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APPLIED MECHANICS

Couples-stresses in the Theory of Thermoelasticity. I

by

W. NOWACKI

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1. Introduction. In the last few years considerable attention is given by researchers to the Cosserat medium. The theory of this medium, initiated by W. Voigt [1] and developed by brothers E. and F. Cosserat [2], takes into account an additional assumption that on the surface of a deformable body there act, besides surface forces, also couples. This theory finds practical application as regards granular media.

Truesdell and Toupin [3] and Aero and Kuvshinskiĭ [4] gave modern formulation of Cosserat medium equations. A number of theorems and methods of essential importance may be found in the works of Mindlin and Tiersten [5], and of Koiter [6].

The aim of the present paper was to extend the theory of Cosserat medium to problems of coupled thermoelasticity. We have derived constitutive equations on the basis of thermodynamics of irreversible processes, and fundamental differential equations of thermoelasticity. In part two of this paper we will present variational theorems, theorem on reciprocity, and conclusions resulting from it.

2. Equations of motion and boundary conditions. Equations of motion in the Cosserat medium lead to the system of two equations: the equation of balance of momentum and equation of momentum of momentum [2], [5], [6]

$$\sigma_{ji,j}+X_i-\varrho\ddot{u}_i=0, \tag{2.1}$$

$$\epsilon_{ijk}\sigma_{jk}+\mu_{ji,j}+Y_i=0, \quad i=1,2,3. \tag{2.2}$$

In these equations σ_{ij} is the force-stress tensor, μ_{ji} — the couple-stress tensor, X_i denotes components of the body force vector referred to a body unit, and Y_i — components of the body couple vector. Further, u_i denotes components of displacement vector, ϱ — the density, and ϵ_{ijk} is the well-known Cartesian ε-tensor.

Denoting by s_{ij} the symmetric, and by r_{ij} the asymmetric part of the force-stress tensor

$$s_{ij}=\frac{1}{2}(\sigma_{ij}+\sigma_{ji}), \quad r_{ij}=\frac{1}{2}(\sigma_{ij}-\sigma_{ji}), \quad \sigma_{ij}=s_{ij}+r_{ij}, \tag{2.3}$$

and representing the couple-stress tensor μ_{ij} as the sum of a spherical tensor and deviator

$$\mu_{ij} = \mu_0 \delta_{ij} + m_{ij}, \quad \mu_0 = \frac{1}{3}\mu_{ii}, \quad m_{ii} = 0, \tag{2.4}$$

we obtain after multiplying (2.2) by ϵ_{imn} the following expression for the asymmetric part of the tensor

$$r_{mn} = -\frac{1}{2}\epsilon_{imn}(\mu_{0,i} + m_{ji,j} + Y_i). \tag{2.5}$$

Inserting (2.3) – (2.5) into the first equation of motion (2.1), we get the final form of the equations of motion

$$s_{mn,m} - \frac{1}{2}\epsilon_{imn}[m_{ji,jm} + Y_{i,m}] + X_n = \varrho \ddot{u}_n. \tag{2.6}$$

To Eqs. (2.6) we should still add the boundary conditions and initial conditions. We shall represent the boundary conditions here in the form given by Koiter [6]:

$$\begin{aligned} &\left[s_{hl} + \frac{1}{2}\varepsilon_{hkl}(m_{jk,j} - m_{(nn),k} + Y_k)\right] n_h = \bar{p}_l, \\ &m_{jh} n_j - m_{(nn)} n_h = \bar{g}_h. \end{aligned} \tag{2.7}$$

We have introduced here the notation $m_{(nn)} = m_{ji} n_j n_j$ (no summation should be made after n).

Further, $\bar{p}_1$ denotes three reduced force tractions

$$\bar{p}_l = p_l - \frac{1}{2}\epsilon_{hkl} g_{(n),k} n_h, \quad g_{(n)} = g_l n_l, \tag{2.8}$$

and $\bar{g}_h$ – two tangential couple tractions

$$\bar{g}_h = g_h - g_{(n)} n_h. \tag{2.9}$$

The quantities p_i and g_i are connected with tensors σ_{ij} and μ_{ij} by the following relations

$$p_i = \sigma_{ji} n_j, \quad g_i = \mu_{ji} n_j, \tag{2.10}$$

where n_i are the components of the unit normal vector.

