}^{(k)}$ we understand stresses, by $m_{ji}^{(k)}$ — couple-stresses caused by a concentrated force $X_i' = \delta(x - \xi)\ \delta(t)\ \delta_{ij}$. The integration operations under the sign of surface integrals are carried out with respect to the variable ξ. The formula (5.7) gives us the relation between the function $\dot{u}_k(x, t)$, $x \in V$, $t > 0$ and functions u_i, p_i, ω_i, g_i, θ, $\theta_{,n}$ on the surface A.

Now let us assume in the system with "primes" the action of a concentrated and instantaneous body couple $Y_i' = \delta(x - \xi)\,\delta(t)\,\delta_{ik}$ acting along the axis x_k. The body couple will cause in an infinite medium the displacement $u_i' = V_i^{(k)}(x, \xi, t)$, vector $\omega_i' = \Lambda_i^{(k)}\ (x, \xi, t)$ and temperature $\theta' = \vartheta^{(k)}\ (x, \xi, t)$. From the theorem on reciprocity (4.13), at $X_i = X_i' = 0$, $Y_i = 0$, $Q = Q' = 0$ we obtain the following formula:

$$(5.8) \qquad \dot{\omega}_k(x, t) = \int_A dA(\xi) \int_0^t d\tau \left\{ p_i(\xi, \tau) \frac{\partial V_i^{(k)}(\xi, x, t-\tau)}{\partial \tau} - \hat{p}_i^{(k)}(\xi, x, t-\tau) \frac{\partial u_i(\xi, \tau)}{\partial \tau} + g_i(\xi, \tau) \frac{\partial \Lambda_i^{(k)}(\xi, x, t-\tau)}{\partial \tau} - \hat{g}_i^{(k)}(\xi, x, t-\tau) \frac{\partial \omega_i(\xi, \tau)}{\partial \tau} + \frac{\beta}{\eta_0} [\theta(\xi, \tau) \vartheta_{,n}^{(k)}(\xi, x, t-\tau) - \vartheta^{(k)}(\xi, x, t-\tau) \theta_{,n}(\xi, \tau)] \right\}, \quad x \in V, \quad \xi \in A.$$

Here

$$\hat{p}_i^{(k)} = \hat{\sigma}_{ij}^{(k)}(x, \xi, t) n_j(x), \qquad \hat{g}_i^{(k)} = \hat{m}_{ji}^{(k)}(x, \xi, t) n_j(x).$$

We denote by $\hat{\sigma}_{ij}^{(k)}$ and $\hat{m}_{ji}^{(k)}$ the force-stress tensor and the couple-stress tensor. Also here the function $\dot{\omega}_k(x, t)$, $x \in V$, $t > 0$ is expressed by the functions u_i, p_i, ω_i, g_i, θ, $\partial\theta/\partial n$ on the surface A. Let now the system with "primes" in an infinite medium be limited to the action of a concentrated and instantaneous heat source $Q' = \delta(x-\xi)\delta(t)$ causing displacements $u_i' = U_i(x, \xi, t)$, rotation vector $\omega_i' = \Omega_i(x, \xi, t)$ and temperature $\theta' = \Theta(x, \xi, t)$.

From (4.13), assuming that $X_i = X_i' = 0$, $Y_i = Y_i' = 0$, $Q = 0$, we get the formula for temperature at the point $x \in V$.

$$(5.9) \qquad \theta(x, t) = \varkappa \int_A dA(\xi) \int_0^t d\tau \left\{ \theta_{,n}(\xi, \tau) \Theta(\xi, x, t-\tau) - \theta(\xi, \tau) \Theta_{,n}(\xi, x, t-\tau) - \frac{\eta_0}{\beta} \left[p_i(\xi, \tau) \frac{\partial U_i(\xi, x, t-\tau)}{\partial \tau} - p_i^*(\xi, x, t-\tau) \frac{\partial u_i(x, \tau)}{\partial \tau} + g_i(\xi, \tau) \frac{\partial \Omega_i(\xi, x, t-\tau)}{\partial \tau} - g_i^*(\xi, x, t-\tau) \frac{\partial \omega_i(x, \tau)}{\partial \tau} \right] \right\},$$

$$x \in V, \quad \xi \in A.$$

Here

$$p_i^* = \sigma_{ij}^*(x, \xi, t) n_j(x), \quad g_i^* = m_{ji}^*(x, \xi, t) n_j(x).$$

We denote by σ_{ji}^* and m_{ji}^* the force-stress tensor and couple-stress tensor caused by the action of an instantaneous and concentrated heat source Q'.

The formulae (5.7)–(5,9) may be treated as an extension of Somiglian's formulae [5], to the problems of thermoelasticity. Some simplifications of these formulae can be obtained by taking into account the reciprocity relation (5.4)–(5.6).

If the Green functions $U_i^{(k)}$, $\Omega_i^{(k)}$, $\Theta^{(k)}$, etc. are selected in such a way as to satisfy on the surface A the homogeneous boundary conditions for displacements angular vector and temperature, then the formulae (5.7)–(5.9) will yield the solution of the first boundary problem, when at the boundary there are given functions p_i, g_i and θ.

Similarly, if we select the Green functions $U_i^{(k)}$, $\Omega_i^{(k)}$ $\Theta^{(k)}$, etc., in such a way that the border is free from loads and temperature, then the formulae (5.7)—(5.9) will yield the solution of the second boundary problem, when on A are given loads p_i, g_i and temperature θ. Let us consider still the stationary problems. Let the body contained within the region V and bounded by the surface A be subjected to the action of heating. Let on the part A_u of the surface A, equalling zero appear displacements u_i and vector of rotation angle ω_i, and on the part A_σ of the surface A, equal to zero, appear functions p_i and g_i. Moreover, let $X_i = 0$.

For determining the displacement $u_i(x)$, $x \in V$ consider a body of the same shape and the same boundary conditions. Let in this body $\theta' = 0$ and let at the point ξ be acting a concentrated force $X_i' = \delta(x-\xi)\,\delta_{ik}$ which is, consequently, directed along the axis x_k. This force will cause displacements $U_i^{(k)}(x, \xi)$ assuming that the functions $U_i^{(k)}(x, \xi)$ are so selected as to satisfy homogeneous boundary conditions on A_u and A_σ.

Making use of the formula (4.14), we obtain

$$u_k(x) = \beta \int_V \theta(\xi)\, U_{i,i}^{(k)}(\xi, x)\, dV, \quad x \in V, \quad k = 1, 2, 3. \tag{5.10}$$

Here $U_{i,i}^{(k)}(\xi, x)$ should be treated as a dilatation caused at the point ξ by a concentrated force X_i applied at the point x. The formula (5.10) may be treated as a generalization of known W. M. Maysel's formula [6], for the problem of thermoelasticity in the Cossérat medium.

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В. НОВАЦКИЙ, **МОМЕНТОВЫЕ НАПРЯЖЕНИЯ В ТЕОРИИ ТЕРМОУПРУГОСТИ.** II.

В работе даются: основа виртуальных работ, энергетическая теорема, теорема об однозначности решений, а также теорема взаимности для термоупругих проблем в среде Коссерата.

Кроме того обсуждаются выводы, вытекающие из теоремы взаимности, приводящие к методу решения дифференциальных уравнений термоупругости при использовании функций Грина.