The magnitude μ_0 from (2.4) does not appear neither in the equations of motion nor in the boundary conditions (2.7) and (2.8). μ_0 is assumed to equal zero for reasons explained in [6].

The aforegoing equations and relations hold regardless of the mechanical and thermal properties of the material.

3. Energy equation and entropy balance. The principle of conservation of energy written for an arbitrary volume V of the body, bounded by a surface A, has the form

$$\begin{aligned} \frac{d}{dt}\int_V \left(\frac{1}{2}\varrho v_i v_i + U\right) dV = \int_V (X_i v_i + Y_i w_i)\, dV + \\ + \int_A (p_i v_i + g_i w_i)\, dA - \int_A q_i n_i\, dA. \end{aligned} \tag{3.1}$$

Here $v_i = \dot{u}_i$, $w_i = \dot{\omega}_i$, where $\omega_i = \frac{1}{2} \epsilon_{ijk} u_{k,j}$ is the component of the angular vector. By U we denote the internal energy, and by q_i — the component of the heat-flux vector.

The term on the left of (3.1) represents the rate of increase of the kinetic and internal energy of the volume. The first term on the right-hand side of the equation represents the rate of work of the body forces and couple-forces, the second term represents the rate of work of the surface tractions and couples. Finally, the last integral on the right-hand side of Eq. (3.1) denotes the heat transferred to the volume by heat conduction. Taking into consideration (2.1), (2.2) and (2.10), we obtain, making use of the divergence theorem, the following equation

$$\int_V (\dot{U} - s_{ij} v_{i,j} - \mu_{ji} w_{i,j} + q_{i,i})\, dV = 0. \tag{3.2}$$

This expression holds for each volume V. If the integrand is continuous, then the relation

$$\dot{U} = s_{ij} v_{i,j} + \mu_{ji} w_{i,j} - q_{i,i}, \tag{3.3}$$

holds locally. Introducing the strain tensor γ_{ij} and torsion-flexure tensor $\varkappa_{ij}$, where

$$\gamma_{ij} = \gamma_{ji} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varkappa_{ij} = \omega_{i,j} \tag{3.4}$$

we can represent (3.3) in the form

$$\dot{U} = s_{ij} \dot{\gamma}_{ij} + m_{ji} \dot{\varkappa}_{ij} - q_{i,i}. \tag{3.5}$$

The equation of entropy balance can be written in the form ([7], p. 29)

$$\int_V \dot{S} dV = - \int_A \frac{q_i n_i}{T} dA + \int_V \Theta dV. \tag{3.6}$$

The left-hand side of this equation represents the rate of increase of entropy. The first term on its right-hand side is the rate at which entropy is supplied to the volume across the surface. The second term on the right-hand side of (3.6) denotes the rate of production of entropy, due to heat conduction. Using the divergence theorem we have

$$\int_V \left(\dot{S} - \Theta - \left(\frac{q_i}{T} \right)_{,i} \right) dV = 0, \tag{3.7}$$

hence, because of assumed arbitrariness in assumption of volume, we get

$$\dot{S} = \Theta - \frac{q_{i,i}}{T} + \frac{q_i T_{,i}}{T^2}, \tag{3.8}$$

which holds for each point of the body. In accordance with the postulate of thermodynamics of irreversible processes we have $\Theta \geqq 0$.

Eliminating $q_{i,i}$ from (3.5) and (3.8), we get

$$\dot{U} = s_{ji}\dot{\gamma}_{ij} + m_{ji}\dot{\varkappa}_{ij} + T\dot{S} - T\left(\Theta + \frac{q_i T_{,i}}{T^2}\right). \tag{3.9}$$

Introducing the expression for the Helmholtz free energy $F = U - ST$, we obtain

$$\dot{F} = s_{ji}\dot{\gamma}_{ij} + m_{ji}\dot{\varkappa}_{ij} - \dot{T}S - T\left(\Theta + \frac{q_i T_{,i}}{T^2}\right). \tag{3.10}$$

Since the free energy is a function of independent variables γ_{ij}, $\varkappa_{ij}$, T there is

$$\dot{F} = \frac{\partial F}{\partial \gamma_{ij}}\dot{\gamma}_{ij} + \frac{\partial F}{\partial \varkappa_{ij}}\dot{\varkappa}_{ij} + \frac{\partial F}{\partial T}\dot{T}. \tag{3.11}$$

Assuming that the functions Θ, $q_i, \ldots, s_{ij}$, m_{ji} do not explicitly depend on time derivatives of the function γ_{ij}, $\varkappa_{ij}$, T, and defining the entropy as $S = -\dfrac{\partial F}{\partial T}$ we obtain, comparing (3.10) with (3.11), the following relations

$$s_{ij} = \frac{\partial F}{\partial \gamma_{ij}}, \quad m_{ji} = \frac{\partial F}{\partial \varkappa_{ij}}, \quad S = -\frac{\partial F}{\partial T}, \quad \Theta + \frac{q_i T_{,i}}{T^2} = 0. \tag{3.12}$$

The second law of thermodynamics will be satisfied if $\Theta \geqslant 0$ or if

$$\frac{-T_{,i}\, q_i}{T^2} \geqslant 0. \tag{3.13}$$

This inequality satisfies the Fourier law of thermal conductivity

$$-q_i = k_{ij} T_{,j} \quad \text{or} \quad -q_i = k_{ij}\theta_{,j}, \quad T = T_0 + \theta. \tag{3.14}$$

Here T_0 denotes the temperature for the natural state in which stresses and deformations are equal zero (i.e. for $\gamma_{ij} = 0$, $\varkappa_{ij} = 0$, $T = T_0$).

The quantities k_{ij} are coefficients of thermal conductivity and according to the principle of Onsager form a symmetric tensor. Function $k_{ij}T_{,i}T_{,j}$ is a positively determined quadratic function.

From Eq. (3.8) — taking into account the last relation of the group (3.12) — we have

$$T\dot{S} = -q_{i,i} = k_{ij}\theta_{,ij}. \tag{3.15}$$

For an isotropic and homogeneous body we get

$$T\dot{S} = k\theta_{,ji}, \tag{3.16}$$

where k is a constant value.

4. Constitutive equations. Let us develop the free energy $F(\gamma_{ij}, \varkappa_{ij}, T)$ in a natural state environment ($\gamma_{ij} = 0$, $\varkappa_{ij} = 0$, $T = T_0$) into a Mac Laurin series, omitting therein powers higher than 2. This development has, for an isotropic body, the form

$$F = \mu\gamma_{ij}\gamma_{ij} + \frac{\lambda}{2}(\gamma_{kk})^2 + \mu'\varkappa_{ij}\varkappa_{ij} + \mu''\varkappa_{ij}\varkappa_{ji} - \beta\gamma_{kk}\theta - \frac{m}{2}\theta^2. \tag{4.1}$$

This form results from the following considerations. Since the free energy is a scalar, each term at the right-hand side of the expression (4.1) should be a scalar, too. But, from the components of the symmetric tensor γ_{ij} we can construe two independent quadratic invariants, namely $\gamma_{ij}\gamma_{ij}$ and $\gamma_{(kk)}^2$. Since the first invariant of the torsion flexure tensor ($\varkappa_{kk} = 0$) vanishes, we have only three additional quadratic invariants $\gamma_{ij}\varkappa_{ij}$, $\varkappa_{ij}\varkappa_{ij}$ and $\varkappa_{ij}\varkappa_{ji}$. However, in Eq. (4.1) the term $\gamma_{ij}\varkappa_{ij}$ cannot appear, since it is contradictory with the postulate of isotropy. In the last but one term on the right-hand side of (4.1) appears the invariant γ_{kk}. This results from the fact that from the components of tensor γ_{ij} one can form only one invariant of the first kind, namely γ_{kk}. The term $\varkappa_{kk}\theta$ will not appear in (4.1) since $\varkappa_{kk} = \omega_{k,k} = \frac{1}{2}\varepsilon_{kmn}\omega_{n,mk} = 0$.

Making use of relations (3.12), we have

$$s_{ij} = 2\mu\gamma_{ij} + (\lambda\gamma_{kk} - \beta\theta)\,\delta_{ij}, \tag{4.2}$$

$$m_{ji} = 4\mu l^2(\varkappa_{ij} + \eta\varkappa_{ji}), \quad \mu' = 2\mu l^2, \quad \mu'' = 2\mu l^2\eta. \tag{4.3}$$

The quantities μ, λ are Lamé's constants, and l^2, η — new material constants.

These constants refer to the isothermal state. Since the function F is determined positively, $\mu > 0$, l is a real quantity, $0 < \nu < \frac{1}{2}$ and $-1 < \eta < 1$. Here $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ is the Poisson's ratio. Solving (4.2) with respect to γ_{ij} and (4.3) with respect to $\varkappa_{ij}$, we obtain

$$\gamma_{ij} = \frac{1}{2\mu}s_{ij} - \frac{\lambda s_{kk}}{2\mu(2\mu+3\lambda)}\delta_{ij} + \frac{\beta}{3K}\theta\delta_{ij}, \tag{4.4}$$

$$\varkappa_{ij} = \frac{1}{4\mu l^2(1-\eta^2)}(m_{ji} - m_{ij}). \tag{4.5}$$

Here $K = \lambda + \dfrac{2}{3}\mu$.

Let us contract the relation (4.4), Then

$$\gamma_{kk} = \frac{s_{kk}}{3K} + \frac{\beta\theta}{K}. \tag{4.6}$$

The physical meaning of the quantity β/K will become clear when considering the thermal deformation of the volume element, free from stresses on its surface. Namely, $\dfrac{\beta}{K} = \alpha$, where α is the coefficient of thermal expansion. After introduction of the coefficient of linear thermal expansion $\alpha_t = \dfrac{\alpha}{3}$ Eq. (4.2) will take the form

$$s_{ij} = 2\mu\gamma_{ij} + (\lambda\gamma_{kk} - \beta\theta)\,\delta_{ij}, \quad \beta = 3K\alpha_t. \tag{4.7}$$

Notice, moreover, that

$$\left(\frac{\partial s_{ij}}{\partial T}\right)_\gamma = -\beta\delta_{ij}, \quad \left(\frac{\partial\gamma_{kk}}{\partial s_{kk}}\right)_T = \frac{1}{3K}, \quad \left(\frac{\partial\gamma_{kk}}{\partial T}\right)_\gamma = 3\alpha_t. \tag{4.8}$$

Let us determine also the internal energy U and entropy S. For this purpose we shall make use of the differential relation resulting from the second law of thermodynamics

(4.9) $$dU = s_{ij}\,d\gamma_{ij} + m_{ji}\,d\varkappa_{ij} + TdS.$$

Substituting into (4.9) the relation

(4.10) $$dS = \left(\frac{\partial S}{\partial \gamma_{ij}}\right)_{\varkappa,T} d\gamma_{ij} + \left(\frac{\partial S}{\partial \varkappa_{ij}}\right)_{\gamma,T} d\varkappa_{ij} + \left(\frac{\partial S}{\partial T}\right)_{\gamma,\varkappa} dT,$$

and taking into account completness conditions of dU, we arrive at the dependence

(4.11) $$\left(\frac{dS}{\partial \gamma_{ij}}\right)_{\varkappa,T} + \left(\frac{\partial s_{ij}}{\partial T}\right)_{\gamma,\varkappa} = \left(\frac{\partial S}{\partial \gamma_{ij}}\right)_{\varkappa,T} - \beta\delta_{ij} = 0, \qquad \left(\frac{\partial S}{\partial \varkappa_{ij}}\right)_{\gamma,T} = 0.$$

Substituting the above formula into (4.9) and (4.10) and taking into account $\left(\frac{\partial S}{\partial T}\right)_{\gamma,\varkappa} = \frac{c_\varepsilon}{T}$, where c_ε denotes specific heat at constant deformation, we get

(4.12) $$dU = s_{ji}\,d\gamma_{ij} + m_{ji}\,d\varkappa_{ij} + T\beta d\gamma_{kk} + c_\varepsilon\,dT,$$

(4.13) $$dS = \beta d\gamma_{kk} + \frac{c_\varepsilon}{T}\,dT.$$

Substituting (4.2) into (4.12) and integrating this expression with the assumption that $U = 0$ for the natural state of the body ($\gamma_{ij} = \varkappa_{ij} = 0,\ T = T_0$) we obtain

(4.14) $$U = W + \gamma_{kk}\,\beta T_0 + c_\varepsilon\,\theta,$$

wherein

$$W = \mu\gamma_{ij}\,\gamma_{ij} + \frac{\lambda}{2}(\gamma_{kk})^2 + 4\mu l^2 [\varkappa_{ij}\,\varkappa_{ij} + \eta\varkappa_{ij}\,\varkappa_{ji}].$$

Similarly, integrating (4.13) with the assumption that $S = 0$ for the natural state, we have

(4.15) $$S = \beta\gamma_{kk} + c_\varepsilon \log\frac{T}{T_0}.$$

In the internal energy formula (4.14) the first term represents deformation work, the last — heat content, whereas the last but one term — interaction of the deformation field and temperature. In the expression for entropy the purely elastic term is lacking. Note that in the expression for entropy there are no invariants of the tensor $\varkappa_{ij}$. The formula for free energy has the form:

(4.16) $$F = U - ST = W - \gamma_{kk}\,\beta\theta + c_\varepsilon\,\theta - Tc_\varepsilon \log\frac{T}{T_0}.$$

Assuming $\left|\frac{\theta}{T}\right| \ll 1$ and developing the logarithm into a series, as well as retaining two terms of this solution, we have

$$F = W - \gamma_{kk}\beta\theta - \frac{c_\varepsilon}{2T_0}\theta^2. \tag{4.17}$$

From comparison of (4.1) and (4.17) it results that $m = \frac{c_\varepsilon}{T_0}$.

5. Differential equations of thermoelasticity. The constitutive relations (4.2) and (4.3) enable to express the equations of motion (2.6) by displacements. Expressing s_{ij} by γ_{ij}, and m_{ji} by $\varkappa_{ij}$, and $\varkappa_{ji}$ and functions γ_{ij} and $\varkappa_{ij}$ by displacements, we represent Eq. (2.6) in the form

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - \mu l^2[u_{i,jj} - u_{j,ji}]_{,kk} + X_i - \tfrac{1}{2}\epsilon_{ijk}Y_{j,k} = \varrho\ddot{u}_i + \beta\theta_{,i}. \tag{5.1}$$

The effect of temperature is characterized here by the term $\beta\theta_{,i}$ at the right-hand side of the equation. For $l = 0$, $Y_i = 0$ Eqs. (5.1) undergo transformation into known equations of thermoelasticity in an elastic medium without couple-stresses. Note that in Eq. (5.1) there appears no coefficient η. Integrating Eq. (5.1) with respect to x_i and applying contraction, we obtain

$$\Box_1^2 e + \frac{1}{c_1^2\varrho}X_{i,i} = m\nabla^2\theta, \quad m = \frac{\beta}{c_1^2\varrho}, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad e = \gamma_{kk}. \tag{5.2}$$

To Eqs. (5.1) we have to add the equation of heat conductivity. For this purpose consider the relations (3.16) and (4.15). Here we have

$$T\dot{S} = k\theta_{,jj}, \tag{5.3}$$

$$T\dot{S} = T\beta\dot{\gamma}_{kk} + c_\varepsilon\dot{T}. \tag{5.4}$$

It results from comparison of these relations that

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta_0\left(1+\frac{\theta}{T_0}\right)\dot{u}_{k,k} = 0, \quad \varkappa = \frac{k}{c_\varepsilon}, \quad \eta_0 = \frac{\beta T_0}{k}. \tag{5.5}$$

Linearizing this equation, we assume that θ/T_0 is a small quantity as compared with unity. Taking, moreover, into account heat sources within the body and denoting by W the quantity of heat generated in the volume and time unit, we obtain the following extended equation of thermal conductivity

$$\theta_{,jj} - \frac{1}{\varkappa}\dot{\theta} - \eta_0\dot{\gamma}_{kk} = -\frac{Q}{\varkappa}, \quad Q = \frac{W}{k}. \tag{5.6}$$

Interesting here is the fact that in Eq. (5.5) appears only the term derived from dilatation, consequently connected with the first deformation invariant γ_{ij} and with the invariant of symmetric tensor s_{ij}.

Eqs. (5.1) and (5.5) constitute a set of equations of linear coupled thermoelasticity in the Cosserat medium. To the equations of thermoelasticity we should add boundary conditions (2.8) and (2.9) and boundary conditions connected with thermal conductivity (the temperature or heat flow on the surface A bounding the body being prescribed).

Dynamic equations of thermoelasticity (5.1) and (5.6) can be separated by decomposition of displacement and body force vector into potential and selenoidal part

$$u_i = \Phi_{,i} + \epsilon_{ijk}\psi_{k,j}, \qquad X_i = \varrho(\vartheta_{,i} + \epsilon_{ijk}\chi_{k,j}). \tag{5.7}$$

Inserting (5.7) into (5.1) and (5.6), we obtain a system of five equations:

$$\square_1^2 \Phi = m\theta - \frac{1}{c_1^2}\vartheta. \tag{5.8}$$

$$\mathring{\square}_2^2 \psi_i = -\frac{1}{c_2^2}\left(\chi_i + \frac{1}{2}Y_i\right), \tag{5.9}$$

$$D\theta - \eta_0 \nabla^2 \dot{\Phi} = -\frac{Q}{\varkappa}. \tag{5.10}$$

We have introduced here the following notations

$$\square_1^2 = \nabla^2 - \frac{1}{c_1^2}\partial_t^2, \qquad \mathring{\square}_2^2 = (1 - l^2\nabla^2)\nabla^2 - \frac{1}{c_2^2}\partial_t^2,$$

$$D = \nabla^2 - \frac{1}{\varkappa}\partial_t, \qquad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}, \qquad m = \frac{\beta}{\varrho c_1^2}.$$

If we eliminate temperature from Eqs. (5.8) and (5.10) we get equations characterizing propagation of longitudinal and transversal waves

$$\left[\square_1^2 D - \frac{\varepsilon}{\varkappa}\partial_t \nabla^2\right]\Phi = -\frac{mQ}{\varkappa} - \frac{1}{c_1^2}\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right)\vartheta, \qquad \varepsilon = \eta m\varkappa. \tag{5.11}$$

$$\mathring{\square}_2^2 \psi_i = -\frac{1}{c_2^2}\left(\chi_i + \frac{1}{2}Y_i\right). \tag{5.12}$$

Consider propagation of thermoelastic waves in an unbounded medium. Let in an unbounded space be no forces Y_i and components $X_i = \epsilon_{ijk}\chi_{k,j}$ and let the initial conditions for the function ψ_i be homogeneous. In this case in the Cosserat medium propagates only the longitudinal wave expressed by Eq. (5.11). It may be caused by heat sources, body forces $X_i = \varrho\vartheta_{,i}$ and non-homogeneous initial conditions of the function Φ. The longitudinal wave is accompanied by heat generation, the temperature θ being determined from Eq, (5.8)

$$\theta = \frac{1}{m}\left(\square_1^2 \Phi + \frac{1}{c_1^2}\vartheta\right). \tag{5.13}$$

In this case we have

$$u_i = \Phi_{,i}, \qquad \gamma_{ij} = \Phi_{,ij}, \qquad \omega_i = \frac{1}{2}\epsilon_{ijk}u_{k,j} = 0, \qquad \varkappa_{ij} = 0. \tag{5.14}$$

and

$$s_{ij} = 2\mu(\Phi_{ij} - \delta_{ij}\nabla^2\Phi) + \varrho(\ddot{\Phi} - \vartheta), \qquad m_{ji} = 0, \tag{5.15}$$

The longitudinal wave causes changes of volume of elements ($\gamma_{kk} = \nabla^2 \Phi \neq 0$) and symmetric components of force-stress tensor.

The longitudinal wave equation (5.11) is identical with the wave equation of elastic medium without couple-stresses.

If in the unbounded Cosserat medium $\vartheta = 0$, $Q = 0$ and the initial conditions of the function Φ are homogeneous, then in every point of the medium is $\Phi \equiv 0$. In the medium will arise only transversal waves characterized by Eq. (5.12). Since in this case $u_i = \epsilon_{ijk}\,\psi_{k,j}$, $u_{i,i} = \gamma_{kk} = 0$, the elements of the body undergo the change of form. In this medium we have

$$s_{ij} = \mu(u_{i,j} + u_{j,i}), \qquad m_{ji} = 4\mu l^2(\varkappa_{ij} + \eta\varkappa_{ji}). \tag{5.16}$$

Propagation of transversal waves causes no change of temperature; this follows from Eq. (5.13), which, at $\Phi = 0$, $\vartheta = 0$, gives $\theta = 0$. In the case $Y_i = 0$, $l = 0$, Eqs. (5.12) are transformed into known equations of elastokinetics. Note that Eqs. (5.12) were discussed in [5].

If we have to do with a bounded medium, the solution of Eqs. (5.11) and (5.9) will be composed of two parts, namely of particular integrals Φ' and ψ_i' of non--homogeneous equations and of general solutions Φ'', ψ_i'' satisfying homogeneous equations

$$[\Box_1^2 D - \eta_0 m \partial_t \nabla^2]\Phi'' = 0, \tag{5.17}$$

$$\mathring{\Box}_1^2 \psi_i'' = 0. \tag{5.18}$$

The system of fundamental integral equations (5.1) and (5.3) may be separated in a different way.

Let us represent the equation of thermoelasticity in a different form, convenient for further considerations,

$$L_{ij}(u_j) + L_{i4}(\theta) = -F_i, \tag{5.19}$$

$$L_{4i}(u_i) + L_{44}(\theta) = -\frac{Q}{\varkappa}, \tag{5.20}$$

where

$$L_{ij} = \mathring{\Box}_2^2 \delta_{ij} + aG\partial_i\partial_j, \quad L_{i4} = -\beta_0\partial_i, \quad L_{4i} = -\eta_0\partial_t\partial_i, \quad L_{44} = D,$$

$$\mathring{\Box}_2^2 = \nabla^2(1 - l^2\nabla^2) - \frac{1}{c_2^2}\partial_t^2, \quad D = \nabla^2 - \frac{1}{\varkappa}\partial_t, \quad G = 1 + \frac{l^2}{a}\nabla^2,$$

$$F_i = \frac{1}{\mu}\left(X_i - \frac{1}{2}\epsilon_{ijk}Y_{j,k}\right), \quad \beta_0 = \frac{\beta}{\mu}, \quad a = \frac{\lambda+\mu}{\mu}.$$

Let us introduce the vector function φ_i and scalar function ζ connected with displacements u_i and with temperature θ by the following relations,

$$u_i = (\Omega\delta_{ij} - \Gamma\partial_i\partial_j)\varphi_j + \beta_0\partial_i\zeta, \tag{5.21}$$

$$\theta = \eta_0\partial_t\partial_j\mathring{\Box}_2^2\varphi_j + H\zeta, \tag{5.22}$$

where

$$\Omega = DH - \beta_0 \eta_0 \partial_t \nabla^2, \quad \Gamma = aGD - \beta_0 \eta_0 \partial_t, \quad H = \mathring{\Box}_2^2 + aG\nabla^2.$$

Inserting (5.21) and (5.22) into Eqs. (5.19) and (5.20), we obtain a system of four already separated equations

$$\mathring{\Box}_2^2 (DH - \eta_0 \beta_0 \partial_t \nabla^2) \varphi_i + F_i = 0, \tag{5.23}$$

$$(DH - \eta_0 \beta_0 \partial_t \nabla^2) \zeta + \frac{Q}{\varkappa} = 0, \tag{5.24}$$

or also

$$\mathring{\Box}_2^2 (\Box_1^2 D - \eta_0 m \partial_t \nabla^2) \varphi_i + \frac{1}{c_1^2 \varrho} \left(X_i - \frac{1}{2} \epsilon_{ijk} Y_{j,k} \right) = 0, \tag{5.25}$$

$$(\Box_1^2 D - \eta_0 m \partial_t \nabla^2) \zeta + \frac{Q\mu}{c_1^2 \varrho \varkappa} = 0, \tag{5.26}$$

where

$$\Box_1^2 = \nabla^2 - \frac{1}{c_1^2} \partial_t^2.$$

Functions φ_i may be treated as a Galerkin's vector function generalized to thermoelastic problems. For $Y_i = 0$, $l = 0$ Eqs. (5.25) and (5.26) transform into known coupled thermoelasticity equations of elastic medium without couples.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA, INSTITUTE OF FUNDAMENTAL TECHNICAL PROBLEMS, POLISH ACADEMY OF SCIENCES

(ZAKŁAD MECHANIKI OŚRODKÓW CIĄGŁYCH, INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI, PAN)

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В. НОВАЦКИЙ, **МОМЕНТОВЫЕ НАПРЯЖЕНИЯ В ТЕОРИИ ТЕРМОУПРУГОСТИ. I.**

В работе выведены конститутивные уравнения термоупругости для упругой среды Коссерата на базисе термодинамики необратимых процессов. Дается полный состав дифференциальных уравнений термоупругости, а также приводятся два метода разъединения этой системы уравнений.